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Multiple functors. III. The cartesian closed category $\text{Cat}_n$


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MULTIPLE FUNCTORS

III. THE CARTESIAN CLOSED CATEGORY $\mathbb{Cat}_n$

by Andrée and Charles EHRESMANN

INTRODUCTION.

This paper is Part III of our work on multiple functors [4, 5] and it is a direct continuation of Part II. It is devoted to an explicit description of the cartesian closed structure on $\mathbb{Cat}_n$ (= category of $n$-fold categories) which will be «laxified», in the Part IV [6] (this is a much more general result than that announced in Part I). The existence of such structures might be deduced from general theorems on sketched structures [7,14], but this does not lead to concrete definitions. Here the construction uses the monoidal closed category $(\Pi\mathbb{Cat}_n, \bullet, \text{Hom})$ of multiple categories defined in Part II.

In the cartesian closed category $\mathbb{Cat}$, the internal Hom functor maps $(A, C)$ onto the category of natural transformations from $A$ to $C$, which is identified with the category $\text{Hom}(A, \Box C)$, where $\Box C$ is the double category of squares of $C$.

To generalize this situation, the idea is to construct a functor $\Box_n$ from $\mathbb{Cat}_n$ to $\mathbb{Cat}_{2n}$ (which reduces for $n = 1$ to the functor $\Box : \mathbb{Cat} \to \mathbb{Cat}_2$), whose composite with the functor $\text{Hom}(A, \cdot) : \mathbb{Cat}_{2n} \to \mathbb{Cat}_n$ gives, for each $n$-fold category $A$, the partial internal Hom functor of the cartesian closed structure of $\mathbb{Cat}_n$. In fact, we first define a pair of adjoint functors $\text{Square}$ and $\text{Link}$ between $\mathbb{Cat}_n$ and $\mathbb{Cat}_{n+1}$, which has also some interest of its own; iteration of this process leads to a functor $\Box_n : \mathbb{Cat}_n \to \mathbb{Cat}_{2n}$ whose left adjoint maps $B \Box A$ onto the product $B \times A$, for each $n$-fold category $B$. Hence the functor

$\text{Hom}(A, \Box_n \cdot) : \mathbb{Cat}_n \to \mathbb{Cat}_n$

is a right adjoint of the product functor $\cdot \times A$, as desired.
The delicate point is the explicit construction of $Link$, which is a left inverse of $Square$. The category of components of a 2-category, as well as the crossed product category associated to the action [8] of a category on a category, appear as examples of $LinkA$.

Finally $Cat_n$ is embedded as the category of 1-morphisms in the $(n+1)$-category $Nat_n$ of hypertransformations (or natural transformation between natural transformations, between...), whose $n$ first categories form the $n$-fold category coproduct of $Hom_n(A, B)$, for any $n$-fold categories $A, B$. The construction of $Nat_n$ uses the equivalence (see Appendix) between categories enriched in a category $V$ with commuting coproducts (in the sense of [21]) and categories internal to $V$ whose object of objects is a coproduct of copies of the final object.

**NOTATIONS.**

The notations are those introduced in Part II. In particular, if $B$ is an $n$-fold category, $B^i$ denotes its $i$-th category for each integer $i < n$, and $B^{i_0, \ldots, i_{p-1}}$, for each sequence $(i_0, \ldots, i_{p-1})$ of distinct integers $i_j < n$, is the $p$-fold category whose $j$-th category is $B^{i_j}$.

Let $A$ be an $m$-fold category. The square product $B \square A$ is the $(n+m)$-fold category on the product set $B \times A$ (where $B$ always denotes the set of blocks of $B$) whose $i$-th category is:

$$B^{dis} \times A^i \quad \text{for} \quad i < m, \quad B^{i-m} \times A^{dis} \quad \text{for} \quad m \leq i < n + m$$

($B^{dis}$ is the discrete category on $B$).

If $m < n$, then $Hom(A, B)$ is the $(n-m)$-fold category on the set of multiple functors $f: A \to B$ (i.e., on the set of $m$-fold functors $f$ from $A$ to $B^{0, \ldots, m-1}$) whose $j$-th composition is deduced pointwise from that of $B^{m+j}$, for each integer $j < n - m$.

The category $\Pi_n Cat_n$ of (all small) multiple categories, equipped with $\square$ and $Hom$ is monoidal closed (Proposition 7 [5]), i.e., the partial functor $Hom(A, -); Cat_{n+m} \to Cat_n$ is right adjoint to $\square A; Cat_n \to Cat_{n+m}$. 
A. The adjoint functors Square and Link.

This Section is devoted to the construction of the functor Square from \( \text{Cat}_n \) to \( \text{Cat}_{n+q} \), and of its left adjoint, the functor Link. For \( n = 1 \), the functor Square reduces to the functor \( \Box : \text{Cat} \rightarrow \text{Cat}_2 \), whose definition is first recalled to fix the notations.

2 is always the category

\[
\begin{array}{ccc}
1 & \rightarrow & (1,0) & \rightarrow & 0 \\
\end{array}
\]

so that \( 2 \times 2 \) is represented by the commutative diagram:

\[
\begin{array}{ccc}
(1,1) & \rightarrow & (z,1) & \rightarrow & (0,1) \\
(1,z) & \rightarrow & (z,z) & \rightarrow & (0,z) \\
(1,0) & \rightarrow & (z,0) & \rightarrow & (0,0) \\
\end{array}
\]

( where \( z = (0,1) \)).

Let \( C \) be a category. A functor \( f : 2 \times 2 \rightarrow C \) is entirely determined by the (commutative) square of \( C \):

\[
\begin{array}{ccc}
f(z,1) & = & f(0,z) \\
f(1,z) & = & f(z,0) \\
\end{array}
\]

( since \( f(z,z) \) is the «diagonal» of this square:

\[
f(z,1) f(0,z) = f(1,z) f(z,0)
\]

and every square \( (c', \hat{c}', \hat{c}, c) \)

of \( C \) is obtained in this way. So we shall identify the set \( \text{Hom}(2 \times 2 , C) \) of
functors from $2 \times 2$ to $C$ with the set of squares of $C$.

On this set, the «vertical» and the «horizontal» compositions:

$$(\hat{c}', \hat{c}^*, \hat{c}', \hat{c}) \sqcup (c', \hat{c}'', \hat{c}, c) = (\hat{c}' c', \hat{c}'', \hat{c}, \hat{c} c)$$

$$(c^*, \hat{c}', \hat{c}, c') \boxtimes (c', \hat{c}'', \hat{c}, c) = (c'', \hat{c}' \hat{c}', \hat{c} \hat{c}, c)$$

define categories $\boxtimes C$ and $\sqsubset C$ (which are both isomorphic, and also called by some authors category of arrows of $C$). The couple $(\boxtimes C, \sqsubset C)$ is the double category $\boxtimes C$ of squares of $C$.

The functor $\Box : \text{Cat} \to \text{Cat}_2$, maps $g : C \to C'$ onto $\Box g : \Box C \to \Box C' : (c', \hat{c}', \hat{c}, c) \mapsto (g(c'), \hat{g}(\hat{c}'), \hat{g}(\hat{c}), \hat{g}(c)).$

Now let $n$ be an integer, $n > 1$. Let $B$ be an $n$-fold category. Taking for $C$ above the $0$-th category $B^0$ of $B$, we have, on the set of squares of $B^0$ (to which are identified the functors $2 \times 2 \to B^0$), not only the double category $\Box B^0$, but also the $(n-1)$-fold category $\text{Hom}(2 \times 2, B)$, whose $i$-th composition (deduced pointwise from that of $B^{i+1}$) is written with squares:

$$(b'_i, \hat{b}'_i, \hat{b}_i, b_i)_{\circ i} (b'_i, \hat{b}'_i, \hat{b}_i, b_i) = (b'_{i+1} \circ_i b'_i, \hat{b}'_{i+1} \circ_i \hat{b}'_i, \hat{b}_{i+1} \circ_i \hat{b}_i, b_{i+1} \circ_i b_i),$$

iff the four composites are defined in $B^{i+1}$.

**DEFINITION.** The multiple category of squares of $B$, denoted by $\text{Sq}B$, is the $(n+1)$-fold category on the set of squares of $B^0$ such that:

$$(\text{Sq}B)^0, \ldots, n-2 = \text{Hom}(2 \times 2, B), (\text{Sq}B)^{n-1} = \Box B^0, (\text{Sq}B)^n = \sqsubset B^0$$

(the $(n-1)$ first compositions are those of $\text{Hom}(2 \times 2, B)$, the two last ones being the vertical and the horizontal compositions of squares).

To «visualize» this multiple category $\text{Sq}B$, we shall also represent a square $b' \overset{\hat{b}'}{\underset{\hat{b}}{b}}$ of $B^0$ by $b' \overset{\hat{b}'}{\underset{\hat{b}}{b}}$;
then the compositions of \( SqB \) are represented by:

\[
\begin{array}{c}
\hat{b}_1' \quad i \quad \hat{b}' \\
\hat{b}' \quad \downarrow^{n-1} \quad \hat{b} \\
\hat{b} \quad \quad \downarrow^n \quad b
\end{array}
\]

\[
\begin{array}{c}
\hat{b}'' \\
\hat{b}' \\
\hat{b}'
\end{array}
\]

REMARK (not used afterwards). The construction of \( SqB \) may be interpreted in terms of sketched structures. To each category \( \phi : \sigma \to V \) internal to a category \( V \) with pullbacks, it is associated a category \( \partial \phi : \sigma \to V \) internal to \( V^\sigma \) (Proposition 28 [7]). If \( \phi : \sigma \to Cat_{n-1} \) is the category in \( Cat_{n-1} \) canonically associated to \( B^1, \ldots, n-1,0 \) (Appendix, Part II [5]), then

\[
\sigma \xrightarrow{\partial \phi} Cat_{n-1} \xrightarrow{\sim} Cat_n
\]

is the category in \( Cat_n \) associated to \( SqB \).

There is a functor from \( Cat_n \) to \( Cat_{n+1} \), called the functor \( Square \), and denoted by

\[
Sq_{n,n+1} : Cat_n \to Cat_{n+1}
\]

which maps an \( n \)-fold functor \( g : B \to B' \) onto the \((n+1)\)-fold functor

\[
Sqg : SqB \to SqB' : (b', \hat{b}', \hat{b}, b) \mapsto (g(b'), g(\hat{b}'), g(\hat{b}), g(b))
\]

(defined by \( \Box g : \Box B^0 \to \Box B^0 \)).

**Proposition 1.** The functor \( Sq_{n,n+1} : Cat_n \to Cat_{n+1} \) admits a left adjoint \( Lk_{n+1,n} : Cat_{n+1} \to Cat_n \).

**Proof.** The proof, quite long, will be decomposed in several steps. Let \( A \) be an \((n+1)\)-fold category, \( a^i \) and \( \beta^i \) the maps source and target of \( A^i \) for each integer \( i \leq n \).

1° We define an \( n \)-fold category, called the multiple category of \((n-1, n)\)-links of \( A \), denoted by \( LkA \) (later on, it will be proved that \( LkA \) is the free object generated by \( A \) with respect to the functor \( Square \)).

a) Consider the graph \( G \) whose vertices are those blocks \( e \) of \( A \) which are objects for the two last categories \( A^{n-1} \) and \( A^n \), and whose edges \( a : e \to e' \) from \( e \) to \( e' \) are the blocks \( a \) of \( A \) such that:
b) Let \( P^0 \) be the set \( P \) of all paths of the graph \( G \) (i.e. sequences \((a_k, \ldots, a_0)\), where \( a_i : e_i \rightarrow e_{i+1} \) in \( G \)), equipped with the concatenation:

\[
(a^I_1, \ldots, a^I_0) \circ_0 (a_k, \ldots, a_0) = (a^I_1, \ldots, a^I_0, a_k, \ldots, a_0)
\]

iff \( a^n a^{n-1}(a^I_0) = \beta^n \beta^{n-1}(a_k) \).

\( P^0 \) is an associative but non-unitary category (called a quasi-category in [10], where \( P^0 \) is shown to be the free quasi-category generated by \( G \)).

c) For each integer \( i \) with \( 0 \leq i < n-1 \), there is a category \( P^{i+1} \) on \( P \) whose composition is deduced «pointwise» from that of \( A^i \), which means:

\[
(\hat{a}_I, \ldots, \hat{a}_0) \circ_{i+1} (a_k, \ldots, a_0) = (\hat{a}_k \circ_i a_k, \ldots, \hat{a}_0 \circ_i a_0)
\]

iff \( l = k \) and the composites \( \hat{a}_j \circ_i a_j \) are defined in \( A^i \), for \( j \leq k \).
d) Consider on the set $P$ of all the paths of $G$ the relation $r$ defined as follows:

$$(R1) \quad (a) \sim (\beta^n a, a^{n-1} a) \sim (\beta^{n-1} a, a^n a) \quad \text{for each block } a \text{ of } A.$$ 

$$(R2) \quad (u', u) \sim (u' \circ_{n-1} u) \quad \text{iff } (u', u) \text{ is a couple of objects of } A^n \text{ whose composite exists in the category } A^{n-1}.$$ 

$$(R3) \quad (\hat{u}', \hat{u}) \sim (\hat{u}' \circ_n \hat{u}) \quad \text{iff } (\hat{u}', \hat{u}) \text{ is a couple of objects of } A^{n-1} \text{ whose composite exists in the category } A^n.$$ 

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (beta) at (1,0) {$\beta^n a$};
  \node (an) at (2,0) {$a^n a$};
  \node (R2) at (-1,-0.5) {$R2$};
  \node (R3) at (3,-0.5) {$R3$};
  \draw[->] (a) -- (beta); \draw[->] (beta) -- (an);
  \node (u) at (0,-2) {$u$};
  \node (u_hat) at (1,-2) {$\hat{u}'$};
  \node (u_prime) at (2,-2) {$\hat{u}$};
  \node (u_prime_prime) at (0,-3) {$u' \circ_{n-1} u$};
  \node (u_hat_hat) at (1,-3) {$\hat{u}' \circ_n \hat{u}$};
  \draw[->] (u) -- (u_prime_prime); \draw[->] (u_prime_prime) -- (u_hat); \draw[->] (u_hat) -- (u_hat_hat);
\end{tikzpicture}
\end{center}

e) According to the proof of Proposition 3 [5], there exists an $n$-fold category (called the multiple category of $(n-1,n)$-links of $A$, denoted by $LkA$) quasi-quotient of $P = (P^0, P^1, \ldots, P^{n-1})$ by $r$ and such that the canonical morphism $\tilde{r}: P \to LkA$ defines a quasi-functor $\tilde{r}: P^0 \to LkA^0$ and a functor $\tilde{r}: P^i \to LkA^i$ for $1 \leq i < n$. The image $\tilde{r}(a_k, \ldots, a_0)$ is denoted by $[a_k, \ldots, a_0]$; those blocks generate $LkA$ ($\tilde{r}$ may not be onto).

There is an $(n+1)$-fold functor $l: A \to Sq(LkA)$ which maps a block $a$ of $A$ onto the square $l(a)$ of $(LkA)^0$ such that

$$l(a) = \begin{bmatrix}
\beta^{n-1} a \\
[\beta^n a] \\
a^{n-1} a
\end{bmatrix}$$

(intuitively, $l(a)$ is the frame of $a$ in the double category $(A^{n-1}, A^n)$).

a) The map $l$ is well-defined: The relation $(R1)$ has been introduced so that $l(a)$ be a commutative square of $(LkA)^0$, since

$$[\beta^{n-1} a]_0 \circ_0 [a^n a] = [(\beta^{n-1} a, a^n a)] = [R1] [\beta^n a]_0 \circ_0 [a^{n-1} a].$$
b) For \(0 \leq i < n-1\) the map \(l\) defines a functor \(l: A^i \to Sq(LkA)^i\):
The \(i\)-th composition of \(Sq(LkA)\) is deduced "pointwise" from the \((i+1)\)-th composition of \(LkA\), which is itself deduced "pointwise" from the composition of \(A^i\). Suppose the composite \(a' \circ_i a\) defined in \(A^i\); as \(a^n: A^i \to A^i\) is a functor, we have

\[
[a^n(a' \circ_i a)] = [(a^n(a')) \circ_i (a^n a)] = [a^n a'] \circ_{i+1} [a^n a];
\]
similar equalities are valid if we replace \(a^n\) by \(\beta^n\), by \(a^{n-1}\) or by \(\beta^{n-1}\).

Hence:

\[
l(a' \circ_i a) = [\beta^n(a' \circ_i a)] = [a^n(a' \circ_i a)] = [\beta^{n-1}(a' \circ_i a)] = [a^{n-1}(a' \circ_i a)]
\]

\[
= [\beta^n a'] \circ_i [a^n a] = [\beta^{n-1} a] \circ_i [a^{n-1} a] = \beta^{n-1}(a') \circ_i \beta^n(a).
\]

c) The relation (R2) implies that \(l: A^{n-1} \to (Sq(LkA))^{n-1}\) is a functor: By definition,

\[(Sq(LkA))^{n-1} = (LkA)^0.\]

Suppose \(a^n \circ_{n-1} a\) defined in \(A^{n-1}\). As \(a^n: A^{n-1} \to A^{n-1}\) is a functor,

\[
[a^n(a^n \circ_{n-1} a)] = [a^n(a^n) \circ_{n-1} a^n(a)] = [(a^n a^n, a^n a)] = R_2
\]

\[
= [a^n a^n] \circ_0 [a^n a];
\]

and similarly with \(a^n\) replaced by \(\beta^n\). Moreover:

\[
[a^{n-1}(a^n \circ_{n-1} a)] = [a^{n-1} a], \quad [\beta^{n-1}(a^n \circ_{n-1} a)] = [\beta^n a^n].
\]
It follows that
\[ l(a'' \circ_{n-1} a) = l(a'') \circ l(a). \]

d) Using the relation \((R3)\) instead of \((R2)\) it is proved analogously that \(l: A^n \to (Sq(LkA))^n = \triangle LkA\) is a functor.

3° \(l: A \to Sq(LkA)\) is the liberty morphism defining \(LkA\) as the free object generated by \(A\) with respect to \(Sq_{n,n+1}: \text{Cat}_n \to \text{Cat}_{n+1}\).

Indeed, let \(B\) be an \(n\)-fold category and \(g: A \to SqB\) an \((n+1)\)-fold functor.

a) The «diagonal map» \(d\) sending a square \(s\) of \(B^0\) onto its diagonal defines an \((n-1)\)-fold functor
\[ d: (SqB)^0,\ldots,n-2 \to B^1,\ldots,n-1; \]
This map \(d\) sends the square
\[
\begin{array}{c}
\hat{b}' \\
\Downarrow \\
\hat{b} \\
\Updownarrow \\
\hat{b}' \\
\end{array}
\]
of \(B^0\) onto
\[ d(s) = \hat{b}' \circ_0 b = b' \circ_0 \hat{b}. \]
For each integer \(i < n-1\), the composition of \((SqB)^i\) is deduced pointwise from that of \(B^{i+1}\). As \(B\) is an \(n\)-fold category, the \(0\)-th and \((i+1)\)-th compositions of \(B\) satisfy the permutability axiom \((P)\). Hence, if \(s_1 \circ_i s\) is defined in \((SqB)^i\), then
\[
\begin{array}{c}
\hat{b}' \circ_{i+1} \hat{b}' \\
\Downarrow \\
\hat{b}_1 \circ_{i+1} \hat{b}_1 \\
\Updownarrow \\
\hat{b}_1 \circ_{i+1} \hat{b}_1 \\
\end{array}
\]
d\((s_1 \circ_i s)\) = \((\hat{b}' \circ_{i+1} \hat{b}') \circ_0 (\hat{b}_1 \circ_{i+1} \hat{b}_1) = (\hat{b}' \circ_0 b_1) \circ_{i+1} (\hat{b}' \circ_0 b_1) \]
\[ = d(s_1) \circ_{i+1} d(s). \]
b) There is a unique morphism \(h: P \to B\) extending the composite
(n-1)-fold functor
\[ A^0, \ldots, n^{-2} \xrightarrow{g} (SqB)^0, \ldots, n^{-2} \xrightarrow{d} B^1, \ldots, n^{-1} : \]
The edge \( a : e \to e' \) of the graph \( G \) is mapped by \( dg \) onto the morphism
\[ dg(a) : dg(e) \to dg(e') \] of \( B^0 \).
There is a unique quasi-functor \( h : P^0 \to B^0 \) extending \( dg \) (by the universal property of \( P^0 \)) and \( h \) sends the path \( p = (a_k, \ldots, a_0) \) onto the composite:
\[ h(p) = dg(a_k) \circ \cdots \circ dg(a_0). \]
For \( 0 \leq i < n-1 \), the composition of \( P^{i+1} \) is deduced pointwise from that of \( A^i \) and \( dg : A^i \to B^{i+1} \) is a functor; it follows that \( h : P^{i+1} \to B^{i+1} \) is a functor. Hence, \( h : P \to B \) is a morphism.

c) \( h : P \to B \) is compatible with the relation \( r \) used to define \( LkA : \)
If \( a \) is a block of \( A \), the square \( g(a) \) of \( B^0 \) will be denoted by:
\[ g(a) = \]
\[ \hat{b}_a' \]
\[ \hat{b}_a \]
\[ b_a \]
\[ b_a' \]

As \( g : A^{n-1} \to \boxtimes B^0 \) is a functor, \( g(a^{n-1}a) \) is the vertical source of the square \( g(a) \), and its diagonal \( h(a^{n-1}a) \) is equal to \( \hat{b}_a \). Similarly, \( h(\beta^n a) = b_a' \), since \( g : A^n \to \boxtimes B^0 \) is a functor. Therefore,
\[ h(a) = \hat{b}_a' \circ \hat{b}_a = h(\beta^n a) \circ h(a^{n-1}a) = h(\beta^n a, a^{n-1}a). \]
In an analogous way, we get

\[ h(a) = \hat{b}_a \circ \circ_0 b_a = h(\beta^{n-1}a, a^n a). \]

This proves that \( h \) is compatible with (R1).

- Let the composite \( u' \circ_{n-1} u \) be defined in \( A^{n-1} \), with \( u \) and \( u' \) objects of \( A^n \). Applying the functor \( g: A^{n-1} \to \square B^0 \), we have

\[ g(u' \circ_{n-1} u) = g(u') \oplus g(u). \]

As \( g: A^n \to \square B^0 \) is a functor, it maps the objects \( u \) and \( u' \) of \( A^n \) onto objects of \( \square B^0 \) whose diagonals are

\[ h(u) = b_u \quad \text{and} \quad h(u') = b_{u'}. \]

The composite \( g(u' \circ_{n-1} u) = g(u') \oplus g(u) \) is also an object of \( \square B^0 \) whose diagonal is \( b_{u' \circ_0 b_u} \). It follows that

\[ h(u' \circ_{n-1} u) = d(g(u') \oplus g(u)) = b_u \circ_0 b_u = h(u') \circ_0 h(u) = h(u', u). \]

Hence \( h \) is compatible with (R2). The compatibility with (R3) is proved by a similar method.

d) By the universal property of the quasi-quotient \( LkA \) of \( P \) by \( r \), there exists a unique \( n \)-fold functor \( \hat{g}: LkA \to B \) factorizing the morphism \( h: P \to B \) compatible with \( r \) through the canonical morphism \( \hat{r}: P \to LkA \):

\[ \begin{array}{c}
    \text{B} \\
    \hat{g} \\
    \text{LkA} \\
    \text{P}
\end{array} \]

It maps the block \([a_k, \ldots, a_0]\) of \( LkA \) onto \( h(a_k) \circ_0 \cdots \circ_0 h(a_0) \). In particular, for each block \( a \) of \( A \), we have

\[ \hat{g}([a^n a]) = h(a^n a) = b_a, \quad \hat{g}([\beta^n a]) = b'_a, \]

\[ \hat{g}([a^{n-1} a]) = \hat{b}_a, \quad \hat{g}([\beta^{n-1} a]) = \hat{b}'_a. \]

These equalities imply that \( Sq \hat{g}: Sq(LkA) \to SqB \) maps
e) Suppose that $\hat{g}': LkA \to B$ is an $n$-fold functor such that

$$(S\hat{g}'(l(a))) = g(a)$$

for each block $a$ of $A$.

In particular, this implies that $\hat{g}'(u) = b_u$ for each object $u$ of $A^n$, and $\hat{g}'(\hat{u}) = \hat{b}_u$ for each object $\hat{u}$ of $A^{n-1}$. Then:

$$\hat{g}'\hat{f}(a) = \hat{g}'[\beta^n a, a^{n-1} a] = \hat{g}'[\beta^n a] \circ_0 \hat{g}'[a^{n-1} a] = \hat{b}_{a} \circ_0 b_a = h(a),$$

i.e., the two morphisms

$$h: P \to B \quad \text{and} \quad (P \xrightarrow{f} LkA \xrightarrow{\hat{g}'} B)$$

have the same *restriction* to the graph $G$. By the unicity of $h$ (see b), it follows that they are equal, and $\hat{g}: LkA \to B$ is their unique factor through
\( \tilde{f} \). Hence, \( \hat{g}' = \hat{g} \).

f) This proves that \( LkA \) is the free object generated by \( A \). The corresponding left adjoint of \( Sq_{n,n+1} : \text{Cat}_n \to \text{Cat}_{n+1} \), denoted by

\[ Lk_{n+1,n} : \text{Cat}_{n+1} \to \text{Cat}_n, \]

maps the \((n+1)\)-fold functor \( f : A \to A' \) onto \( Lkf : LkA \to LkA' \) such that

\[ (Lkf)[a_k, \ldots, a_0] = [f(a_k), \ldots, f(a_0)]. \]

DEFINITION. The functor \( Lk_{n+1,n} : \text{Cat}_{n+1} \to \text{Cat}_n \) defined above is called the \text{Link} functor from \( \text{Cat}_{n+1} \) to \( \text{Cat}_n \).

COROLLARY 1. The functor \( \Box : \text{Cat} \to \text{Cat}_2 \) admits as a left adjoint the \text{Link} functor from \( \text{Cat}_2 \) to \( \text{Cat} \). \( \Box \)

By iteration, for each integer \( m \), we define the functor \( Sq_{n,n+m} = \)

\[ \text{Cat}_n \xrightarrow{Sq_{n,n+1}} \text{Cat}_{n+1} \ldots \text{Cat}_{n+m-1} \xrightarrow{Sq_{n+m-1,n+m}} \text{Cat}_{n+m} \].

COROLLARY 2. The functor \( Sq_{n,n+m} \) admits as a left adjoint the functor \( Lk_{n+m,n} = \)

\[ \text{Cat}_{n+m} \xrightarrow{Lk_{n+m,n+m-1}} \text{Cat}_{n+m-1} \ldots \text{Cat}_{n+1} \xrightarrow{Lk_{n+1,n}} \text{Cat}_n \]. \( \Box \)

DEFINITION. \( Sq_{n,n+m} \) will be called the \text{Square} functor, from \( \text{Cat}_n \) to \( \text{Cat}_{n+m} \), and \( Lk_{n+m,n} \) the \text{Link} functor from \( \text{Cat}_{n+m} \) to \( \text{Cat}_n \).

These functors (for \( n = m \)) will be used as essential tools in Section C to describe the cartesian closed structure on \( \text{Cat}_n \).

B. Some examples concerning double categories.

1° The category of links of a double category.

By Corollary 1, Proposition 1, the functor \( \Box : \text{Cat} \to \text{Cat}_2 \) admits as a left adjoint the functor \( \text{Link} \) from \( \text{Cat}_2 \) to \( \text{Cat} \). If \( A \) is a double category \((A^0, A^1)\), the category of its links \( LkA \) may also be described as follows:

Let \( G \) be the graph associated to \( A \) in Proposition 1, whose vertices are the vertices \( e \) of \( A \) and whose edges \( a : e \to e' \) are the blocks
Let $L$ be the free category generated by this graph; its objects are the vertices of $A$ and its other morphisms are the «reduced» (i.e., with no factor a vertex) paths $(a_k, \ldots, a_0)$ of $G$. Let $R$ be the equivalence relation compatible with the composition of $L$ generated by the relation $r$ (introduced in Proposition 1):

$$(a) - (\beta^0 a, a^1 a) - (\beta^1 a, a^0 a),$$

for each block $a$ of $A$ which is not a vertex,

$$(u', u) - u' \circ_0 u, \quad \text{for } u' \text{ and } u \text{ objects of } A^1,$$

$$(\hat{u}', \hat{u}) - \hat{u}' \circ_{-1} \hat{u}, \quad \text{for } \hat{u}' \text{ and } \hat{u} \text{ objects of } A^0.$$
A\textsuperscript{0} or A\textsuperscript{1} and two successive factors are not objects of the same category. Any morphism \((a_k, \ldots, a_0)\) of \(L\) is equivalent modulo \(R\) to at least one simple path. Indeed,

\[ (a_k, \ldots, a_0) \rightarrow (\beta_1 a_k, a^0 a_k, \ldots, \beta_1 a_0, a^0 a_0) \]

if this path is reduced; otherwise, there exist successive factors of this path, \((v_{j+m}, \ldots, v_j)\), which are objects of the same category \(A_i\); in this case, we replace \((v_{j+m}, \ldots, v_j)\) by its composite \(v_{j+m} \circ \ldots \circ v_j\). The sequence thus obtained is a simple path, equivalent to \((a_k, \ldots, a_0)\) modulo \(R\). Hence the morphisms of \(L = LkA\) are of the form \([v_1, \ldots, v_0]\), where \((v_1, \ldots, v_0)\) is a simple path. Remark that two different simple paths may be equivalent modulo \(R\), as shows the example of the double category \(2 \htp 2\) which has only one non-degenerate block \(a\): 

and in which

\((\beta_1 a, a^0 a)\) and \((\beta_0 a, a^1 a)\) are two simple paths which are equivalent modulo \(R\).

Remark. With the general hypotheses of Proposition 1, to each path \(p\) of \(G\) is also associated a «simple path» defined as above (with \(A^0\) and \(A^1\) replaced by \(A^{n-1}\) and \(A^n\)), and which is mapped by \(\tilde{r}: P \rightarrow LkA\) onto the same block than \(p\). But the compositions of \(LkA\) other than the 0-th one are not expressed easily on these simple paths.

2° Fibrations as categories of links.

Let \(F: C \rightarrow \text{Cat}\) be a functor, where \(C\) is a small category (\(F\) is also called «une espèce de morphismes» [8]).
a) \( F \) determines an action \( \kappa' \) of the category \( C \) on the category \( S \) coproduct of the categories \( F(u) \), for all objects \( u \) of \( C \), defined by:
\[
\kappa'(c, s) = F(c)(s) \text{ (written } cs \text{ )}
\]
iff \( c: u \to u' \) in \( C \) and \( s \) in \( F(u) \).

Conversely, each action of a (small) category on a (small) category corresponds in this way to a functor toward \( \text{Cat} \) (see Chapter II [8]).

b) To \( F \) (or to the action \( \kappa' \) of \( C \) on \( S \)) is also associated a double functor \( h: \Sigma \to (\mathcal{C}^{\text{dis}}, \mathcal{C}) \) defined as follows:
- Let \( h: \Sigma^1 \to \mathcal{C} \) be the discrete fibration (or «foncteur d'hypermorphisme» in the terminology of [8]) associated to the action \( \kappa' \) of \( C \) on the set \( \mathcal{S} \) of morphisms of \( S \): the morphisms of \( \Sigma^1 \) are the couples \( (c, s) \) such that the composite \( \kappa'(c, s) = cs \) is defined; the composition of \( \Sigma^1 \) is:
\[
(c', s') \circ_1 (c, s) = (c'c, s) \text{ iff } s' = cs.
\]

\[\begin{array}{ccc}
\downarrow h & & \downarrow S_0 \\
\Sigma & \xrightarrow{\kappa'} & S \\
\end{array}\]

The object \((u, s)\) of \( \Sigma^1 \) is identified with the morphism \( s \) of \( S \). The functor \( h: \Sigma^1 \to \mathcal{C} \) maps \((c, s)\) onto \( c \).
- There is another category \( \Sigma^0 \) with the same set \( \Sigma \) of morphisms than \( \Sigma^1 \), whose composition is:
\[
(\hat{c}, \hat{s}) \circ_0 (c, s) = (\hat{c}, \hat{s}s) \text{ iff } c = \hat{c} \text{ and } \hat{s}s \text{ defined in } S.
\]
The couple \((\Sigma^0, \Sigma^1)\) is a double category \( \Sigma \), and \( h: \Sigma \to (\mathcal{C}^{\text{dis}}, \mathcal{C}) \) is a double functor.

c) By the construction of b, we obtain every double functor \( f: T \to K \) satisfying the two conditions:
(F 1) The 0-th category of \( K \) is discrete;
(F 2) The functor $f : T^1 \to K^1$ is a discrete fibration.

A double functor $f : T \to K$ satisfies (F 2) iff it is a discrete fibration internal to $\text{Cat}$ (i.e., a realization in $\text{Cat}$ of the sketch of discrete fibrations given in $0$-$D$ [4]), and then it is in 1-1 correspondence with a category action in $\text{Cat}$ (in the sense of [4], page 22).

The category actions in $\text{Cat}$ have been introduced in 1963 [9] under the name «catégories $\mathcal{F}$-structurées d'opérateurs» or «$\mathcal{F}$-espèces de morphismes»; in this Note, it was also indicated that the actions of a category on a category (or the functors toward $\text{Cat}$) are in 1-1 correspondence with the discrete fibrations internal to $\text{Cat}$ over a double category whose 0-th category is discrete.

d) To $F$ (or to the action $\kappa'$ of $C$ on $S$) is also associated the (non-discrete) fibration $h' : X \to C$, where $X$ is the crossed product category defined as follows (see Chapter II [8]);

- The morphisms of $X$ are the triples $(s, c, e)$ such that $e$ is an object of $S$, the composite $ce = \kappa'(c, e)$ is defined and $s : c e \to e'$ is a morphism of $S$. The composition of $X$ is:

$$ (s', c', e')(s, c, e) = (s'(c's), c'c', e) \text{ iff } s : c e \to e'. $$

- The category $X$ is generated by the morphisms of one of the forms:

$(e', s, \hat{e})$, where $s : \hat{e} \to e'$ in $S$, identified with $s$,

$(c e, c, e)$, denoted by $(c, e)$.

The functor $h' : X \to C$ maps $(s, c, e)$ onto $c$.

Different characterizations of $X$ have been indicated [15,16,17], and fibrations are of a great actuality [20,2]. Another characterization of $X$ is given now:
PROPOSITION 2. Let $h: \Sigma \to \left(C^{\text{dis}}, C\right)$ be the discrete fibration internal to $\text{Cat}$ associated (in b) to the action $\kappa'$ of $C$ on the category $S$. Then $Lk\Sigma$ is isomorphic with the crossed product category $X$.

PROOF. 1° Each morphism of the category $Lk\Sigma$ is of the form $[s, (c, e)]$, where $(s, c, e)$ is a morphism of $X$.

Indeed, the objects of $\Sigma^1$ are the morphisms of $S$, those of $\Sigma^0$ are the couples $(c, e)$, where $e$ is an object of $S$. So a simple path $p$ is of the form $p = (s_k, (c_k, e_k), \ldots, s_0, (c_0, e_0))$, where $s_i : c_i e_i \to e_{i+1}$ in $S$, for each $i \leq k$. We have

$$((c_{i+1}, e_{i+1}), s_i) - (c_{i+1}, s_i) - (c_{i+1}, s_i, (c_{i+1}, c_i e_i))$$

in the equivalence relation $R$ defining $Lk\Sigma$ as a quotient of the category of paths (we use the «simplified» construction of $Lk\Sigma$ given in 1-B above). Moreover, in $R$, we have also:

$$(s_{i+1}, c_{i+1} s_i, (c_{i+1}, c_i e_i), (c_i, e_i)) - (s_{i+1}, (c_{i+1}, s_i), (c_{i+1}, c_i, e_i))$$

By iteration it follows that $p = (s, (c, e_0))$ where

$$s = s_k(c_k s_{k-1}) \ldots (c_k \ldots c_1)s_0, \quad c = c_k \ldots c_0.$$ 

Since each morphism of $Lk\Sigma$ is of the form $[p]$ for some simple path $p$, it is also of the form $[s, (c, e)]$, as announced.

2° There is a double functor $g: \Sigma \to \Box X$ mapping $(c, s)$ onto the square
whose diagonal \( d(g(c, s)) \) is \( (c s, c, e) \). Since \( Lk \Sigma \) is a free object generated by \( \Sigma \) with respect to \( \square: Cat \to Cat_2 \), there corresponds to \( g \) a unique functor \( \tilde{g}: Lk \Sigma \to X \) which maps \( [s,(c,e)] \) onto

\[
\begin{align*}
\text{This functor is 1-1 and onto, hence it is an isomorphism, whose inverse} \\
\tilde{g}^{-1}: X \to Lk \Sigma \text{ maps } (s,c,e) \text{ onto } [s,(c,e)].
\end{align*}
\]

**Corollary.** With the hypotheses of Proposition 2, \( X \) is a free object generated by \( \Sigma \) with respect to \( \square: Cat \to Cat_2 \).

**Remark.** The category of links of \( (\zeta \text{dis}, C) \) is identified with \( C \), so that

\[
Lk h: Lk \Sigma \to Lk(\zeta \text{dis}, C) = C
\]

is a fibration isomorphic with \( h': X \to C \). This suggests the following generalization of Chapter II [8]: Let \( f: T \to K \) be any discrete fibration internal to \( Cat \). The functor \( Lkf: Lk T \to Lk K \) «plays the role» of the fibration associated to the action of a category on a category. In particular, the equivalence classes of the sections of the functor \( Lkf \) could be called «classes of cohomology of \( f \) of order 1».

3° The multiple category of links of an \((n+1)\)-category.

An \((n+1)\)-fold category is called an \((n+1)\)-category \( A \) if the objects of \( A^n \) are also objects of \( A^{n-1} \). For \( n = 1 \), this reduces to the usual notion of a 2-category. For \( n = 2 \), an example of a 3-category is provided by the 3-category of cylinders of a 2-category [1].

Let \( A \) be an \((n+1)\)-category. Those blocks of \( A \) which are objects for \( A^{n-1} \) define an \( n \)-fold subcategory of \( A^{0,\ldots,n-2,n} \), denoted by
There exists (Proposition 3 [5]) an n-fold category quasi-quotient of $|A^{n-1}|$ by the relation:

$$a^{n-1}a - \beta^{n-1}a$$

for each block $a$ of $A$;

it will be called the $n$-fold category of components of $A$, denoted by $\Gamma A$.

The canonical n-fold functor $\tilde{\rho}: |A^{n-1}| \to \Gamma A$ may not be onto, but its image generates the n-fold category $\Gamma A$. Remark that two objects of $A^{n-1}$ which are in the same component of $A^{n-1}$ have the same image by $\tilde{\rho}$.

EXAMPLE. Let $A$ be a 2-category; then $|A^0|_1 = |A^0|$ is the category of 1-morphisms of $A$; the equivalence relation $\rho$ generated on it by the relation (considered above):

$$v^0a - \beta^0a$$

for each block (or 2-cell) $a$ of $A$

is defined by:

$$v - v' \text{ iff } v \text{ and } v' \text{ are in the same component of } A^0.$$

Since $\rho$ is compatible with the composition of $|A^0|^1$, the category $\Gamma A$ of components of $A$ is then the category quotient of $|A^0|$ by $\rho$. So its morphisms are the components of $A^0$, and $\tilde{\rho}: |A^0| \to \Gamma A$ is onto. It is this example which explains the name given to $\Gamma A$.

PROPOSITION 3. Let $A$ be an $(n+1)$-category, $\Gamma A$ the n-fold category of its components. Then $LkA$ is isomorphic to $(\Gamma A)^{n-1,0,...,n-2}$, which is deduced from $\Gamma A$ by a permutation of the compositions.

PROOF. 1° The n-fold category $LkA$ is generated by those blocks $[v]$, where $v$ is an object of $A^{n-1}$: With the notations of Proposition 1, Proof, 1°, $LkA$ is generated by the blocks $[a]$, where $a$ is a block of $A$, and

$$(a) \sim (\beta^n a, a^{n-1}a) \sim (\beta^n a \circ a^{n-1}a) = (a^{n-1}a),$$

or more simply $|A^{n-1}|$. 

$\square$
since $\beta^n a$ is also an object of $A^{n-1}$; so $[a] = [a^{n-1} a]$. 

20 There exists an $n$-fold functor $\tilde{\Psi}: lkA \to (\Gamma A)^{n-1,0,...,n-2,n}$ such that $\tilde{\Psi}(v) = \tilde{\rho}(v)$ for each object $v$ of $A^{n-1}$, where $\tilde{\rho}: |A^{n-1}| \to \Gamma A$ is the canonical $n$-fold functor.

a) For each $n$-fold category $B$, the $n$-fold subcategory of the $n$-fold category $(SqB)^0,...,n-2,n$ formed by the objects of $(SqB)^{n-1} = \square B^0$ (which are degenerate squares) is isomorphic with $B^1,...,n-1,0$, by the isomorphism mapping $b: e \to e'$ in $B^0$ onto the degenerate square

$$b = \begin{array}{c} e' \square \varepsilon \varepsilon \end{array}$$

(since the composition of $(SqB)^i$, for $i < n-1$, is deduced pointwise from that of $B^{i+1}$ and $(SqB)^n = \square B^0$).

In particular, let $B$ be the $n$-fold category $(\Gamma A)^{n-1,0,...,n-2}$; then $B^1,...,n-1,0 = \Gamma A$ and $B^0 = (\Gamma A)^{n-1}$, so that the map

$$b \mapsto b = \begin{array}{c} e' \square \varepsilon \varepsilon \end{array}$$

(where $b: e \to e'$ in $(\Gamma A)^{n-1}$) defines an $n$-fold functor

$$\tilde{\Psi}: \Gamma A \to (SqB)^0,...,n-2,n.$$

b) There is an $(n+1)$-fold functor $g: A \to SqB: a \mapsto \tilde{\rho}(a^{n-1} a)^\square$.

Indeed, the composite $n$-fold functor
A^{0,\ldots,n-1,n} \overset{a^{n-1}}{\longrightarrow} |A^{n-1}|^{0,\ldots,n-2,n} \overset{\tilde{\rho}}{\longrightarrow} \Gamma A \overset{\equiv}{\longrightarrow} (\text{Sq}B)^{0,\ldots,n-2,n}

is defined by the map \( g: a \mapsto \tilde{\rho}(a^{n-1} a) \equiv \). The map \( \tilde{\rho} a^{n-1} \) is constant on each component of \( A^{n-1} \) (by definition of \( \tilde{\rho} \)) and \( \equiv \) takes its values in the set of objects of \( (\text{Sq}B)^{n-1} = \equiv(\Gamma A)^{n-1} \); whence the functor

\[
g: A^{n-1} \to (\text{Sq}B)^{n-1}.
\]

c) To \( g: A \to \text{Sq}B \) is canonically associated (by the adjunction between the \textit{Square} and \textit{Link} functors) the \( n \)-fold functor \( \hat{g}: \text{Lk}A \to B \) which maps \([v]\) onto the diagonal \( \tilde{\rho}(v) \) of \( g(v) = \tilde{\rho}(v) \equiv \) for each object \( v \) of \( A^{n-1} \) (Proof, Proposition 1).

\[
\begin{array}{ccc}
\hat{g} & & g \\
\downarrow & & \downarrow \\
\text{Lk}A & \to & \text{B} \\
\end{array}
\]

\[
\begin{array}{ccc}
\hat{g} & & g \\
\downarrow & & \downarrow \\
\text{SqLkA} & \to & \text{SqB} \\
\end{array}
\]

\[
3^o \hat{g}: \text{Lk}A \to B \text{ is an isomorphism and its inverse is constructed as follows, using the universal property of } \Gamma A:
\]

a) There is an \( n \)-fold functor

\[
g': |A^{n-1}|^{0,\ldots,n-2,n} \to (\text{Lk}A)^{1,\ldots,n-1,0}: v \mapsto [v]:
\]

the \((i+1)\)-th composition of \( \text{Lk}A \) being deduced pointwise from that of \( A^i \), the map \( g': v \mapsto [v] \) defines a functor from the \( i \)-th category \( |A^{n-1}|^i \) of \( |A^{n-1}|^{0,\ldots,n-2,n} \) to \( (\text{Lk}A)^{i+1} \) for \( i \leq n-2 \); it defines also a functor from \( |A^{n-1}|^n \) to \((\text{Lk}A)^0\), since

\[
\begin{array}{ccc}
A^i & \to & |v'| \\
\downarrow & & \downarrow \quad |v'\ |
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
\Lambda^n & \to & (\text{Lk}A)^0 \\
\end{array}
\]

\[
(v_1, v) \sim (v_1 \circ_n v) \text{ for } v \text{ and } v_1 \text{ objects of } A^{n-1}
\]

implies

\[
g'(v_1 \circ_n v) = [v_1 \circ_n v] = [v_1] \circ_0 [v] = g'(v_1) \circ_0 g'(v).
\]

b) There is an \( n \)-fold functor \( \hat{g}' : B \to \text{Lk}A \) such that \( \hat{g}' \tilde{\rho}(v) = [v] \)
for each object $v$ of $A^{n-1}$: For each block $a$ of $A$, we have
\[ g'(a^{n-1}a) = [a^{n-1}a] = [\beta^{n-1}a] = g'(\beta^{n-1}a). \]
So $g'$ is compatible with the relation by which $\Gamma A$ is the quasi-quotient of $|A^{n-1}|$. It follows that $g'$ factors uniquely through $\overline{\rho}$ into an $n$-fold functor
\[
\Gamma A \xrightarrow{g'} (LkA)^{1,\ldots,n-1,0} \xrightarrow{g'} |A^{n-1}|.
\]
\[ \hat{g}' : \Gamma A \to (LkA)^{1,\ldots,n-1,0}. \]
After permutation of the compositions, we have also the $n$-fold functor
\[ \tilde{g}' : B = (\Gamma A)^{n-1,0,\ldots,n-2} \to LkA. \]

(c) $\hat{g}'$ is the inverse of $\tilde{g}$: For each object $v$ of $A^{n-1}$,
\[ \hat{g}' \tilde{g}(v) = \tilde{g}'\overline{\rho}(v) = g'(v) = [v] \]
and
\[ \tilde{g} \hat{g}' \overline{\rho}(v) = \tilde{g}'[v] = \overline{\rho}(v). \]
As the blocks $[v]$ generate $LkA$, it follows that $\hat{g}'\tilde{g}$ is an identity; similarly, $\tilde{g}\hat{g}'$ is an identity, since the image of $\overline{\rho}$ generates $\Gamma A$ (and a fortiori $B$). Hence $\tilde{g}' = \tilde{g}^{-1}$. \( \Box \)

**COROLLARY.** If $A$ is a 2-category, $LkA$ is isomorphic to the category $\Gamma A$ of components of $A$.

In this case, the preceding proof may be simplified: as $\Gamma A$ is the quotient category of $|A^0|^1$ by the relation «in the same component», it follows directly that $\tilde{g}$ is 1-1 and onto (hence an isomorphism). \( \Box \)

4° **The category of links of a multiple category of squares.**

If $A$ is a 2-category, $Q(A)$ denotes the double category of its up-squares (Section 2 [4])

\[
s = u' \xrightarrow{a} v' \xrightarrow{b} u \quad (a : u' @ v \to v' @ u \text{ in } A^0),
\]
the 0-th and 1-st compositions being the vertical and horizontal compositions of up-squares:

\[
\begin{array}{c}
\text{The objects of } Q(A)^{\sqcap} \text{ and } Q(A)^{\sqcup} \text{ are respectively the degenerate squares} \\
v^{\sqcap} = \begin{array}{c}
\begin{array}{c}
\text{v} \\
\text{v}
\end{array}
\end{array} \text{ and } v^{\sqcup} = \begin{array}{c}
\begin{array}{c}
\text{v} \\
\text{v}
\end{array}
\end{array}
\end{array}
\]

where \(v\) is a 1-morphism of \(A\).

**Proposition 4.** Let \(A\) be a 2-category; then \(LkQ(A)\) is isomorphic to the category \(\Gamma A\) of components of \(A\).

**Proof.** 1° Each morphism of \(LkQ(A)\) is of the form \([v^{\sqcap}]\), where \(v\) is a 1-morphism of \(A\). Indeed, \(Q(A)\) being a double category, \(LkQ(A)\) may be constructed by the method of 1-B as the quotient category \(L/ R\) of a category \(L\) of paths by the equivalence \(R\) defined in 1-B. For each up-square

\[
s = \begin{array}{c}
\begin{array}{c}
u' \\
v
\end{array}
\end{array}
\]

we have successively:

\[
(s) \sim (u^{\sqcup}, v^{\sqcap}) \sim (u, v) \sim ((u \circ_1 v)^{\sqcap}, e^{\sqcup}) \sim ((u \circ_1 v)^{\sqcap})
\]

modulo \(R\), since \(e^{\sqcup} = e^{\sqcap}\) is an object for the two categories of \(Q(A)\). Each morphism of \(LkQ(A) = L/ R\) is of the form \([s_k, \ldots, s_0]\), and

\[
(s_k, \ldots, s_0) \sim ((u_k \circ_1 v_k)^{\sqcap}, \ldots, (u_0 \circ_1 v_0)^{\sqcap}) \sim ((u_k \circ_1 v_k)^{\sqcap} \sqcup \ldots \sqcup (u_0 \circ_1 v_0)^{\sqcap}) = (w^{\sqcap}),
\]
where \( w = u_0 \circ_1 v_k \circ_1 \cdots \circ_1 u_0' \circ_1 v_0 \); hence \([s_k, \ldots, s_0] = [w^{\Box}]\).

2° a) There is a functor \( g : Q(A) \rightarrow \Gamma A : \)

\[
\begin{array}{ccc}
| & & \downarrow a \\
\downarrow u' & & \downarrow <u'> \\
\downarrow v & & \downarrow <u> \\
\end{array}
\]

where \( <u> \) denotes the component of \( u \) in \( A^0 \); indeed, \( u' \circ_1 v \) and \( v' \circ_1 u \), being the source and target of \( a \) in \( A^0 \), are in the same component of \( A^0 \), so that, in \( \Gamma A \),

\( <u'> <v> = <u' \circ_1 v> = <v' \circ_1 u> = <v'> <u> \).

b) To \( g : Q(A) \rightarrow \Gamma A \) corresponds (by the adjunction between the functors \( Link \) and \( \Box \)) the functor

\[
\begin{array}{ccc}
\Gamma A \\
\downarrow \hat{g} \\
LkQ(A) \\
\end{array}
\]

\[
\begin{array}{ccc}
\Box \Gamma A \\
\downarrow \hat{g} \\
\Box LkQ(A) \\
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma A \\
\downarrow g \\
Q(A) \\
\end{array}
\]

This functor is onto, each morphism of \( \Gamma A \) being of the form \( <v> \) for some 1-morphism \( v \) of \( A \) (by Example 3-B). It is also 1-1, since \( \hat{g}[v^{\Box}] = \hat{g}[v'^{\Box}] \) means \( <v> = <v'> \), which implies \( (v^{\Box}) = (v'^{\Box}) \) modulo \( R \), hence \([v^{\Box}] = [v'^{\Box}]\). This proves that \( \hat{g} : LkQ(A) \rightarrow \Gamma A \) is an isomorphism. \( \nabla \)

**Corollary.** If \( C \) is a category, \( Lk(\Box C) \) is isomorphic to \( C \).

**Proof.** \( \Box C \) is the double category of up-squares of the (trivial) 2-category \( (\mathbb{C}^{dis}, C) \), whose category of components is (identified with) \( C \). So, the corollary is a particular case of the Proposition 3. \( \nabla \)

This Corollary means that each double functor \( g : \Box C \rightarrow \Box C' \), where \( C \) and \( C' \) are categories, is of the form \( \Box f \), for a unique functor \( f : C \rightarrow C' \). We use this result to generalize the Corollary as follows:
PROPOSITION 4. Let $B$ be an $n$-fold category; then $Lk(SqB)$ is isomorphic to $B$.

PROOF. It suffices to prove that $B$ is also a free object generated by $SqB$ with respect to the functor

$$Sq_{n,n+1}: Cat_n \to Cat_{n+1},$$

the liberty morphism being $id: SqB \to SqB$. For this, let $H$ be an $n$-fold category and $g: SqB \to SqH$ an $(n+1)$-fold functor.

a) As $g$ defines a double functor

$$g: (SqB)^{n-1,n} = □B^0 \to (SqH)^{n-1,n} = □H^0,$$

by the Corollary there exists a unique functor $f: B^0 \to H^0$ such that

$$g = □f: □B^0 \to □H^0.$$

In particular, $g(b)\equiv = f(b)\equiv$ for each block $b$ of $B$.

b) Let us prove that $f: B \to H$ is an $n$-fold functor. Indeed, denote by $|SqB|^{n-1}$ the $n$-fold subcategory of $(SqB)^0,\ldots,n-2,n$ formed by the objects of $(SqB)^{n-1} = □B^0$ (i.e., formed by the degenerate squares $b\equiv$).

There is an isomorphism

$$(-\equiv)_B: B^1,\ldots,n-1,0 \simeq |(SqB)^{n-1}|: b \mapsto b\equiv$$

(see Proof, Proposition 3). The composite functor

$$B^1,\ldots,n-1,0 \xrightarrow{(-\equiv)_B} |(SqB)^{n-1}| \xrightarrow{|g|} |(SqH)^{n-1}| \xrightarrow{(-\equiv)^{-1}_H} H^1,\ldots,n-1,0$$

where $|g|$ is a restriction of $g$, maps $b$ onto $f(b)$, since $g(b)\equiv = f(b)\equiv$.

Hence it is defined by $f$, and this implies (after a permutation of compositions) that $f: B \to H$ is an $n$-fold functor. It is the unique $n$-fold functor such
that \( Sqf = g: SqB \to SqH \). \( \Box \)

**COROLLARY.** Whatever be the integers \( n \) and \( m \), the Link functor from \( \text{Cat}_{n+m} \) to \( \text{Cat}_n \) is equivalent to a left inverse of \( S_{q_n,n+m}: \text{Cat}_n \to \text{Cat}_{n+m} \).

**Proof.** Proposition 4 implies that the composite functor

\[
\text{Cat}_n \xrightarrow{S_{q_n,n+1}} \text{Cat}_{n+1} \xrightarrow{L_{k_{n+1,n}}} \text{Cat}_n
\]

is equivalent to the identity. By iteration, the same result is valid for the functors \( L_{k_{n+m,n}} \) and \( S_{q_n,n+m} \), due to their definition (end of Section A) as composites of functors \( L_{k_{p+1,p}} \) and \( S_{q_{p,p+1}} \) respectively. \( \Box \)

**C. The cartesian closed structure of \( \text{Cat}_n \).**

Let \( n \) be an integer, \( n > 1 \). In this section we are going to show that the category \( \text{Cat}_n \) of \( n \)-fold categories is cartesian closed, by constructing the partial internal Hom functor \( \text{Hom}_n(A, \cdot) \), for an \( n \)-fold category \( A \), as the composite

\[
\text{Cat}_n \xrightarrow{S_{q_n,2n}} \text{Cat}_{2n} \xrightarrow{\tilde{\gamma}} \text{Cat}_{2n} \xrightarrow{\text{Hom}(A, \cdot)} \text{Cat}_n,
\]

where \( \text{Hom}(A, \cdot) \) is the Hom functor associated to the partial monoidal closed structure of \( \text{MCat} \) (defined in [5] and recalled on page 2) and where \( \tilde{\gamma} \) is the isomorphism "permutation of compositions" associated to the permutation \( \gamma: \)

\[
(0, \ldots, 2n-1) \mapsto (0, 2, \ldots, 2n-2, 1, 3, \ldots, 2n-1)
\]

(which maps \( g: H \to K \) onto \( g: H^Y \to K^Y \), where

\[
H^Y = H^{0,2,\ldots,2n-2,1,3,\ldots,2n-1}.
\]

The necessity of introducing this isomorphism \( \tilde{\gamma} \) is best understood on the Example here after and on the following Proposition.

**Example:** The 4-fold category \( \text{SqSqB} \), where \( B \) is a double category.

By definition, \( \text{SqB} \) is the 3-fold category whose 1-st and 2-nd categories are the vertical and horizontal categories \( \text{SqB}_0 \) and \( \text{SqB}_1 \) of squares of the 0-th category \( B^0 \) of \( B \), and whose 0-th composition is «deduced pointwise» from that of \( B^1 \).
The 4-fold category $Sq_{2,4}(B)$ is constructed as follows:
- The set of its blocks is $\Box(SqB)^0$, i.e., the blocks are the squares so that

$$s_i = \begin{cases} b_i \end{cases}$$

is a square of $B^0$ for $i = 1, 2, 3, 4$, and

$$s_3 \circ_0 s_1 = b_3 \circ_1 b_1 = b_4 \circ_1 b_2 = s_4 \circ_0 s_2.$$

Such a block will be represented by the «frame»

- The 0-th and 1-st compositions are deduced «pointwise» from that of

$$(SqB)^1 = \square B^0 \quad \text{and} \quad (SqB)^2 = \square B^0,$$

so that they consist in putting «one frame behind the other» and «one frame inside the other».

- The 2-nd and 3-rd compositions are the vertical and horizontal com-
positions of squares of \((SqB)^0\) (whose composition is deduced from that of \(B^1\)) so that they consist in putting «one frame above the other» and «one frame beside the other» (the common border being «erased»).

- The sources and targets of \((s_4, s_3, s_2, s_1)\) are respectively the degenerate frames:

\[
(Sq\, Sq\, B)^0
\]

\[
(Sq\, Sq\, B)^1
\]

\[
(Sq\, Sq\, B)^2
\]

\[
(Sq\, Sq\, B)^3
\]

- The sources and targets of \((s_4, s_3, s_2, s_1)\) are respectively the degenerate frames:

for the 0-th category,

\[
\begin{align*}
&b_4^{\square} \quad b_2^{\square} \\
&b_1^{\square} \quad b_3^{\square}
\end{align*}
\]

and

\[
\begin{align*}
&b'_4^{\square} \quad b'_2^{\square} \\
&b'_1^{\square} \quad b'_3^{\square}
\end{align*}
\]

for the 1-st category,

\[
\begin{align*}
&\hat{b}_4^{\square} \quad \hat{b}_2^{\square} \\
&\hat{b}_1^{\square} \quad \hat{b}_3^{\square}
\end{align*}
\]

and

\[
\begin{align*}
&\hat{b}'_4^{\square} \quad \hat{b}'_2^{\square} \\
&\hat{b}'_1^{\square} \quad \hat{b}'_3^{\square}
\end{align*}
\]

for the 2-nd category,

\[
\begin{align*}
&s_2^{\square} \quad s_3^{\square}
\end{align*}
\]

and

\[
\begin{align*}
&s_1^{\square} \quad s_4^{\square}
\end{align*}
\]

for the 3-rd category.

Hence, the two first compositions are deduced from that of \(B^0\), the two last ones being deduced from that of \(B^1\).

More generally, if we consider \(Sq_{n, 2n}(B)\) for an \(n\)-fold category \(B\), its \((2i)\)-th and \((2i+1)\)-th compositions are deduced from that of \(B^i\), for each \(i < n\). Therefore, \(Sq_{n, 2n}(B)^\gamma\) has its compositions deduced respectively of that of \(B^0, \ldots, B^{n-1}, B^0, \ldots, B^{n-1}\).
The following proposition will be an essential tool to describe the cartesian closed structure of $\text{Cat}_n$.

**Proposition 6.** Let $A$ and $B$ be two $n$-fold categories; then the $n$-fold category product $B \times A$ is isomorphic to $\text{Lk}((B \square A)^{\gamma^{-1}})$, where $B \square A$ is the square product and $\gamma^{-1}$ the permutation

$$(0, \ldots, 2n-1) \mapsto (0, n, \ldots, i, n+i, \ldots, n-1, 2n-1).$$

**Proof.** Remark firstly that $\gamma^{-1}$ is the permutation inverse of the permutation (considered above) $\gamma$:

$$(0, \ldots, 2n-1) \mapsto (0, 2, \ldots, 2n-2, 1, 3, \ldots, 2n-1).$$

We denote by $H$ the $(2n)$-fold category $(B \square A)^{\gamma^{-1}}$, so that:

$H^{2i} = B^{\text{dis}} \times A^i$ and $H^{2i+1} = B^i \times A^{\text{dis}}$, for each $i < n$.

1° $\text{Lk}H$ is isomorphic to the $(2n-1)$-fold category $K$ on $B \times A$ such that

$K^0 = B^{n-1} \times A^{n-1}$ and $K^{j+1} = H^j$ for $0 \leq j < 2n-2$.

(hence $K = (B^{n-1} \times A^{n-1}, B^{\text{dis}} \times A^0, B^0 \times A^{\text{dis}}, \ldots, B^{\text{dis}} \times A^{n-2}, B^{n-2} \times A^{\text{dis}})$).

a) There exists a $(2n)$-fold functor $g : H \to \text{Sq}K$:

$$(b, a) \mapsto (\beta_B^{n-1}b, a) \quad (a, b) \mapsto (\beta_A^{n-1}a, b)$$

where $a^{n-1}_A$ and $\beta^{n-1}_A$ denote the source and target maps of $A^{n-1}$:

(i) $g(b, a)$ is a square of $K^0 = B^{n-1} \times A^{n-1}$, for any blocks $a$ of $A$ and $b$ of $B$.

(ii) For $0 \leq j < 2n-2$, the $j$-th composition of $\text{Sq}K$ is deduced pointwise from that of $K^{j+1} = H^j$; to prove that $g : H^j \to (\text{Sq}K)^j$ is a functor, it suffices to show that the four maps:

$\beta_B^{n-1} \times \text{id}_A : (b, a) \mapsto (\beta_B^{n-1}b, a), \quad \beta_B^{n-1} \times \text{id}_A : (b, a) \mapsto (\beta_B^{n-1}b, a),$

$\text{id}_B \times a^{n-1}_A : (b, a) \mapsto (b, a^{n-1}_A), \quad \text{id}_B \times \beta_A^{n-1} : (b, a) \mapsto (b, \beta_A^{n-1}a)$

define functors from $H^j$ to $H^j$. This comes from the following facts:
- \(H^i = \mathbb{B}^{dis \times \mathbb{A}^i}\) if \(j = 2i\) and \(H^i = \mathbb{B}^i \times \mathbb{A}^{dis}\) if \(j = 2i + 1\),
- \(\alpha_{A}^{n-1}\) and \(\beta_{A}^{n-1}\) define functors \(\mathbb{A}^{i} \to \mathbb{A}^{i}\) and \(\mathbb{A}^{dis} \to \mathbb{A}^{dis}\),
- \(\alpha_{B}^{n-1}\) and \(\beta_{B}^{n-1}\) define functors \(\mathbb{B}^{i} \to \mathbb{B}^{i}\) and \(\mathbb{B}^{dis \times \mathbb{B}^{dis}}\).

(iii) \(g: H^{2n-2} = \mathbb{B}^{dis \times \mathbb{A}^{n-1}} \to (\mathbb{S}q\mathbb{K})^{2n-2} = \mathbb{K}^{0}\) is a functor. Indeed, if \(a: x \to x'\) and \(a': x' \to x^*\) in \(\mathbb{A}^{n-1}\) and \(b: y \to y'\) in \(\mathbb{B}^{n-1}\), then

\[(b, a') \circ_{2n-2} (b, a) = (b, a'^{n-1}a)\]

and \(g(b, a') \sqcup g(b, a) =
\[
\begin{array}{ccc}
(b, x^*) & (y, a') & (b, x') \\
(b, x') & (y, a) & (b, x)
\end{array}
\]
\[
= g(b, a'^{n-1}a).
\]

(iv) A similar method gives the functor

\[g: H^{2n-1} = \mathbb{B}^{n-2} \times \mathbb{A}^{dis} \to (\mathbb{S}q\mathbb{K})^{2n-1} = \mathbb{K}^{0}\]

\[
\begin{array}{ccc}
(b', a) & (b, a) & (b', a') \\
(b', a') & (b', a) & (b', x')
\end{array}
\]
\[
= (b'^{n-1}b, x')
\]

(b': \(y' \to y^*\) in \(\mathbb{B}^{n-1}\)).

b) To \(g: \mathbb{H} \to \mathbb{S}q\mathbb{K}\) is canonically associated (by the adjunction between the Link and Square functors) a \((2n-1)\)-fold functor \(\hat{g}: Lk\mathbb{H} \to \mathbb{K}\) such that \(\hat{g} \{b, a\} = (b, a)\) for any \((b, a)\) in \(\mathbb{B} \times \mathbb{A}\), since \((b, a)\) is the diagonal of the square \(g(b, a)\) of \(\mathbb{K}^{0} = \mathbb{B}^{n-1} \times \mathbb{A}^{n-1}\) (Proof, Proposition 1).

c) There exists a \((2n-1)\)-fold functor
(i) For $0 \leq j < 2n-2$, since $K_{j+1} = H^j$ and the composition of $(LkH)_{j+1}$ is deduced pointwise from that of $H^j$, it follows that $\hat{g}' : K_{j+1} \to (LkH)_{j+1}$ is a functor.

(ii) It remains to prove that $\hat{g}' : K^0 \to (LkH)^0$ is a functor. For this, let the composite

$$(b', a')_0(b, a) = (b'_0 \circ_{n-1} b, a'_0 \circ_{n-1} a)$$

be defined in $K^0 = B^{n-1} \times A^{n-1}$, so that

- $a : x \to x'$ and $a' : x' \to x''$ in $A^{n-1}$,
- $b : y \to y'$ and $b' : y' \to y''$ in $B^{n-1}$.

Since $H^{2n-2} = B^{dis} \times A^{n-1}$ and $H^{2n-1} = B^{n-1} \times A^{dis}$, in the relation on paths used to define $LkH$ (Proof, Proposition 1), we have successively

This implies

$$\hat{g}'(b', a')_0 \circ_0 \hat{g}'(b, a) = [b', a']_0[b, a] = [(b', a'), (b, a)] =$$
Hence \( g' : \mathcal{K} \to (\mathcal{L}_k \mathcal{H})^0 \) is also a functor.

d) \( \hat{g}' \) is the inverse of \( \hat{g} : \mathcal{L}_k \mathcal{H} \to \mathcal{K} \). Indeed,

\[
\hat{g} \hat{g}'(b, a) = \hat{g}[b, a] = (b, a)
\]

for each block \((b, a)\) of \( \mathcal{K} \), so that \( \hat{g} \hat{g}' \) is an identity. On the other hand, the equalities

\[
\hat{g}' \hat{g}[b, a] = \hat{g}'(b, a) = [b, a]
\]

imply that \( \hat{g}' \hat{g} \) is also an identity, the blocks \([b, a]\) generating (by definition) the \((2n-1)\)-fold category \( \mathcal{L}_k \mathcal{H} \). So \( \hat{g} : \mathcal{L}_k \mathcal{H} \to \mathcal{K} \) is an isomorphism.

2° Let us suppose proven that \( \mathcal{L}_k 2n_{2n-m} \mathcal{H}, \) for \( 1 \leq m \leq n-1 \), is isomorphic to the \((2n-m)\)-fold category \( \mathcal{K}_m \) such that

\[
(\mathcal{B}^{n-m} \times \mathcal{A}^{n-m}, ..., \mathcal{B}^{n-1} \times \mathcal{A}^{n-1}, \mathcal{B}^{dis} \times \mathcal{A}^0, ..., \mathcal{B}^{dis} \times \mathcal{A}^{n-m-1}, \mathcal{B}^{n-m-1} \times \mathcal{A}^{dis}).
\]

Then a proof similar to the preceding one proves that \( \mathcal{L}_k \mathcal{K}_m \), and a fortiori

\[
\mathcal{L}(\mathcal{L}_k 2n_{2n-m} \mathcal{H}) = \mathcal{L}_k 2n_{2n-m-1} \mathcal{H}
\]

is isomorphic to the \((2n-m-1)\)-fold category \( \mathcal{K}_{m+1} \). By induction, it follows that \( \mathcal{L}_k 2n_n \mathcal{H} \) is isomorphic to

\[
\mathcal{B} \times \mathcal{A} = (\mathcal{B}^0 \times \mathcal{A}^0, ..., \mathcal{B}^{n-1} \times \mathcal{A}^{n-1}).
\]

COROLLARY. For each \( n \)-fold category \( \mathcal{A} \), the «partial» product functor

\[
\times \mathcal{A} : \mathcal{C}at_n \to \mathcal{C}at_n
\]

is equivalent to the composite functor

\[
\mathcal{C}at_n \xrightarrow{\Box} \mathcal{C}at_{2n} \xrightarrow{\tilde{y}^{-1}} \mathcal{C}at_{2n} \xrightarrow{\mathcal{L}_k 2n_n} \mathcal{C}at_n.
\]

DEFINITION. The composite functor

\[
\mathcal{C}at_n \xrightarrow{\mathcal{S}q_n 2n} \mathcal{C}at_{2n} \xrightarrow{\tilde{y}} \mathcal{C}at_{2n}
\]

will be called the \( n \)-square functor, denoted by \( \Box_n : \mathcal{C}at_n \to \mathcal{C}at_{2n} \).

PROPOSITION 7. \( \mathcal{C}at_n \) is a cartesian closed category whose internal Hom functor \( \mathcal{H}om_n : (\mathcal{C}at_n)^{op} \times \mathcal{C}at_n \to \mathcal{C}at_n \) is such that, for any \( n \)-fold category \( \mathcal{A} \), the partial functor \( \mathcal{H}om_n(\mathcal{A}, -) \) is equal to the composite:

\[
\mathcal{C}at_n \xrightarrow{\Box_n} \mathcal{C}at_{2n} \xrightarrow{\mathcal{H}om(\mathcal{A}, -)} \mathcal{C}at_n.
\]
PROOF. 1° Since $\text{Cat}_n$ admits (finite) products, to prove that it is cartesian closed it suffices to show that the partial product functor $- \times A : \text{Cat}_n \to \text{Cat}_n$ admits a right adjoint \([13]\). By the Corollary of Proposition 6, this functor is equivalent to the composite of three functors:

- $A : \text{Cat}_n \to \text{Cat}_{2n}$ who has a right adjoint $\text{Hom}(A, -)$ (due to the partial monoidal closed structure of $M\text{Cat}$, Proposition 7 \([5]\))
- $\bar{y}^{-1} : \text{Cat}_{2n} \to \text{Cat}_{2n}$ whose inverse $\bar{y}$ is a right adjoint,
- $Lk_{2n,n} : \text{Cat}_{2n} \to \text{Cat}_n$ who admits $Sq_{n,2n}$ as a right adjoint.

By transitivity of adjunctions, this implies that $- \times A$ admits as a right adjoint the composite $\text{Hom}_n(A, -) =$

$$\begin{array}{c}
\text{Cat}_n \\
\xrightarrow{\text{Sq}_{n,2n}} \text{Cat}_{2n} \\
\xrightarrow{\bar{y}} \text{Cat}_{2n} \\
\xrightarrow{\text{Hom}(A, -)} \text{Cat}_n
\end{array}$$

2° The corresponding internal Hom functor (or closure functor)

$$\text{Hom}_n : (\text{Cat}_n)^{op} \times \text{Cat}_n \to \text{Cat}_n$$

maps the couple of $n$-fold functors $(f : A' \to A$, $g : B \to B')$ onto the $n$-fold functor

$$\text{Hom}_n(f, g) : \text{Hom}_n(A, B) = \text{Hom}(A, \square_n B) \to \text{Hom}_n(A', B')$$

mapping $h : A \to \square_n B$ onto

$$\begin{array}{c}
A' \\
\xrightarrow{f} A \\
\xrightarrow{h} \square_n B \\
\xrightarrow{\square_n g} \square_n B'
\end{array}$$

3° Let us describe more explicitly the adjunction between $- \times A$ and $\text{Hom}_n(A, -) : \text{Cat}_n \to \text{Cat}_n$. Let $B$ be an $n$-fold category.

a) There is a map $\partial : \square_n B \to B$ (it is not a multiple functor, but a map between the sets of blocks) which maps an $n$-square of $B$ onto its diagonal* defined as follows: For each $i < n$, there is the diagonal map

$$d_i : \text{Sq}_{n,n+i+1} B = \text{Sq}(\text{Sq}_{n,n+i} B) \to \text{Sq}_{n,n+i} B$$

which maps the square

$$\begin{array}{c}
\text{Sq}_{n,n+i} B \\
\xrightarrow{d_i} \text{Sq}_{n,n+i+1} B
\end{array}$$

onto its diagonal $\hat{s}' \circ_0 s = s' \circ_0 \hat{s}$. Then $\partial$ is the composite map $d_0 \ldots d_{n-1}$:
b) The 1-1 correspondence due to the adjunction between \(- \times A\) and \(\text{Hom}_n(A, -)\) maps the \(n\)-fold functor \(h : A' \to \text{Hom}_n(A, B)\) onto the \(n\)-fold functor

\[
\hat{h} : A' \times A \to B : \ (a', a) \mapsto \partial(h(a')(a)).
\]

Indeed, the adjunction between \(\text{Hom}(A, -)\) and \(- \cdot A\) associates to \(h\) the \(n\)-fold functor

\[
h_0' : A' \cdot n \to \triangleq_n B : \ (a', a) \mapsto h(a')(a),
\]

and therefore the \(n\)-fold functor

\[
h_0 : (A' \cdot n)^{\vee, n} \to (\triangleq_n B)^{\vee, n} = \text{Sq}_{n, 2n} B;
\]

we write \(H\) instead of \((A' \cdot n)^{\vee, n}\). By induction, we define

\[
h_{i+1} : Lk_{2n, 2n-i-1} H = Lk(Lk_{2n, 2n-i} H) \to \text{Sq}_{n, 2n-i-1} B,
\]

for each \(i < n\), as the \((2n-i-1)\)-fold functor associated (by the adjunction between \(Lk_{2n-i, 2n-i-1}\) and \(\text{Sq}_{2n-i-1, 2n-i} : \text{Cat}_{2n-i} \to \text{Cat}_{2n-i}\) ) to

\[
h_i : Lk_{2n, 2n-i} H \to \text{Sq}_{n, 2n-i} B = \text{Sq}(\text{Sq}_{n, 2n-i-1} B);
\]

by construction, \(h_{i+1}\) maps a block of \(Lk_{2n, 2n-i-1} H\) of the form \([a', a]\) (see Proof, Proposition 6) onto the diagonal \(d_{n-i-1} h_i [a', a]\) of the square

\[
\hat{h} = (A' \times A \xrightarrow{g} Lk_{2n, n} H \xrightarrow{h_n} B),
\]
where \( \hat{g}' \) is the canonical isomorphism \((a', a) \rightarrow [a', a]\) (see Proof, Proposition 6); \( \hat{h} \) maps \((a', a)\) onto
\[
d_0 \ldots d_{n-1} h(a')(a) = \partial h(a')(a).
\]

c) The coliberty morphism defining \( \text{Hom}_n(A, B) \) as a cofree object generated by \( B \) is the «evaluation»:
\[
ev: \text{Hom}_n(A, B) \times A \rightarrow B: (f, a) \mapsto \partial f(a),
\]
since it corresponds to the identity of \( \text{Hom}_n(A, B) \). In particular, if \( A \) is the \( n \)-fold category \( 2D^n \) (see [5]), with only one non-degenerate block \( z \), then
\[
\text{Hom}_n(2D^n, B) = \text{Hom}(2D^n, \square_n B)
\]
is identified with \((\square_n B)^{2D^n, \ldots, 2D^{n-1}}\), and the evaluation becomes the \( n \)-fold functor \( ev: (\square_n B)^{n, \ldots, 2D^{n-1}} \times 2D^n \rightarrow B \) such that the map
\[
ev(-, z): \square_n B \rightarrow B: s \mapsto \partial s
\]
is the diagonal map \( \partial \) defined in a. \( \checkmark \)

**Corollary 1.** The vertices of \( \text{Hom}_n(A, B) \) are identified with the \( n \)-fold functors from \( A \) to \( B \).

**Proof.** The final object \( I_n \) of \( \text{Cat}_n \) is the unique \( n \)-fold category on the set \( I = \{0\} \). The vertices of \( \text{Hom}_n(A, B) \) are identified [5] with the \( n \)-fold functors \( I_n \rightarrow \text{Hom}_n(A, B) \), which are in 1-1 correspondence (by adjunction) with the \( n \)-fold functors from \( I_n \times A = A \) to \( B \). To \( f: A \rightarrow B \) corresponds the vertex of \( \text{Hom}_n(A, B) \) mapping \( a \) onto the degenerate \( n \)-square (vertex of \( \square_n B \)) determined by \( f(a) \). \( \checkmark \)

**Corollary 2.** There is a canonical isomorphism
\[
\lambda: \text{Hom}_n(A', \text{Hom}_n(A, B)) \rightarrow \text{Hom}_n(A' \times A, B)
\]
extending the 1-1 correspondence (Proof above):
(\ h: A' \to \text{Hom}_n(A, B)) \mapsto (\hat{h}: A' \times A \to B: (a', a) \mapsto \partial h(a')(a')).

**Proof.** It is a general result on cartesian (as well as monoidal) closed categories \cite{13}; it means that \(\text{Hom}_n(A, -): \text{Cat}_n \to \text{Cat}_n\) is a \(\text{Cat}_n\)-right adjoint of \(- \times A\). \(\blacksquare\)

**Corollary 3.** There is a canonical \(n\)-fold «composition» functor

\[ \kappa_{A,B,B'}: \text{Hom}_n(A, B) \times \text{Hom}_n(B, B') \to \text{Hom}_n(A, B') : (f, f') \mapsto (f'': A \to \square_n B') \text{ with } \partial_B f'' = \partial_B f' \partial_B f : A \to B. \]

**Proof.** This is also a general result on cartesian closed categories; in fact, \(\kappa_{A,B,B'}\) corresponds to the composite \(n\)-fold functor:

\[
\begin{array}{c}
\text{Hom}_n(A, B) \times \text{Hom}_n(B, B') \\
\downarrow \text{id} \times \text{ev}_{A,B}
\end{array}
\begin{array}{c}
\text{Hom}_n(B, B') \times B \\
\text{ev}_{B,B'} \text{ev}_B
\end{array}
\]

mapping \((f, f', a)\) onto \(\partial_B f'(\partial f(a))\). \(\blacksquare\)

This Corollary 3 means that \(\text{Cat}_n\) is a \(\text{Cat}_n\)-category (i.e., a category enriched in the cartesian closed category \(\text{Cat}_n\)) and it will be used in Proposition 8.

**Remark.** The existence of a cartesian closed structure on \(\text{Cat}_n\) may also be deduced, by induction, from Corollary 3, Proposition 23 \cite{7}, as follows: since \(\text{Cat}\) is cartesian closed, the sketch \(\sigma\) of categories is cartesian \cite{7}; so, if \(\text{Cat}_i\) is cartesian closed, the category \(\text{Cat}_i^0\) of categories in \(\text{Cat}_i\) is cartesian closed by this Corollary, as well as the equivalent category \(\text{Cat}_{i+1}\) (see Appendix \cite{5}). However the explicit construction of \(\text{Hom}_n\) cannot be deduced from this (or from another) existence result.

**Example.** The cartesian closed category \(\text{Cat}_2\):

Let \(A\) and \(B\) be double categories. Then \(\square_2 B\) is the \(4\)-fold category deduced from \(\text{Sq} \text{Sq} B\) (described in the Example above) by permutation of the 1-st and 2-nd compositions. Hence, \(\text{Hom}_2(A, B)\) is constructed as follows:

- Its blocks are the double functors from \(A\) to the double category

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of "frames" whose compositions are "one frame behind the
the other" and "one frame above the other".

- Its compositions are deduced pointwise from the compositions "one frame inside the other" and "one frame beside the other".

- Its vertices are the double functors \( f : A \to B \).

- The 3-fold subcategory \( (\text{SqSqB})^3 \) of \( (\text{SqSqB})^{0,2,1} \) formed by the objects of \( (\text{SqSqB})^3 \) is identified with \( (\text{SqB})^{1,0,2} \) by the isomorphism

\[
\Phi : (\text{SqB})^{1,0,2} \to (\text{SqSqB})^3 : s \mapsto s^0.
\]

Then an object of \( \text{Hom}_2(A,B)^1 \) (which is a double functor \( A \to (\text{SqSqB})^{0,2} \) taking its values in \( (\text{SqSqB})^3 \)) will be identified with a double functor \( \phi : A \to (\text{SqB})^{1,0} \), and the subcategory of \( \text{Hom}_2(A,B)^0 \) formed by these objects is \( \text{Hom}(A,(\text{SqB})^{1,0,2}) \). The objects of this last category are themselves identified with the double functors \( f : A \to B \). With the terminology of [7], a double functor \( \phi : A \to (\text{SqB})^{1,0} \) is called a double natural transformation (i.e., a natural transformation internal to \( \text{Cat} \)) from \( f \) to \( f' \), if \( \phi : f \to f' \) in \( \text{Hom}(A,(\text{SqB})^{1,0,2}) \). This may suggest to call the block \( \Phi : A \to \mathbb{2} B \) of \( \text{Hom}_2(A,B) \) a hypertransformation from \( \phi \) to \( \phi' \) where \( \Phi : \phi \to \phi' \) in \( \text{Hom}_2(A,B)^1 \).

- If \( h : A' \to \text{Hom}_2(A,B) \) is a double functor, the double functor canonically associated (by adjunction) \( \hat{h} : A' \times A \to B \) maps \( (a',a) \) onto the diagonal of the frame \( h(a')(a) \), which is equal to

\[
(b_0' \circ_0 b_1') \circ_1 (b_2' \circ_0 b_2')
\]
APP LIC ATION. The \((n+1)\)-category \(\text{Nat}_n\) of hypertransformations.

The following Proposition 8 shows that \(\text{Cat}_n\) is the category of 1-morphisms of an \((n+1)\)-category \(\text{Nat}_n\) which, for \(n = 1\), is the 2-category of natural transformations. It is based on the Lemma, whose proof is given in the Appendix:

**LEMMA.** Let \(V\) denote a cartesian category with commuting coproducts (in the sense of Penon [21]) and \(A\) be a \(V\)-category. If \(V\) admits coproducts indexed by the class of objects of \(A\), then there is a category in \(V\) whose object of morphisms is the coproduct of \(\lambda(e, e')\), for any objects \(e\) and \(e'\) of \(A\), and whose composition «glues together» the composition morphisms

\[\kappa_{e, e', e''}: \lambda(e, e') \times \lambda(e', e'') \to \lambda(e, e'').\]

**PROPOSITION 8.** There is an \((n+1)\)-fold category \(\text{Nat}_n\) satisfying the following conditions:

1. \((\text{Nat}_n)^0, \ldots, n-1\) is the \(n\)-fold category coproduct of the \(n\)-fold categories \(\text{Hom}_n(A, B)\), for any (small) \(n\)-fold categories \(A, B\).

2. Its \(n\)-th composition \(\kappa_n\) is (notations Corollary 3, Proposition 7):

\[(f, f') \mapsto \kappa_{A, B, B'}(f, f') \text{ iff } f \text{ in } \text{Hom}_n(A, B) \text{ and } f' \text{ in } \text{Hom}_n(B, B').\]

3. \(\text{Cat}_n\) is the category of 1-morphisms of \(\text{Nat}_n\).

**PROOF.** 1° Let \(\hat{\text{Cat}}_n\) be the category of \(n\)-fold categories associated to a universe \(\hat{U}\) to which belongs the universe \(U\) of small sets, and a fortiori the class of objects of \(\text{Cat}_n\). Then \(\hat{\text{Cat}}_n\) is also cartesian closed. The faithful functor «forgetting all the compositions» from \(\hat{\text{Cat}}_n\) toward the category \(\hat{\text{Set}}\) (of sets associated to the universe \(\hat{U}\)) preserves coproducts and limits, and it reflects isomorphisms (an \(n\)-fold functor defined by a 1-1 and onto map is an isomorphism); hence Corollary 1, Proposition 1-6 of Penon...
[21] asserts that $\text{Cat}_n$ has commuting coproducts (in [21] "small" is now to be replaced by: belonging to $\hat{U}$).

2° As $\text{Cat}_n$ is cartesian closed, it is a $\text{Cat}_n$-category [3], and it determines also a $\text{Cat}_n$-category, the insertion functor $\text{Cat}_n \hookrightarrow \text{Cat}_n$ preserving the cartesian closed structure. More precisely, we have the $\text{Cat}_n$-category $H_n$ defined as follows:

- its objects are the small $n$-fold categories $A, B, \ldots$, and

$$H_n(A, B) = \text{Hom}_n(A, B);$$

- the "unitarity" morphisms are of the form $j_A : 1_n \to H_n(A, A)$, where $j_A(0)$ is the vertex of $\text{Hom}_n(A, A)$ identified with $id : A \to A$;

- the "composition" morphisms $\kappa_{A,B,B'}$ are those defined in Corollary 3, Proposition 7.

3° The Lemma associates to $H_n$ a category $H_n$ in $\text{Cat}_n$ defined as follows:

- its object of morphisms $H_n(2)$ is the $n$-fold category $\coprod_{A,B} \text{Hom}_n(A, B)$ coproduct of the $n$-fold categories $\text{Hom}_n(A, B)$, for any (small) $n$-fold categories $A, B$ (as the sets $\text{Hom}_n(A, B)$ are disjoint, this coproduct is on their union);

- its object of objects $H_n(1)$ is the "discrete" $n$-fold category on $\text{Cat}_n$ (since it is the coproduct of $\text{Cat}_n$ copies of the final object $1_n$);

- the morphisms source $\alpha^n$ and target $\beta^n$ send a block $f : A \to \square_n B$ of $\text{Hom}_n(A, B)$ onto $A$ and $B$ respectively;

- the composition morphism $\kappa^n$ is the union of the $n$-fold "composition" functors $\kappa_{A,B,B'}$ (Corollary 3, Proposition 7).

4° By the equivalence between categories in $\text{Cat}_n$ and $(n+1)$-fold categories (see Appendix [5]), $H_n : \sigma \to \text{Cat}_n$ is the realization associated to the $(n+1)$-fold category $\text{Nat}_n$ such that:

$$(\text{Nat}_n)^0, \ldots, n = H_n(2) = \coprod_{A,B} \text{Hom}_n(A, B),$$
\((\text{Nat}_n)^n\) is the category whose composition is \(\kappa^n\).

\(\text{Nat}_n\) is, in fact, an \((n+1)\)-category, its vertices being the objects of the \(n\)-th category \((\text{Nat}_n)^n\). A block of \(\text{Nat}_n\) will be called an hypertransformation (as suggested by the Example above). The hypertransformations being objects of the \(n\) first categories \((\text{Nat}_n)^i\) are the vertices of the \(n\)-fold categories \(\text{Hom}_n(A,B)\), hence (Corollary 1, Proposition 7) they are the \(n\)-fold functors; the subcategory of \((\text{Nat}_n)^n\) that they form is so identified with \(\text{Cat}_n\), by definition of \(\kappa^n\). \(\mathbb{V}\)

**DEFINITION.** The \((n+1)\)-fold category \(\text{Nat}_n\) defined in Proposition 8 is called the \((n+1)\)-category of hypertransformations between \(n\)-fold (small) categories.

For \(n = 1\), the 2-category \(\text{Nat}_1\) is the (usual) 2-category \(\text{Nat}\) of natural transformations (introduced in 1963 in [11]).

**REMARK.** The functors

\[\text{Sq}_{n,n+1} : \text{Cat}_n \to \text{Cat}_{n+1}\]

for all integers \(n\), do not extend into an endofunctor of the category \(\text{MCat}\) of multiple categories (considered in [5]). This comes from the fact that in \(\text{SqB}\), we have put the vertical and horizontal compositions of squares at the two last places, the compositions deduced pointwise from that of \(B\) being first indicated. However, it exists a functor \(\text{MCat} \to \text{MCat}\) which maps an \(n\)-fold category \(B\) onto the \((n+1)\)-fold category \((\text{SqB})^{n-1,n,0,\ldots,n-2}\), deduced from \(\text{SqB}\) by permutation of compositions, for \(n \geq 1\), and a set \(E\) onto the discrete category \(E^{\text{dis}}\).

This functor admits a «partial» left adjoint (it is not defined on \(\text{Set} \subseteq \text{MCat}\)) which associates to an \((n+1)\)-fold category \(A\) the \(n\)-fold category \(\text{Lk}(A^2,\ldots,n,0,1)\) of \((0,1)\)-links of \(A\).

We have not considered these functors, because their iterates are not interesting, while the iterates of the \textit{Square} and \textit{Link} functors have played in important role in this Section, since they consider successively all the compositions of a multiple category.
APPENDIX

Enriched categories as internal categories

The aim of this Appendix is to prove that, under mild enough conditions on a cartesian category $V$, the $V$-categories (categories enriched in $V$, in the sense of [13,3]) are those categories internal to $V$ whose object of objects is discrete, i.e., is a coproduct of copies of the final object $I$.

The main condition is that $V$ is a category with commuting coproducts; this notion, due to Penon [21], means that:

- $V$ admits finite limits and (small) coproducts preserved by pull-backs (in fact, Penon requires the existence of all small limits, but only finite limits are used);
- if the coproduct of two morphisms of $V$ is an isomorphism, both are isomorphisms.

It implies (Proposition 2-3 [21]) that the partial product functors

$$\times: V \times V \to V$$

preserve coproducts, for each object $V$ of $V$.

Let $V$ be a category with commuting coproducts. A fortiori, it is a cartesian category (i.e., it admits finite products), and we may consider the $V$-categories (for this cartesian structure). We denote by:

- $V\text{-Cat}$ the category of $V$-categories $A$ whose class $A_0$ of objects is small;
- $\text{Cat} V$ the category of categories internal to $V$.

**PROPOSITION A.** Let $V$ be a category with commuting coproducts. Then there exists a functor $\Gamma: V\text{-Cat} \to \text{Cat} V$ associating to the $V$-category $A$ a category in $V$ whose object of morphisms is the coproduct of $A(e,e')$ for every couple $(e,e')$ of objects of $A$.

**PROOF.** We will use the following assertion: If $(V_\lambda)_{\lambda \in \Lambda}$ and $(V_\mu')_{\mu \in M}$ are families of objects of $V$, if $\phi: \Lambda \to M$ is a map and if $v_\lambda: V_\lambda \to V_{\phi\lambda}$ is a morphism of $V$, for each $\lambda \in \Lambda$, then there exists a unique morphism $v$...
between coproducts such that, for each \( \lambda \in \Lambda \), the diagram

\[
\begin{array}{ccc}
\Pi_{\mu \in \mathcal{M}} V'_{\mu} & \xrightarrow{v} & \Pi_{\lambda \in \Lambda} V'_\lambda \\
\downarrow j_{\phi \lambda} & & \downarrow \lambda \\
V_{\phi \lambda} & \xrightarrow{v_{\lambda}} & V_{\lambda}
\end{array}
\]

commutes, where \( j_{\lambda} \) and \( j_{\phi \lambda} \) always denote the injections into the coproducts. Indeed, \( v \), called the factor of \( (v_{\lambda})_{\lambda} \) with respect to \( \phi \), is defined as follows:

\[
v = (\Pi_{\lambda \in \Lambda} V_{\lambda}) \xrightarrow{\sim} \Pi_{\mu} (\Pi_{\lambda \in \phi^{-1}(\mu)} V_{\lambda}) \xrightarrow{\Pi_{\mu} v_{\mu}} (\Pi_{\mu} V'_{\mu}),
\]

where \( v_{\mu} : \Pi_{\lambda \in \phi^{-1}(\mu)} V_{\lambda} \rightarrow V'_{\mu} \) is the factor of \( (v_{\lambda})_{\lambda} \) through the coproduct \( \Pi_{\lambda \in \phi^{-1}(\mu)} V_{\lambda} \).

2° Construction of the category \( \Gamma A \) in \( V \), for a \( V \)-category \( A \) such that there exist in \( V \) coproducts indexed by the class \( \mathcal{A}_0 \) of objects of \( A \).

a) Since \( \mathcal{A}_0 \) is finite or equipotent with \( \mathcal{A}_0 \times \mathcal{A}_0 \) and \( \mathcal{A}_0 \times \mathcal{A}_0 \times \mathcal{A}_0 \), there exist in \( V \) coproducts:

- \( S_1 \) of the family \( (I_e)_e \) indexed by \( \mathcal{A}_0 \), where \( I_e \) is equal to the final object \( I \) of \( V \) for each object \( e \) of \( A \),

- \( S_2 \) of the family \( (A(e, e'))_{e, e'} \) indexed by \( \mathcal{A}_0 \times \mathcal{A}_0 \), where \( A(e, e') \) is the object of morphisms from \( e \) to \( e' \) in \( A \),

- \( S_3 \) of the family \( (A(e, e') \times A(e', e''))_{e, e', e''} \) indexed by \( \mathcal{A}_0 \times \mathcal{A}_0 \times \mathcal{A}_0 \).

b) (i) There exist unique morphisms \( S_{\alpha}, S_{\beta}, S_{\nu_1} \) rendering commutative the *cube*:

![Diagram](image_url)
where $p_{e',e''}^i$ are the projections of the product, $\alpha_{e',e''}$ and $\beta_{e,e'}$ are the unique morphisms toward the final object $I$ (the name of such a morphism will often be omitted). Indeed, $S\alpha$, $S\beta$, $S\nu_i$ are respectively the factors of:

- $(\alpha_{e',e''})$ with respect to the projection $A_0 \times A_0 \to A_0 : (e', e'') \mapsto e'$,
- $(\beta_{e,e'})$ with respect to the map $A_0^2 \to A_0 : (e, e') \mapsto e'$,
- $(p_{e',e'',e''}^i)$ with respect to the maps $q_i : A_0 \times A_0 \times A_0 \to A_0 \times A_0$ with $q_1(e, e', e'') = (e, e')$ and $q_2(e, e', e'') = (e', e'')$.

Since the down face of the cube commutes (there is only one morphism $I_{e',e''} : A(e,e') \times A(e',e'') \to I$), by unicity of the factor of $(I_{e',e''})$ with respect to the projection $A_0 \times A_0 \times A_0 \to A_0 : (e, e', e'') \mapsto e'$, the up face of the cube also commutes.

(ii) The square

\[
\begin{array}{ccc}
S1 & \xrightarrow{S\beta} & S2 \\
\downarrow{S\alpha} & & \downarrow{S\nu_1} \\
S2 & \xrightarrow{S\nu_2} & S3
\end{array}
\]

is a pullback. Indeed, for each object $e'$ of $A$ we have the pullback

\[
\begin{array}{ccc}
I & \xrightarrow{\beta_{e',e''}} & \coprod_{e''} A(e,e') \\
\downarrow{\alpha_{e',e''}} & & \downarrow{p_{i}^{e'}} \\
\coprod_{e''} A(e',e'') & \xrightarrow{\coprod_{e'} A(e',e'')} & (\coprod_{e'} A(e',e'')) \times (\coprod_{e'} A(e',e''))
\end{array}
\]

where $p_{i}^{e'}$ are projections of the product, since $I$ is a final object. Having commuting coproducts, the theorem of commutation of Penon (Corollary 3, Proposition 1-8 [21]) asserts that the square $(D')$ coproduct of the squares $(D_{e'})$ is also a pullback. Now $(D')$ is the down face of the cube
The vertical edges of this cube are canonical isomorphisms between co-products (the existence of $\delta$ follows from the preservation of coproducts by the partial product functors in $\mathcal{V}$). By construction of the factors $S_\alpha$, $S\beta$, $S\nu_i$, this cube commutes, so that its up face (D) is also a pullback.

(iii) There exist unique morphisms $S_\iota$ and $S\kappa$ rendering commutative the squares

\[
\begin{array}{ccc}
S_1 & S_\iota & S_2 \\
\downarrow j_e & \downarrow i_{e,e} & \downarrow j_e,e'' \\
1 & \downarrow u_e & A(e,e) \\
& \downarrow A(e,e') & \downarrow A(e,e') \times A(e',e'') \\
S_2 & S_\kappa & S_3 \\
\downarrow j_e,e'' & \downarrow i_{e',e''} & \downarrow j_{e',e''} \\
\end{array}
\]

where $u_e$ and $\kappa_{e,e',e''}$ are the «identity» morphisms and the «composition» morphisms of the $\mathcal{V}$-category $A$. Indeed, $S_\iota$ and $S\kappa$ are respectively the factors of

$(u_e)_e$ with respect to the map $A_0 \rightarrow A_0 \times A_0 : e \mapsto (e,e)$,

$(\kappa_{e,e',e''})$ with respect to $A_0 \times A_0 \times A_0 \rightarrow A_0 \times A_0 : (e,e',e'') \mapsto (e,e')$.

c) This defines a category $S$ in $\mathcal{V}$, i.e., a realization $S : \sigma \rightarrow \mathcal{V}$ of the sketch $\sigma$ of categories (see [4] and [5] Appendix):

\[
\begin{array}{ccc}
1 & \overset{\nu_1}{\rightarrow} & 2 \\
\downarrow \iota & \overset{\beta}{\rightarrow} & \downarrow \kappa \\
3 & \leftarrow \nu_2 & \end{array} \quad \begin{array}{ccc}
S_1 & S_\iota & S_2 \\
\downarrow S\nu_1 & \downarrow S_\iota \rightarrow S\nu_2 & \downarrow S\nu_1 \\
S_2 & S_\kappa & S_3 \\
\end{array}
\]

(i) For a couple $(e,e')$ of objects of $A$, let $u_{e,e'}$ be equal to

\[
A(e,e') \xrightarrow{\sim} I \times A(e,e') \xrightarrow{u_e \times id} A(e,e) \times A(e,e')
\]

and $Su_\alpha$ be the factor of $(u_{e,e'})_{e,e'}$ with respect to the map

\[
A_0 \times A_0 \rightarrow A_0 \times A_0 \times A_0 : (e,e') \mapsto (e,e,e').
\]

Then

\[
S\nu_1 \cdot Su_\alpha = S_\iota \cdot S\alpha, \quad S\nu_2 \cdot Su_\alpha = id_{S2} = S\kappa \cdot Su_\alpha
\]

(«source» unitarity axiom of an internal category). Indeed, by unicity of the factors and by definition of $u_{e,e'}$, for every objects $e$ and $e'$ of $A$ the two following diagrams commute, so that the two first equalities are valid.
The validity of the third equation is deduced from the commutativity of the diagram

(iv) A similar proof shows that $S$ satisfies the "target" unitarity axiom of an internal category.

(iii) $S$ also satisfies the associativity axiom of an internal category. Indeed, for objects $e, e', e'', e'''$ of $A$, there exists a commutative cube

where $j_{e, e', e'', e'''}$ is the injection toward the coproduct $S_4$ of the family $(A(e, e') \times A(e', e'') \times A(e'', e'''))_{e, e', e'', e''''}$ indexed by $A_0^4$, and where $S_{\nu_i}'$ is the factor of the family $(p^i_{e, e', e'', e''''})$ of projections with respect to the map $q_i': A_0^4 \rightarrow A_0^3$ defined by

$$q_i'(e, e', e'', e''') = (e, e', e''), \quad q_2'(e, e', e'', e''') = (e', e'', e''').$$

As the down face of this cube is a pullback, a proof analogous to that of Part b proves that the up face of this cube is a pullback. Now, let us de-
note by \( \kappa_1^1 e, e', e'', e''' \) the composite
\[
A(e, e') \times A(e', e'') \times A(e'', e''') \rightarrow (A(e, e') \times A(e', e'') \times A(e'', e'''))
\]
\[
A(e, e') \times A(e', e'') \times e'' \times \text{id}
\]

\( S_{K_1} \) factor of the family \( (\kappa_1^1 e, e', e'', e''') \) with respect to the projection
\[
A_0^4 \rightarrow A_0^3 : (e, e', e'', e''') \rightarrow (e, e'', e''')
\]
renders commutative the cubes

(by definition of \( \kappa_1^1 e, e', e'', e'''' \) and of the different factors), so that
\[
S_{V_1} \cdot S_{K_1} = S_{K} \cdot S_{V_1}' \quad \text{and} \quad S_{V_2} \cdot S_{K_1} = S_{V_2} \cdot S_{V_2}'.
\]

In the same way, there is a factor \( S_{K_2} : S_4 \rightarrow S_3 \) of the family of composites \( \kappa_1^2 e, e', e'', e''' \) =
\[
A(e, e') \times A(e', e'') \times A(e'', e''') \rightarrow A(e, e') \times (A(e', e'') \times A(e'', e'''))
\]
\[
A(e, e') \times (id \times \kappa_1^2 e, e'', e''')
\]
with respect to the projection
\[
A_0^4 \rightarrow A_0^3 : (e, e', e'', e''') \rightarrow (e, e', e'''),
\]
and \( S_{K_2} \) satisfies the equalities
\[
S_{V_1} \cdot S_{K_2} = S_{V_1} \cdot S_{V_1}' \quad \text{and} \quad S_{V_2} \cdot S_{K_2} = S_{K} \cdot S_{V_2}'.
\]
The associativity axiom \( S_{K} \cdot S_{K_1} = S_{K} \cdot S_{K_2} \) then follows from the unicity of factors and from the following cube, whose down face commutes due to the associativity axiom satisfied by the \( \mathcal{V} \)-category \( A \) and whose lateral faces are commutative, by definition of the different factors. Hence, \( S \) defines a realization \( S : \sigma \rightarrow \mathcal{V} \) of \( \sigma \) in \( \mathcal{V} \), i.e., a category internal to \( \mathcal{V} \), which will be denoted by \( \Gamma A \).
2° a) Let $F : A \to A'$ be a $V$-functor, $F_0 : A_0 \to A'_0 : e \mapsto Fe$ the map between objects and $F(e, e') : A(e, e') \to A(Fe, Fe')$ the canonical morphism, for every couple $(e, e')$ of objects of $A$. There exist factors $\Gamma F(2) : \Gamma A(2) \to \Gamma A'(2)$ of $(F(e, e'))_{e, e'}$ with respect to $F_0 \times F_0$, $\Gamma F(1) : \Gamma A(1) \to \Gamma A'(1)$ of $(I_e = l)_e$ with respect to $F_0$, $\Gamma F(3) : \Gamma A(3) \to \Gamma A'(3)$ of $(F(e, e') \times F(e', e''))_{e, e', e''}$ with respect to $F_0 \times F_0 \times F_0$.

These factors render commutative the diagrams

whose down faces commute by definition of a $V$-functor. This proves that $\Gamma F : \Gamma A \to \Gamma A'$ is a functor in $V$.

b) This defines a functor $\Gamma : V\text{-Cat} \to \text{Cat } V : F \mapsto \Gamma F$, due to the unicity of the factors defining $\Gamma F(i)$, $i = 1, 2, 3$. $V$

PROPOSITION B. The functor $\Gamma : V\text{-Cat} \to \text{Cat } V$ constructed above admits a right adjoint.

PROOF. Let $B$ be a category in $V$. 

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We define a $\mathbf{V}$-category $\mathbf{B} = \Gamma'B$. The class $\mathbf{B}_0$ of its objects is the set of morphisms $e : I \to B1$. If $e : I \to B1$ and $e' : I \to B1$ are such objects, $\mathbf{B}(e, e')$ is defined by the pullback

$$\begin{array}{ccc} [B\alpha, B\beta] & \rightarrow & B(e, e') \\
\downarrow & & \downarrow \\
B1 \times B1 & \rightarrow & B(e, e') \\
\{e, e'\} & \downarrow & \{e, e'\} \\
& & B(e, e') \\
\end{array}$$

where $\{e, e'\}$ and $[B\alpha, B\beta]$ are the factors of $(e, e')$ and $(B\alpha, B\beta)$ through the product $B1 \times B1$. There exists a factor $u_e : I \to B(e, e')$, through the pullback $(D_{e,e})$, of the diagram

$$\begin{array}{ccc} [B\alpha, B\beta] & \rightarrow & B(e, e') \\
\downarrow & & \downarrow \\
B1 \times B1 & \rightarrow & B(e, e') \\
\{e, e\} & \downarrow & \{e, e\} \\
& & B(e, e') \\
\end{array}$$

(which commutes, since $B\alpha.B\iota$ and $B\beta.B\iota$ are identities). Let $e''$ be another «object» $e'' : I \to B1$. The commutative diagram

$$\begin{array}{ccc} B1 & \rightarrow & B(e, e') \times B(e', e'') \\
\downarrow & & \downarrow \\
B\alpha & \rightarrow & B(e, e') \times B(e', e'') \\
\downarrow & & \downarrow \\
B2 & \rightarrow & B(e, e') \times B(e', e'') \\
\downarrow & & \downarrow \\
\{e, e''\} & \rightarrow & B(e, e') \times B(e', e'') \\
\end{array}$$

factors uniquely through the pullback

$$\begin{array}{ccc} B1 & \rightarrow & B2 \rightarrow B3 \\
\downarrow & & \downarrow \\
B\alpha & \rightarrow & B2 \rightarrow B3 \\
\downarrow & & \downarrow \\
B2 & \rightarrow & B2 \rightarrow B3 \\
\downarrow & & \downarrow \\
\{e, e''\} & \rightarrow & B(e, e') \times B(e', e'') \\
\end{array}$$

into $t_{e,e',e''} : B(e, e') \times B(e', e'') \to B3$, and the diagram

$$\begin{array}{ccc} [B\alpha, B\beta] & \rightarrow & B(e, e') \times B(e', e'') \\
\downarrow & & \downarrow \\
B1 \times B1 & \rightarrow & B(e, e') \times B(e', e'') \\
\{e, e''\} & \downarrow & \{e, e''\} \\
& & B(e, e') \times B(e', e'') \\
\end{array}$$

commutes (this uses the equalities $B\alpha.B\kappa = B\alpha.B\nu_1$ and $B\beta.B\kappa = B\beta.B\nu_2$ of an internal category, and the commutativity of $(D_{e,e})$ and $(D_{e',e''})$). Hence this diagram factors uniquely through the pullback $(D_{e,e''})$ into
b) This defines a $V$-category $B$.

(i) Let us denote by $u_{e,e'}$, the composite

$$B(e,e') \xrightarrow{u_e \times id} B(e,e) \times B(e,e').$$

In the diagrams

all the faces commute, except perhaps the back one; as $B_{\nu_i}$ are projections of a pullback, it follows that this last face also commutes. So, we have the commutative diagram

and the unicity of the factor through the pullback $B(e,e')$ implies that $\kappa_{e,e,e'} u_{e,e'}$ is an identity. Therefore, $B$ satisfies the unitarity axiom.

(ii) A similar method proves that $B$ satisfies the associativity axiom. It uses the fact that there is a cube

in which all the vertical edges are projections of pullbacks and all faces, except perhaps the up face commute; so this up face also commutes.
2° There is an internal functor \( t: \Gamma B \to B \). Indeed, let \( t(2) \) be the factor through the coproduct \( \Gamma B(2) \) (constructed in Proposition A) of the family \( (t_{e,e'}: B(e,e') \to B2)_{e,e'} \) indexed by \( B_o \times B_o \), so that:

\[
\begin{array}{ccc}
\Gamma B(2) & \xrightarrow{j_{e,e'}} & B(e,e') \\
\downarrow t(2) & & \downarrow t_{e,e'} \\
B2 & & B2
\end{array}
\]

commutes. Let \( t(1): \Gamma B(1) \to B1 \) be the factor through the coproduct \( \Gamma B(1) \) of the family \( (e)_{e} \) indexed by \( B_o \), so that

\[
\begin{array}{ccc}
\Gamma B(1) & \xrightarrow{j_{e}} & l \\
\downarrow t(1) & & \downarrow e \\
B1 & & B1
\end{array}
\]

commutes. Then the back face of the diagram

\[
\begin{array}{ccc}
\Gamma B(1) & \xrightarrow{j_{e}} & B(e,e) \\
\downarrow t(1) & & \downarrow t(2) \\
B1 & & B2
\end{array}
\]

commutes, because all the other faces commute and \( \Gamma B(1) \) is a coproduct. Similarly, the back face of the diagram

\[
\begin{array}{ccc}
\Gamma B(2) & \xrightarrow{j_{e,e'}} & B(e,e') \\
\downarrow t(2) & & \downarrow t_{e,e''} \\
B2 & & B3
\end{array}
\]

commutes, where \( t(3) \) is the factor of \( (t_{e,e',e''}) \) through the coproduct \( \Gamma B(3) \). We have so defined an internal to \( V \) functor \( t: \Gamma B \to B \).

3° \( t: \Gamma B \to B \) is the coliberty morphism defining \( B \) as a cofree object generated by \( B \). Indeed, let \( A \) be a \( V \)-category and \( t': \Gamma A \to B \) be a functor in \( V \). We are going to construct a \( V \)-functor \( T: A \to B \).

b) For each object \( a \) of \( A \), let \( Ta \) be the object of \( B \):
where \( j_a \) is always the injection into the coproduct; this defines a map \( T_0 : A_0 \to B_0 \). If \( a \) and \( a' \) are objects of \( A \), the two small squares of the diagram

\[
\begin{array}{c}
\begin{array}{cc}
& A(a, a') \\
\Gamma A(1)^2 & \\
\Gamma A(1)^2 & \\
[1, j_a, j_a] & \\
(1, \Gamma A, \Gamma A) & \\
& \Gamma A(2) \\
\end{array}
\end{array}
\]

are commutative (by definition of \( \Gamma A \) and \( t' \) being an internal functor). Hence the exterior square is commutative, and it factors through the pullback \((D_T a, D_T a')\) into a unique \( T(a, a') : A(a, a') \to B(T a, T a') \).

b) This defines a V-functor \( T : A \to B \). Indeed, for each object \( a \) of \( A \), the up face of the diagram

\[
\begin{array}{c}
\begin{array}{cc}
B(T a, T a') & A(a, a') \\
\Gamma A(2) & \\
\Gamma A(2) & \\
t'(2) & \\
\end{array}
\end{array}
\]

commutes, since all the other faces commute and \( B(T a, T a') \) is a pullback. Similarly, the up face of the following cube

\[
\begin{array}{c}
\begin{array}{cc}
B(T a, a') \times T(a', a') & A(a, a') \\
\Gamma A(2) & \\
\Gamma A(2) & \\
t'(2) & \\
\end{array}
\end{array}
\]
commutes, all the other faces commuting and \( B(\tau a, \tau a') \) being a pull-back. Hence, \( T : A \rightarrow B \) is a \( V \)-functor.

c) The down face of the diagram

```
\[
\begin{array}{ccc}
B(\tau a, \tau a') & \rightarrow & T(a, a') \\
\downarrow & & \downarrow \\
\Gamma(2) & \rightarrow & A(a, a')
\end{array}
\]
```

commutes, whatever be the objects \( a', a \) of \( A \) since the other faces commute and \( \Gamma A(2) \) is the coproduct of \( (A(a, a'))_{a, a'} \). It follows that

\[
(t' : \Gamma A \rightarrow B) = (\Gamma A \xrightarrow{T} \Gamma B \xrightarrow{t} B).
\]

Finally, the unicity of the \( V \)-functor \( T \) satisfying this equality results from the unicity of the morphisms \( T(a, a') \). So \( B \) is a cofree object generated by \( B \) with respect to \( \Gamma : V\text{-Cat} \rightarrow Cat V \).

**DEFINITION.** A category in \( V \) is called **pseudo-discrete** if its object of objects is a coproduct of copies of the final object \( I \).

By the construction of the functor \( \Gamma \) (Proposition A), it takes its values into the full subcategory \( PsCat V \) of \( Cat V \) whose objects are the pseudo-discrete categories in \( V \). Hence it admits as a restriction a functor, also denoted by \( \Gamma : V\text{-Cat} \rightarrow PsCat V \). Remark that the existence of this functor is conjectured (without precise hypotheses) in the Appendix III of the book [8].

**PROPOSITION C.** Let \( V \) be a category with commuting coproducts, \( I \) its final object. If the functor \( \text{Hom}(I, -) : V \rightarrow V \) preserves coproducts, then the functor \( \Gamma : V\text{-Cat} \rightarrow PsCat V \) is an equivalence.

**PROOF.** Let \( \Gamma' : PsCat V \rightarrow V\text{-Cat} \) be the right adjoint of \( \Gamma \) constructed in Proposition B.
The composite

\[
\begin{array}{ccc}
PsCat \mathcal{V} & \xrightarrow{\Gamma'} & \mathcal{V}-\text{Cat} & \xrightarrow{\Gamma} & PsCat \mathcal{V}
\end{array}
\]

is equivalent to the identity:

Indeed, it suffices to prove that, for each pseudo-discrete category \( B \) in \( \mathcal{V} \), the coliberty morphism \( t': \Gamma' \Gamma' B \to B \) is an isomorphism. By hypothesis, \( B \mathcal{I} \) is the coproduct of a family \( (I_\lambda = I)_{\lambda \in \Lambda} \) and

\[
\text{Hom}(I, B \mathcal{I}) \cong \prod_{\lambda} \text{Hom}(I, I_\lambda) \cong \Lambda
\]

since \( \text{Hom}(I, I) \) is reduced to the identity of \( I \); hence \( B \mathcal{I} \) is also a coproduct of the family \( (I_e = I)_e \) indexed by the set \( B_0 = \text{Hom}(I, B \mathcal{I}) \) of morphisms \( e: I \to B \mathcal{I} \), the \( e \)-th injection being \( e \) itself. As the partial product functors preserve coproducts, \( B \mathcal{I} \times B \mathcal{I} \) is the coproduct of the family \( (I_{e,e'} = I)_{e,e'} \) indexed by \( B_0 \times B_0 \), the injections being the factor \( [e,e']: I \to B \mathcal{I} \times B \mathcal{I} \) into the product. We take the pullback

\[
\begin{array}{ccc}
I & \xrightarrow{l} & B(e,e') \\
\| & & \| \\
B \mathcal{I} \times B \mathcal{I} & \xrightarrow{[B_\alpha, B_\beta]} & B \mathcal{I}
\end{array}
\]

used to define \( B = \Gamma B \). The category \( \mathcal{V} \) admitting commuting coproducts, by pulling back along \([B_\alpha, B_\beta]\) the coproduct \( B \mathcal{I} \times B \mathcal{I} \), we get \( B \mathcal{I} \) as a coproduct of \( (B(e,e'))_{e,e'} \), the injections being the morphisms \( t_{e,e'}: B(e,e') \to B \mathcal{I} \). So the factor \( t(2): \Gamma B \mathcal{I} \to B \mathcal{I} \) of \( (t_{e,e'})_{e,e'} \) is an isomorphism. This implies that \( t: \Gamma B \to B \) is an isomorphism.

The composite

\[
\begin{array}{ccc}
\mathcal{V}-\text{Cat} & \xrightarrow{\Gamma} & PsCat \mathcal{V} & \xrightarrow{\Gamma'} & \mathcal{V}-\text{Cat}
\end{array}
\]

is also equivalent to the identity, so that \( PsCat \mathcal{V} \) and \( \mathcal{V}-\text{Cat} \) are equivalent. Indeed, let \( A \) be a \( \mathcal{V} \)-category; by adjunction, there is a \( \mathcal{V} \)-functor \( \Gamma: A \to \Gamma' \Gamma A \) such that \( \Gamma A \) is the injection \( j_a: I \to \Gamma A(1) \) for each object \( a \) of \( A \) and that the following diagram commutes, for each couple \((a,a')\) of objects of \( A \) (we take up the notations of Proposition B, in which we choose \( B = \Gamma A \)).
We are going to prove that $T$ is an isomorphism.

a) $T_0 : A_0 \rightarrow (\Gamma'A)_{0}$ is 1-1 and onto: $\Gamma'A(1)$ is the coproduct of the family $(I = I_a)_a$ indexed by the set $A_o$ of objects of $A$; since $\text{Hom}(I, -) : \mathcal{V} \rightarrow \mathcal{V}$ preserves coproducts, we have

$$\text{Hom}(I, \Gamma'A(1)) = \bigsqcup_a \text{Hom}(I, I_a) = A_0,$$

so that $T_0$ is an isomorphism.

b) For every objects $a$, $a'$ of $A$, there is a pullback

$$I \quad \Gamma'A(Ta, Ta') \quad \Gamma'A(1) \quad \Gamma'A(2)$$

$$\begin{array}{c}
\downarrow [j_a, j_{a'}] \\
\Gamma'A(1) \quad \Gamma'A(1) \quad \Gamma'A(1) \quad \Gamma'A(1)
\end{array}$$

$$t_{Ta, Ta'}$$

defining $\Gamma'A(Ta, Ta')$. We deduce as in Part 1 that $\Gamma'A(2)$ is the coproduct of $(\Gamma'A(Ta, Ta'))_{a, a'}$ with injections $t_{Ta, Ta'}$. But (by definition) $\Gamma'A(2)$ is also the coproduct of $(A(a, a'))_{a, a'}$, and the commutativity of the diagrams defining $T(a, a')$ implies that the identity of $\Gamma'A(2)$ is the coproduct of $(T(a, a'))_{a, a'}$. So, by definition of a category with commuting coproducts, each $T(a, a')$ is an isomorphism. Hence $T : A \rightarrow \Gamma'A$ is an isomorphism.

**COROLLARY.** If $\mathcal{V}$ is a category with commuting coproducts, the functor $\Gamma : \mathcal{V}\text{-Cat} \rightarrow \mathcal{PsCat}\mathcal{V}$ is an equivalence iff the endofunctor $\text{Hom}(I, -)$ preserves coproducts of copies of the final object $I$.

**PROOF.** The preceding proof shows that the condition is sufficient. On the other hand, let us suppose that $\Gamma : \mathcal{V}\text{-Cat} \rightarrow \mathcal{PsCat}\mathcal{V}$ is an equivalence and let $S$ be the coproduct of a family $(I_\lambda = I)_\lambda$. There exists a $\mathcal{V}$-category $A$ (the «$\mathcal{V}$-groupoid of pairs of $\Lambda$») such that $\Lambda$ is the set of its objects and $A(\lambda, \lambda') = I$ for each couple $(\lambda, \lambda')$ of objects. The canonical $\mathcal{V}$-functor

$$T : A(\lambda, \lambda') \rightarrow \Gamma'A(T\lambda, T\lambda')$$
being an isomorphism by hypothesis, its «restriction to the objects»:
\[ T_0 : (A_0 = \Lambda) \rightarrow (\Gamma \Gamma \Lambda)_0 = \text{Hom}(I, S) \]
is an isomorphism, and \( \text{Hom}(I, S) \cong \Lambda = \prod_{\lambda \in \Lambda} \text{Hom}(I, I_{\lambda}) \).

**Examples.**

1º There are many examples of categories \( V \) with commuting coproducts (see Penon [21]):
- the elementary topoi admitting coproducts,
- the categories admitting finite limits and coproducts and equipped with a faithful functor toward \( \text{Set} \) preserving pullbacks and coproducts and reflecting isomorphisms; in particular, the initial structure categories (Wischnewsky [22], or topological categories in the sense of Herrlich [18]), the categories \( \text{Cat}_n \) for any integer \( n \).

The condition that \( \text{Hom}(I, -) : V \rightarrow V \) preserves coproducts means that \( I \) is connected (in the sense of Hoffmann [19], see also Proposition 3-12 of Penon [21]). It is satisfied in the categories of a «topological nature», as well as in \( \text{Cat}_n \). Remark that an \((n+1)\)-fold category \( H \) (considered as a category in \( \text{Cat}_n \), see Appendix [5]) is pseudo-discrete, and therefore «is» a \( \text{Cat}_n \)-category, by Proposition C, iff the objects of the last category \( H^n \) are also objects for the \( n \) first categories \( H^i \) (in an \((n+1)\)-category, the objects of \( H^n \) are only supposed to be objects for \( H^{n-1} \)). The \((n+1)\)-category \( \text{Nat}_n \) constructed in Proposition 8 «is» pseudo-discrete.

2º Proposition C is also valid if \( V \) is the category of \( r \)-differentiable manifolds (modelled on Banach spaces), though only some pullbacks exist in it (the pullbacks used in the proof will exist). Hence categories whose \( \text{Hom} \) are equipped with «compatible \( r \)-differentiable structures «are» those \( r \)-differentiable categories (in the sense of [12]) in which the topology induced on the class of objects is discrete.
REFERENCES