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**THE GENESIS OF MATHEMATICAL STRUCTURES, AS EXEMPLIFIED
IN THE WORK OF CHARLES EHRESMANN**

by Saunders MAC LANE

1. STRUCTURE.

Recently I have become interested in the need to revive the study of the philosophy of mathematics. Such a revival should avoid the well-worn and hackneyed standpoints - logicism, formalism and intuitionism. It must also escape from the inordinately idealistic demands of platonism and the prevalent but mistaken view that sets are the measure and matter of all mathematics. In a recently prepared article [23] I have presented a preliminary description of an alternative view, that mathematics starts from a variety of human activities and goes on to construct many interlocking formal models suggested by various abstractions and generalizations from these human activities. This point of view, to be effective, must rest on an extended observation of the actual process by which different mathematical structures are discovered (or invented). This is intended as a sharp break with the style of most writing on the philosophy of mathematics, which usually involves very little reference to any mathematics beyond counting and elementary geometry.

Thus we hope to examine how mathematical structures arise - not principally how they actually happened to arise in the historical development, but how they logically and genetically must arise. A structure may be algebraic or geometric, described by axioms, definitions, or constructions, set-theoretic or topos-theoretic. Its genesis starts with observations of analogies, parallels, and symmetries, and of various needs for conceptual or invariant description. From these observations various standard processes of abstraction or generalization will lead to the structures at issue. We will be interested not just in the standard forms of abstraction, but in a variety of specific cases where the process of genesis is more subtle.

Our examples will be drawn largely from geometry and algebra, but there should be many striking cases elsewhere in mathematics.

The work of Charles Ehresmann provides splendid examples of the genesis of mathematical structures. From a starting point in Differential Geometry and Topology, he took the initiative in disentangling several decisive geometrical structures; following the logic of these discoveries, he went on to locate many more structures in and around categories. Here we can mention only a few of his contributions to such discovery of structure.

2. FIBRATIONS.

First, consider fiber spaces and fiber bundles. In 1940, algebraic topology was coming of age; the techniques of homology, cohomology, co-products, and homotopy groups were at hand, so that geometers could turn their attention to the use of these tools in handling tangent and cotangent bundles on manifolds, sphere bundles, and similar objects. The notion of a fiber bundle for a group G was ready for development. One of the first formal definitions was that given by Ehresmann and Feldbau in 1941 [18]. Their definition involved four essential objects (E, B, F, G) , where E is a topological space with a projection $p: E \rightarrow B$ to the base space B , the topological space F is the typical fiber, while G is a group which acts (perhaps continuously) on F - all such that locally, over suitable open sets U in B , $p^{-1}U$ has the structure of a product $F \times U$ with the given G action on the first factor. This first paper by Ehresmann and Feldbau was followed by two other brief notes on fiber spaces by Ehresmann alone [7, 8]. Together these papers included the now familiar ideas of induced bundle (or «associated» bundle), principal bundle, and the process of reducing the group G of a fibration to a subgroup. Moreover, the first paper clearly stated a form of the covering homotopy theorem and some of its consequences for the relations between the homotopy groups of base, fiber, and total space. These ideas were in the air at the time; as noted in the review [29] by Weil of Ehresmann's papers, his definition of a fiber space was intermediate in generality to definitions crafted at about the same time by Whitney [33] (who emphasized sphere-bundles) and by Hurewicz-Steenrod [21] as well

as by Eckmann [6]. They too had seen the importance of the covering homotopy theorem; since communications were interrupted by the war, these must count as independent discoveries. Ehresmann's own detailed proof of the covering homotopy theorem was published subsequent to the war [11], as was his extension of his ideas to the consideration of differentiable fiber spaces [10].

The notion of a fiber space is a striking example of the genesis of a complex mathematical structure. This structure involves several objects, both spaces and groups. Examples of the intended structure are manifest, but it is by no means clear how to arrange all these examples under any clear concept. Finding the right concepts must have been hard. I do not know at first hand how it was in France or in Strasbourg (where Ehresmann was then a professor). I do recall that it was hard for topologists in the United States. In 1940 there was a well attended conference on topology at the University of Michigan; there Whitney lectured on sphere bundles. Most of the audience found his comments very hard to grasp. Even subsequently, when Steenrod's book on fiber bundles [26] had served to codify and clarify these ideas, it was hard for students to grasp - especially at the use of a maximal atlas (avoided by Steenrod, loc. cit., page 9, out of needless logical compunctions).

The subtle notion of an atlas (discussed by Ehresmann in several of his papers, for example in [11]) was essential to the conceptual definition of a manifold «in the large». This is another example of a non-obvious structure. For years the intentions of differential geometry were clearly global, but the practice was strictly local; tensors and all other geometric objects were defined only in terms of one coordinate system, supplemented by rules for changes of coordinates (as in the 1927 tract by Veblen on Quadratic Differential forms [27]). The first explicit global definition of a manifold appeared (in cumbersome form) in the subsequent tract by Veblen and Whitehead [28], while the present form of the definition of a manifold in the large first appeared in the paper by Whitney [31] on embeddings of manifolds. One may doubt that this idea was much heeded until it was taken up and exploited succinctly by Chevalley in his first book on Lie groups

[3]. In short, manifolds were hard to define conceptually, and this is one of the reasons for the depth of the derivative idea of a fiber space.

Actually, the idea of a fibration involved two different but related kinds of structures, which were only slowly disentangled. On the one hand, fiber *bundles* are given complete with the action of a specific group G on the fiber, because this is what is really present in examples like the tangent bundles of differential geometry. These bundles were the subject of Ehresmann's work. However, Steenrod and others considered potentially more general fibrations $p: E \rightarrow B$ with no specified group action but with suitable local retractions into the fibers. This latter structure has a variety of definitions, and was disentangled from the notion of a bundle structure only slowly, so that the right definition was found only gradually. First came the recognition of the importance of the covering homotopy theorem, and of its validity both for bundles and in other cases such as the space $E(B)$ of all paths (from a fixed base point) in a space B , with fiber the loop space. Only then was the covering homotopy theorem raised to become a definition: A Serre fibration $p: E \rightarrow B$ is simply any continuous map such that the covering homotopy theorem holds for polyhedra (or similarly, a Hurewicz fibration with the covering homotopy theorem holding for more general spaces).

This is one typical step in the genesis of a mathematical structure - a shift of attention from a previous definition to the recognition that some one property (here the covering homotopy theorem) is central and so can and should be used to define the structure to be studied.

At this point one should also note that the concept of a Serre fibration involves much more than the covering homotopy theorem, since the added structures given by the various associated spectral sequences also enter. Probably, a spectral sequence should be viewed not as an independent kind of structure, but as an (important) algebraic constituent of the structure «fiber space». In many such cases, each major structure carries with it a battery of techniques, devices and subordinate structures. Thus it is that fiber spaces are associated with the half-geometric operations of transgression and suspension, which in their turn are pieces of the more elaborate

structure of a spectral sequence (in homology or cohomology).

This observation of associated structures can apply in many other cases. Thus the structure of a group is a straightforward one, described by simple axioms abstracted from the evident examples of substitution groups. The deeper investigation of finite groups shows that there is a great deal of subordinate structure - for each prime p a Sylow structure, with much associated local structure, as described for example by Alperin [1].

Another geometric structure studied by Ehresmann [14] is that of an almost complex structure on a manifold - this being what might be called a subordinate structure to that of complex structure. Jointly with Wu Wen-Tsun, Ehresmann made several vital contributions to the recognition and analysis of this structure and of related structures such as abstract hermitian structures.

3. FOLIATIONS.

A foliation of a n -dimensional manifold M can be described as a partition of that manifold in a smooth way, into k -dimensional manifolds, the «leaves» of that foliation. In 1944 Ehresmann and Reeb [20], on considering the properties of completely integrable Pfaffian systems, came to the notion of such a foliation, further developed in Reeb's 1948 thesis (at Strasbourg). The need for analysing such structure was clearly then «in the air». In simpler cases the work of Whitney [32] and his student Kaplan [23] dealt with families of curves filling the plane. The Ehresmann-Reeb analysis of foliations is thoroughly developed; in particular Ehresmann observed that a foliation of a manifold M (or a more general space) could be described conceptually in terms of two topologies on the set M - one the topology of the manifold M itself; the other given by the discrete union of the topologies of the leaves, as described systematically in Ehresmann [17]. Here again the path from the examples in differential systems to the resulting conceptual description of the structure was a long one, and one idea stimulated another by analogy. For example, as Ehresmann pointed out in 1951 [12], the projections of a foliated manifold upon its space of leaves (the quotient space) is much like the projection of a fiber space.

4. JETS.

Tangents are special cases of jets. The concept of a k -jet at a point in a manifold is an important geometric structure first recognized by Ehresmann in 1951. As one knows, the k -jet of a real-valued function f defined in the neighborhood of a point p of a manifold M is determined by the first k derivatives of the function f at the point p . The study of jets may well have led Ehresmann to the investigation of category theory; at any rate his typical categorical notation of α and β for source and target of a morphism appear in his work on jets [13].

More generally, the study of jets can be seen as a development of the earlier geometrical idea of studying the «infinitely nearby» points on algebraic curves and manifolds. Presumably it was Ehresmann's initiative which stimulated the 1953 paper by Weil [29] on «points proches»; in their turn, these ideas (Weil algebras) enter into Dubuc's models for synthetic differential geometry [4]. Disentangling structures from geometric phenomena to their categorical formulation is a long process! Can we describe how it is done?

5. ABSTRACTION AND GENERALIZATION.

Ordinarily, the genesis of mathematical structure is not formally described. However, there is an accepted «standard» view of the process. The process starts with *observation* of elementary mathematical phenomena (or premathematical phenomena), geometric, algebraic, or otherwise. One notes analogies - that power series are like congruences modulo p and may suggest p -adic series, or that the projection of a fiber bundle to the base B is like the projection of a foliated manifold onto the space of the leaves of the foliation. One notices phenomena of invariance - this or that geometric object does not really depend on its expression in any one coordinate system. One analyses simple proofs or even fallacious proofs - just as the analysis of the covering homotopy theorem led to a deeper understanding of fiber bundles and spectral sequences. Again, Kummer's analysis of his own failed proof of Fermat's last theorem led him to «ideal» prime factors of algebraic integers, and these ideal factors were subsequently revealed

to be «ideals» in the present structural sense.

After observation comes *generalization*. There are probably several different sorts of generalization. First, there is what I might call generalization by *parallels*. Thus plane, solid, and 4-dimensional geometry lead to n -dimensional geometry and on beyond to infinite dimensional spaces. The observation of the systems of real numbers, complex numbers, and quaternions suggests the study of division algebras and linear algebras more generally. Quadratic number fields, cyclotomic number fields (as with Kummer) and then abelian number fields are the source of the more general consideration of algebraic number fields. The study of algebraic varieties arose from sequential consideration of conics, cubics, quartics, and general plane curves and their algebraic surfaces.

Generalization can also take place by what could be called *modification*. A desirable theorem, true in some cases, breaks down in others. The mathematician searches for ways to restore its original correctness. Thus prime ideals of algebraic integers restored the unique prime factorization valid for rational integers. The Jordan curve theorem, proved with such painful displays of rigor, fails for surfaces more general than the sphere - but the right ideas about homology will restore the *status quo*.

Abstraction is a somewhat different genetic process of generalization. The vital elements of a preceding theory are not just generalized as they stand, but are altered to apply to other, more abstract entities, which may have been there all along - hidden in some specific presentation. Such observations can take place by *analogy*. For example, observations on the multiplicative group of integers prime to n , modulo n and of other structures arising in number theory came together to suggest the much more abstract notion of an abelian group and of the structure theorems for finitely generated such groups. Or the analogy between formal power series and p -adic numbers suggests the abstract notions of a field with a (discrete) valuation. Or the examples of smooth, C^1 , and analytic manifolds suggest more general concepts of manifolds and local structures. Or these specific examples of local structures may, as with the work of Ehresmann [15] suggest a general notion of local structure, as expressed say by the groupoid of allowable coordinate transformations.

Another process we might call abstraction by «subtracting». One takes a given situation - say proofs of a given type - and discovers that everything can be done (the proof remaining valid) with only part of the data. The transition from groups of transformations to abstract groups and that from algebra of sets to Boolean algebra are examples of such abstraction by subtraction. Such cases can give rise to representation theorems which «justify» the process of subtraction by showing that every one of the abstract objects can be presented (perhaps in more than one way) as an unabstracted object. Instances are the Cayley theorem representing every abstract group as a permutation group of some set (*not* necessarily all permutations of that set) and the Stone theorem representing every Boolean algebra as an algebra of subsets of a universe (not necessarily all subsets of that universe).

There can also be abstraction by «shift of attention». On the basis of considerable experience, one discovers that the important features of a situation may be quite different from those which appear on the surface. An example is the discovery of the notion of a topological space. Initially, mathematicians were concerned with quite specific spaces, where one sees nearness in terms of the distances between points or of small neighborhoods of points - and used distance or neighborhoods to define, *inter alia*, the continuity of functions. It requires a massive shift of attention to think primarily of the continuity of functions and to observe the remarkable fact that continuity can be defined by using *only* the knowledge of those subsets of the space which are *open*. The result of this is the appearance of an essentially new mathematical object - a topological space. Recent studies, as yet to be published, of my student Joel Fingerma, have emphasized to me how massive was the shift of attention in this case. It would be difficult to describe in general the process of such a shift, since it is hard to find many really similar cases. For example, continuity requires only open sets, but there appears to be no way to describe the differentiability of a function just in terms of selected subsets of its domain and codomain.

6. MORE COMPLEX STRUCTURES.

This completes our list of the generally accepted processes of generating new mathematical structures; one observes analogies and invariances. One then generalizes by parallel cases or by modifications while abstraction takes place by processes of analogy, subtraction, or shift of attention. A mathematical structure so discovered consists of one or more sets with additional data. If the structure is algebraic, the additional data consists of unary, binary, ..., n -ary operations, satisfying appropriate identities. In more geometric cases, the data may also involve binary or tertiary relations, «specified elements» of double power sets (the open sets of a space) and the like, again subject to suitable conditions. Structures in this sense are the subjects of study in universal algebra, model theory, Bourbaki's early description of structure [2] or the «geometric theories» currently under examination in topos theory.

Other examples, in particular many of those arising in the work of Charles Ehresmann, indicate that actual mathematical structure can be much more complex, and that the genetic process of its discovery is much more convoluted. Thus the notion of a fiber bundle is not really just that of a few sets with some added structure, but it is built up in a complex way out of previous structural concepts (space, group, and group action). Similarly, the notion of a fiber space hides subordinate structures, now revealed only in part by the array of differentials of the associated spectral sequences. A foliation can be described as a pair of topologies on a set - but these two topologies have very specific character, suggested by the intended geometric examples. For group theory, the algebraic structure needed to describe a group is simple and straightforward, but the properties flowing from this apparently simple structure are subtle and manifold. I suggest that this is because the basic group structure carries with it many subordinate or associated structures - all the associated spectral sequences and their differentials for the group extensions involved and the local structure (Sylow structures for each prime p) so heavily used in the current work on finite simple groups (Alperin [1]).

Another instance of a hidden structure is the presence of «prime

spots» in an algebraic number field. Originally, they do not appear with the field. Only later, with the discovery of prime ideals to restore unique factorization or valuations to understand the analogy with algebraic function, do the prime spots gradually stand revealed - described then not as one structure but in many equivalent ways (prime ideal, valuation, a map into a complete field). Mathematics is a network of hidden structures.

7. CATEGORIES AND STRUCTURE.

The complexity of describing an algebraic or a topological structure in terms of things attached to a set is one of the main reasons for the introduction of the notion of a category - where the objects with a given structure are not described by explicit sets, as in Bourbaki [2], but as abstract objects equipped only with morphisms suitably composable. This view, and its expansion, must have motivated much of Ehresmann's work. Studying manifolds and their associated structures (connections, jets, foliations) he shifted attention to the coordinate transformations (on the intersection of allowable neighborhoods in an atlas), noted that they formed a groupoid, and so came to use categories to formulate local structures [15]. Not stopping there, he went on to discover double categories [18] (such is just a category in the category *Cat*) and topological categories [16] (that is, a category in the category of topological spaces). By these two examples he paved the way for the general recognition of the notion of an object with a specified structure in a category. He made vital contributions to the study of such structures by means of sketches (where the structure is described by specified local cones to be preserved). His most recent work, with Madame Ehresmann, on multiple functors [7] and the existence of lax limits, again arises from such study of structure.

The thesis of this paper is this: Just as category theory was needed to analyse flexibly the notion of mathematical structure, so will there be a need for more explicit and precise analysis of the genesis of mathematical structure, going beyond the familiar simple stories about generalizations and abstraction so as to get at the subtle interplay between different structures.

8. AFTERTHOUGHTS.

The original version of this paper was delivered on the first day of Colloquium at Amiens. This allowed time for several of the participants to tell me what I had omitted. They gave me many examples :

(a) There should be more emphasis on the subject matter of mathematics so as to distinguish what is mathematics more clearly, say, from what is physics.

(b) My emphasis on structure does not sufficiently recognize the importance of mathematical problems ; mathematics can better be described as the attempt to solve a sequence of specific problems, with new structures used as a means to get at such problems.

(c) An adequate philosophy of mathematics would surely need to explain why particular items in mathematics are interesting, while others are uninteresting.

(d) Among existing philosophies of mathematics, my list omits empiricism and materialism.

(e) My negative observation on Russell and Hilbert does not note that they were addressing the central philosophic problem facing mathematicians : Now that mathematics has considerably expanded beyond the old and simple confines of number and geometry, how does one explain anew the security and applicability of mathematics ?

(f) My criticism of existing philosophies of mathematics should have observed that the major gap is a failure to adequately explain how formal mathematical structures can be applied.

(g) Lawvere's penetrating analyses of the nature of mathematics clearly show that the study of the philosophy of mathematics is not now dead.

(h) The idea of using «neighborhoods» to define the topology of the plane is due to Hilbert (1902 [21]).

I look forward to future meetings when these and other omissions might be explored.

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THE GENESIS OF MATHEMATICAL STRUCTURES

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