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ALGEBRAIC LOCALIZATIONS AND ELEMENTARY TOPOSES

by Francis BORCEUX

If A is a commutative ring and $\mathfrak{p}$ a prime ideal in A, we can localize A at $\mathfrak{p}$ and get a new ring $A_{\mathfrak{p}}$ provided with a morphism $A \to A_{\mathfrak{p}}$. This induces a functor between the corresponding categories of modules

$$Mod_{A_{\mathfrak{p}}} \longrightarrow Mod_A .$$

This functor is full and faithful and has a left adjoint obtained by tensorization by $A_{\mathfrak{p}}$ over A:

$$- \underset{A}{\otimes} A_{\mathfrak{p}} : Mod_A \longrightarrow Mod_{A_{\mathfrak{p}}} .$$

This last functor is exact because $A_{\mathfrak{p}}$, seen as an A-module, is flat.

This basic example led Gabriel, in his thesis, to study the localizations of an abelian category. If $\mathcal{C}$ is a category with finite limits, a localization of $\mathcal{C}$ is simply a full reflective subcategory of $\mathcal{C}$ whose reflection preserves finite limits. This contains obviously the previous example. In particular, Gabriel has completely solved the problem of the localizations of the category of modules on an arbitrary small additive category. In that case, he classifies the localizations by means of what he calls a «localizing system on $\mathcal{C}$»: roughly speaking, this is the additive version of a Grothendieck topology.

About at the same time, the school of Grothendieck studied the localizations of the categories of set-valued presheaves on a small category. They showed that these localizations are exactly the Grothendieck toposes and they are classified by means of the Grothendieck topologies.

Later, Lawvere and Tierney proved that the localizations of an elementary topos are exactly the subtoposes of this one and these localizations are classified by means of the Lawvere-Tierney topologies on the object Ω of the topos.

Grothendieck, Grillet and other authors also considered the presheaves on a small category with values in a sufficiently regular category. They showed that any Grothendieck topology on the small category gave rise to a localization of the category of presheaves, but there are localizations which are not obtained in this way.

This short list of examples will probably be sufficient to convince of the interest of the problems of localization. For various reasons which, I hope, will become clearer through these pages, I have the feeling that the study of the localizations of algebraic categories will give rise to interesting results and notions, especially when working with respect to an elementary topos. In order to give some content to this very vague conjecture, I would like to study this problem in some details in the case of a topos of set-valued presheaves and an external theory.

Thus I consider a finitary algebraic theory T and a small category $\mathcal{C}$. I consider the topos of presheaves $\mathcal{E}_0 = [\mathcal{C}^{op}, Sets]$ and the category of T-algebras in this topos, that is the category $\mathcal{A} = [\mathcal{C}^{op}, Sets^{\mathrm{T}}]$. I am interested in classifying the localizations of $\mathcal{A}$. First of all, I would like to obtain a localization in terms of generalized Grothendieck topologies. For this, I denote by $U: Sets^{\mathrm{T}} \to Sets$ the forgetful functor and by

$$F: Sets \longrightarrow Sets^{\mathrm{T}}$$

the free algebra functor.

A Grothendieck topology on $\mathcal{C}$ is given by means of subfunctors of the representable presheaves

$$\mathcal{C}(-, X): \mathcal{C}^{op} \longrightarrow Sets.$$

These presheaves are not in $\mathcal{A}$ and thus it seems more reasonable to try to classify the localizations of $\mathcal{A}$ by means of subobjects of the functors

$$F\mathcal{C}(-, X): \mathcal{C}^{op} \longrightarrow Sets^{\mathrm{T}}$$

which are objects of $\mathcal{A}$. So I consider the following definition.

DEFINITION 1. A T-*topology on* $\mathcal{C}$ consists in given, for any object X of $\mathcal{C}$, a family $J(X)$ of subobjects of $F\mathcal{C}(-, X)$ such that:

(1) $F\mathcal{C}(-,X) \in J(X)$.

(2) If $v: F\mathcal{C}(-,Y) \to F\mathcal{C}(-,X)$ and $R \in J(X)$, then $v^{-1}(R) \in J(Y)$.

(3) If $R \in J(X)$ and $R' \rightarrowtail F\mathcal{C}(-,X)$ are such that $v^{-1}(R') \in J(Y)$, for any $v: F\mathcal{C}(-,Y) \to R$, then $R' \in J(X)$.

There is an application:

$$\{\text{ Localizations of } \mathcal{A} \} \longrightarrow \{\text{ T-topologies on } \mathcal{C} \}$$

which sends a localization on the T-topology given by the monomorphisms $R \rightarrowtail F\mathcal{C}(-,X)$ whose reflection is an isomorphism. This application can be seen to be injective and thus a localization of $\mathcal{A}$ can always be classified by a T-topology. Now the question arises: does every T-topology on $\mathcal{C}$ give rise to a localization of $\mathcal{A}$? The answer is: Yes if the theory T is commutative, no in general.

I give first a counterexample in the non-commutative case. I take $\mathcal{C} = 1$ and T the theory of groups. Then $\mathcal{A}$ is simply the category of groups and it can be shown that the category of groups has only two obvious localizations: (0) and the category of groups itself. Indeed, the unique basic functor $F\mathcal{C}(-,X)$ corresponds to the group Z of integers. Now if some non-obvious subgroup $n\mathrm{Z} \subset \mathrm{Z}$ becomes an isomorphism in some localization, then the same holds for

$$n\mathrm{Z} \amalg n\mathrm{Z} \subset \mathrm{Z} \amalg \mathrm{Z}.$$

If x and y denote the two basic generators in $\mathrm{Z} \amalg \mathrm{Z}$, consider the group homomorphism $f: \mathrm{Z} \to \mathrm{Z} \amalg \mathrm{Z}$ which sends 1 on the word $x^{n-1}y^{n-1}$. In the localization, the inclusion $f^{-1}(n\mathrm{Z} \amalg n\mathrm{Z}) \subset \mathrm{Z}$ becomes an isomorphism. But $f^{-1}(n\mathrm{Z} \amalg n\mathrm{Z}) = (0)$. Thus (0) is covering and thus any subgroup of Z is covering: the localization is obvious. On the other hand, there exist non-obvious group-topologies on 1, for example that which is given by all the non-zero subgroups of Z. (This is the topology which, in the case of abelian groups gives as localization the category of rational vector spaces.)

This counterexample shows that in the context of a general theory T, the good notion of T-topology needs additional axioms to the three basic axioms of Grothendieck.

In the commutative case, let us first remark that we have obtained a complete classification of the localizations of $\mathcal{A}$ by means of the T-topologies on $\mathcal{C}$: this improves the results of Grothendieck, Grillet and others. By considering only usual topologies on $\mathcal{C}$, they reduced themselves to consider the subobjects of the form

$$FR \rightarrowtail F\mathcal{C}(-,X)$$

where R is a crible on X. Clearly, there are many other subobjects of $F\mathcal{C}(-,X)$.

Now in order to study in more details the commutative case, it is useful to point out that in this case the category $Sets^{\mathrm{T}}$ is closed. Let us then consider the free category $\hat{\mathcal{C}}$ enriched in $Sets^{\mathrm{T}}$ and generated by $\mathcal{C}$. There is an isomorphism between the two categories

$$\mathcal{A} = [\mathcal{C}^{op}, Sets^{\mathrm{T}}] \approx Sets^{\mathrm{T}}[\hat{\mathcal{C}}^{op}, Sets^{\mathrm{T}}]$$

where, in the second case, the functors are enriched in $Sets^{\mathrm{T}}$. The basic functors $F\mathcal{C}(-,X)$ simply correspond to the $Sets^{\mathrm{T}}$-representable functors $\hat{\mathcal{C}}(-,X)$. The notations will become easier and the situation more general if I replace $\hat{\mathcal{C}}$ by a general small category $\mathcal{C}$ enriched in $Sets^{\mathrm{T}}$ and study the localizations of the category $\mathcal{A} = Sets^{\mathrm{T}}[\mathcal{C}^{op}, Sets^{\mathrm{T}}]$ of presheaves on $\mathcal{C}$ enriched in $Sets^{\mathrm{T}}$. This is what I do now.

Thus I consider a commutative finitary algebraic theory T. The category $Sets^{\mathrm{T}}$ of T-algebras is closed and I denote by

$$U: Sets^{\mathrm{T}} \to Sets \quad \text{and} \quad F: Sets \to Sets^{\mathrm{T}}$$

the forgetful and the free algebra functors. I consider a small $Sets^{\mathrm{T}}$-category $\mathcal{C}$ and the category $\mathcal{A} = Sets^{\mathrm{T}}[\mathcal{C}^{op}, Sets^{\mathrm{T}}]$ of $Sets^{\mathrm{T}}$-presheaves on $\mathcal{C}$. I am interested in the localizations of $\mathcal{A}$.

First of all, I define the notion of a T-topology on $\mathcal{C}$ and the notion of sheaf for such a topology.

DEFINITION 2. A T-*topology on the* $Sets^{\mathrm{T}}$-*category* $\mathcal{C}$ consists in giving, for any object X of $\mathcal{C}$, a family $J(X)$ of subobjects of $\mathcal{C}(-,X)$ such that:

(1) $\mathcal{C}(-,X) \in J(X)$.

(2) If $v: \mathcal{C}(-,Y) \to \mathcal{C}(-,X)$ and $R \in J(X)$, then $v^{-1}(R) \in J(Y)$.

(3) If $R \in J(X)$ and $R' \rightarrowtail \mathcal{C}(-,X)$ are such that $v^{-1}(R') \in J(Y)$ for any $v: \mathcal{C}(-,Y) \to R$, then $R' \in J(X)$.

An element of $J(X)$ is called a *covering subobject of* $\mathcal{C}(-,X)$. More generally, a monomorphism in $\mathcal{A}$ is said *covering* if all its inverse images over the representables $\mathcal{C}(-,X)$ are co.ering.

DEFINITION 3. A *sheaf for a* T-*topology* J on a small $Sets^{\mathrm{T}}$-category $\mathcal{C}$ is a presheaf $P \in \mathcal{A}$ such that, for any object X in $\mathcal{C}$, any $R \in J(X)$ and any morphism $f: R \to P$, f extends uniquely to $\mathcal{C}(-,X)$:

$$\begin{tikzcd} R \arrow[r, tail] \arrow[dr, "f"'] & \mathcal{C}(-,X) \arrow[d, dashed] \\ & P \end{tikzcd}$$

The first basic result is :

THEOREM 1. *The full subcategory of sheaves for the* T-*topology* J *on* $\mathcal{C}$ *is a localization of the category* $\mathcal{A}$ *of presheaves.*

I do not intend to prove here this result, but I would like to give a flash on the proof which shows the impact of the commutativity of T. In the usual *Sets*-case, a topology was given by subobjects of the representable presheaves, but any presheaf could be obtained as a universal colimit of representable presheaves. This is no longer the case in this context.

It is possible to prove that the $\mathcal{C}(-,X)$ form a system of finitely presentable generators. Thus any presheaf in $\mathcal{A}$ can be obtained as a universal colimit of finite coproducts of the $\mathcal{C}(-,X)$. But it remains to pass from the representables to finite coproducts of the representables. For example, let us consider a covering subobject of a finite coproduct of representables and let us sketch the proof that a sheaf possesses the unique extension property along such a covering subobject.

Thus R covers $\coprod \mathcal{C}(-,X_i)$ and by pulling back, R_i covers $\mathcal{C}(-,X_i)$. f is given with P a sheaf and we are looking at a unique g such that $gr = f$. Each $f.s_i'$ extends to $\mathcal{C}(-,X_i)$ by a unique g_i and the morphisms g_i factor through some g. The family $(s_i)_i$ is epimorphic but this is no longer the

$$\begin{tikzcd}
 & R_i \arrow[rr, "s'_i"] \arrow[dd, tail] \arrow[ddrr, phantom, "\text{P.B.}"] & & R \arrow[dd, tail, "r"] \arrow[dddr, "f"] & \\
R_v \arrow[urrr, "s'_v"] \arrow[dd, tail] & & & & \\
 & \mathcal{C}(-, X_i) \arrow[rr, "s_i"] \arrow[drrr, dashed, "g_i"] & & \amalg \mathcal{C}(-, X_i) \arrow[dr, dashed, "g"] & \\
\mathcal{C}(-, Y) \arrow[ur, dashed, "v_{i_j}"] \arrow[urrr, "v"] \arrow[rrrr, dashed, "g_v"'] & & & & P
\end{tikzcd}$$

case for the family $(s'_i)_i$, because of the lack of universality of colimits. In order to prove that $gr = f$, let us consider a general morphism

$$v : \mathcal{C}(-, Y) \to \amalg \mathcal{C}(-, X_i).$$

The construction of colimits in $Sets^{\mathrm{T}}$ and the commutativity of T show that there is an n-ary operation α and morphisms $v_{i_j} : \mathcal{C}(-, Y) \to \mathcal{C}(-, X_{i_j})$ such that

$$v = \alpha(s_{i_1} . v_{i_1}, \dots, s_{i_n} . v_{i_n}).$$

By pulling back r along v, one deduces an extension g_v of $f . s'_v$. The conditions on the T-topology and the fact that g is an homomorphism imply that $g . v = g_v$. But now the family of all the morphisms v is surjective and thus by pulling back along r, the family of all the morphisms s'_v is surjective. But

$$g . r . s'_v = f . s'_v \quad \text{and thus} \quad g . r = f.$$

The unicity follows easily. ■

It can be noticed that, both in the counterexample of groups and in the flash on the proof of Theorem 1, the point seems to be the way to pass from the covering subobjects of the representables to the covering subobjects of finite coproducts of the representables.

But let us now consider the topos $\mathcal{E} = [\mathcal{C}^{op}, Sets]$ of usual presheaves on $\mathcal{C}$. The object Ω of this topos $\mathcal{E}$ is given by

$$\Omega(X) = \text{set of subobjects of } U\mathcal{C}(-, X) \text{ in } \mathcal{E}.$$

The functor $U : Sets^{\mathrm{T}} \to Sets$ preserves monomorphisms and reflects the equivalence of subobjects. Thus we find a subobject Ω_{T} of Ω:

$$\Omega_{\mathrm{T}}(X) = \text{set of subobjects of } \mathcal{C}(-, X) \text{ in } \mathcal{A}.$$

The subobject Ω_T of Ω contains the greatest element and is stable for finite intersections. Thus we have the situation :

$$\begin{tikzcd} 1 \arrow[r, tail, "t_T"] \arrow[dr, tail, "t"'] & \Omega_T \arrow[d, tail] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega_T \times \Omega_T \arrow[r, "\Lambda_T"] \arrow[d, tail] & \Omega_T \arrow[d, tail] \\ \Omega \times \Omega \arrow[r, "\Lambda"] & \Omega \end{tikzcd}$$

We can now state some important results.

THEOREM 2. *There are bijections between (with the above notations):*

(1) localizations of $\mathcal{A}$;

(2) T*-topologies on* $\mathcal{C}$;

(3) morphisms $j : \Omega_T \to \Omega_T$ *in* $\mathcal{E}$ *such that:*

(a) $j . t_T = t_T$;

(b) $j . j = j$;

(c) $\Lambda_T . (j \times j) = j . \Lambda_T$. ■

Moreover Ω_T has also a property of classifying the subobjects in $\mathcal{A}$; more precisely :

THEOREM 3 *(with the above notations). For any object* P *in* $\mathcal{A}$, *there is a bijection between:*

(1) the subobjects of P *in* $\mathcal{A}$;

(2) the morphisms $\phi : UP \to \Omega_T$ *in* $\mathcal{E}$ *such that, for any n-ary operation* a, *we have* $\Lambda_T^n . \phi^n \leq \phi . a$.

$$\begin{tikzcd} UP^n \arrow[r, "a"] \arrow[d, "\phi^n"'] \arrow[dr, phantom, "(\leq)"] & UP \arrow[d, "\phi"] \\ \Omega_T^n \arrow[r, "\Lambda_T^n"'] & \Omega_T \end{tikzcd}$$

Moreover this correspondence gives rise to a pullback in $\mathcal{E}$

$$\begin{tikzcd} UR \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\text{P.B.}"] & 1 \arrow[d, tail, "t_T"] \\ UP \arrow[r, "\phi"'] & \Omega_T \end{tikzcd}$$

and there is a presheaf Ω_T^P *in* $\mathcal{E}$ *which represents the subobjects of* P. ∎

Thus the object Ω_T plays with respect to $\mathfrak{A}$ a role analogous to that of the Ω-object for a topos; the main difference is that Ω_T is not an object in $\mathfrak{A}$ but well an object in the topos $\mathcal{E}$ canonically associated to $\mathfrak{A}$. Moreover, like in the case of toposes, this property of the categories of algebraic presheaves reflects to the categories of algebraic sheaves in the following way:

If $j: \Omega_T \to \Omega_T$ arises from a T-topology on $\mathcal{C}$ and if

$$\begin{tikzcd} \mathfrak{A}_j \arrow[r, hook] & \mathfrak{A} \arrow[l, bend right] \end{tikzcd}$$

is the corresponding localization, consider the image Ω_j of j in $\mathcal{E}$:

$$\begin{tikzcd} \Omega_T \arrow[rr, "j"] \arrow[dr] & & \Omega_T \arrow[r, tail] & \Omega \\ & \Omega_j \arrow[ur, tail] & & \end{tikzcd}$$

Again, Ω_j is stable for finite intersections and greatest element.

THEOREM 4 *(with the above notations). The morphisms* $k: \Omega_j \to \Omega_j$ *satisfying the conditions a-b-c of Theorem 2 classify exactly the localizations of* $\mathfrak{A}_j$. *If* P *is an object in* $\mathfrak{A}_j$, *the morphisms* $\phi: UP \to \Omega_j$ *in* $\mathcal{E}$ *satisfying the condition* $(\leq)$ *of Theorem 3 classify exactly the subobjects of* P *in* $\mathfrak{A}_j$. ∎

The present work opens several problems:

(1) What about the non-commutative case?

(2) Develop an analogous theory with respect to an elementary topos with natural-numbers-object. More precisely, classify the localizations of the algebraic category corresponding to a theory defined internally with respect to an elementary topos with NNO.

(4) Give a characterization of the categories of sheaves of algebras over *Sets*. This would be a combination of the theorem of Giraud for Grothendieck toposes and the theorem of Lawvere for algebraic categories.

(4) Using the objects Ω_T, is it possible to describe a notion of «elementary algebraic topos» which would be to the categories of sheaves of algebras what the elementary toposes are with respect to the categories

of sheaves of sets ?

BIBLIOGRAPHY.

P. GABRIEL, Des catégories abéliennes, *Bull. Soc. Math. France* 90 (1962), 323-448.

P. GABRIEL & F. ULMER, Lokal-präsentierbare Kategorien, *Lecture Notes in Math.* 221, Springer (1971).

P. A. GRILLET, Regular categories, *Lecture Notes in Math.* 236, Springer (1971), 121-222.

A. GROTHENDIECK, Théorie des topos et cohomologie étale des schémas, *Lecture Notes in Math.* 269, Springer (1972).

F. W. LAWVERE, Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sc.* 50 (1963), 869-872.

M. TIERNEY, Axiomatic sheaf theory, *C.I.M.E. Varenna* (1971), Ed. Cremonese (1973), 249-326.

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