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Locally injective $G$-sheaves of abelian groups

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The problem of the existence of enough injective abelian group objects in an elementary topos with a natural number object leads to the construction of the internal (parametrized) coproduct of abelian group objects [4]. From certain properties of this parametrized coproduct we earlier derived some further consequences [5], among them the surprising result that «all internal notions of injectivity for abelian group objects are equivalent».

In the following summary we shall apply this result to $Shv(X)^{G^{op}}$, the topos of set-valued sheaves on a topological space $X$ with a left action of a group-valued sheaf $G$.

We require the following results and definitions [4, 5] (where $\mathcal{E}$ denotes an elementary topos with natural number object and $Ab(\mathcal{E})$ the category of abelian group objects in $\mathcal{E}$).

(0.1) **Theorem and Definition.** For any object $X$ in $\mathcal{E}$ the functor $X^*: Ab(\mathcal{E}) \to Ab(\mathcal{E}/X)$ has a left adjoint $\Theta_X: Ab(\mathcal{E}/X) \to Ab(\mathcal{E})$ which respects monomorphisms and is faithful.

For $A(x) \in Ob\ Ab(\mathcal{E}/X)$ the abelian group object $\Theta_X A(x)$ in $\mathcal{E}$ is called parametrized coproduct of $A(x)$. (We use «parametrized» to emphasize that the indexing object is in general not just a set but for example a set with an action of a group on it.)

A consequence of this theorem is the following proposition [5]:

(0.2) **Proposition.** If $Ab(\mathcal{E})$ has enough injectives, then so does $Ab(\mathcal{E}^A)$ for any internal category $A$ in $\mathcal{E}$.

In the following the internal Hom-functor $Ab(\mathcal{E})^{op} \times Ab(\mathcal{E}) \to Ab(\mathcal{E})$
is denoted by \( \text{Hom}(\cdot, \cdot) \). For \( A, B \) abelian group objects in \( \mathcal{E} \), \( \text{Hom}(A, B) \) is the abelian group object in \( \mathcal{E} \) that internalizes the abelian group of group-morphisms from \( A \) to \( B \).

(0.3) **DEFINITION.** An abelian group object \( B \) in \( \mathcal{E} \) is called

(i) internally injective if for every monomorphism \( A \rightarrow C \) in \( \text{Ab}(\mathcal{E}) \), \( \text{Hom}(C, B) \rightarrow \text{Hom}(A, B) \) is an epimorphism in \( \mathcal{E} \).

(ii) locally injective if for every diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{m} & C \\
\downarrow f & & \\
B & & \\
\end{array}
\]

in \( \text{Ab}(\mathcal{E}) \) there exists a cover \( U \rightarrow 1 \) of \( \mathcal{E} \) and a morphism \( g: U^*C \rightarrow U^*B \) in \( \text{Ab}(\mathcal{E}/U) \) such that

\[
\begin{array}{ccc}
U^*A & \xrightarrow{U^*m} & U^*C \\
\downarrow U^*f & & \downarrow g \\
U^*B & & \\
\end{array}
\]

commutes.

(0.4) **PROPOSITION.** The following conditions on an abelian group object \( B \) in \( \mathcal{E} \) are equivalent (for (iii) we suppose that \( \text{Ab}(\mathcal{E}) \) has enough injectives):

(i) \( B \) is locally injective.

(ii) For every diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{m} & C \\
\downarrow f & & \\
B & & \\
\end{array}
\]

in \( \text{Ab}(\mathcal{E}) \) there exist a cover \( U \rightarrow 1 \) of \( \mathcal{E} \) and a morphism \( g: U \rightarrow \text{Hom}(C, B) \) in \( \mathcal{E} \)

such that the diagram
(iii) There exists a cover \( U \rightarrow 1 \) of \( \mathcal{E} \) such that \( \eta_B : B \rightarrow B^U \) is an injective effacement [2]. Here \( \eta : \text{id}_{\text{Ab}(\mathcal{E})} \rightarrow \pi_U.\pi_U^* \) denotes the unit of the adjunction \( \pi_U^* \dashv \pi_U \), and the abelian group object structure of \( B^U \) is induced by that of \( B \).

(A monomorphism \( m : A \rightarrow C \) in \( \text{Ab}(\mathcal{E}) \) is an injective effacement iff for every monomorphism \( f : A \rightarrow B \) in \( \text{Ab}(\mathcal{E}) \) there exists a morphism \( g : B \rightarrow C \) in \( \text{Ab}(\mathcal{E}) \) such that \( g \cdot f = m \).)

(0.5) Lemma. An abelian group object \( B \) in \( \mathcal{E} \) is internally injective iff for every monomorphism \( m : A \rightarrow C \) in \( \text{Ab}(\mathcal{E}) \) and every generalized element \( f : V \rightarrow \text{Hom}(A, B) \) in \( \mathcal{E} \) there exist an object \( U \), an epimorphism \( h : U \rightarrow V \) and a morphism \( g : U \rightarrow \text{Hom}(C, B) \) in \( \mathcal{E} \) such that

\[
\begin{array}{ccc}
\text{Hom}(C, B) & \xrightarrow{\text{Hom}(m, B)} & \text{Hom}(A, B) \\
g & & f \\
U & \xrightarrow{\text{Hom}(B, U)} & V \\
\end{array}
\]

commutes.

(0.6) Theorem. An abelian group object \( B \) in \( \mathcal{E} \) is locally injective iff \( B \) is internally injective.

In the following we shall study the meaning of this result in the topos \( \mathcal{E} = \text{Shv}(X)^{\text{Gop}} \), where \( X \) denotes a topological space (resp. a locale [7, 9]) and \( G \) a group-valued sheaf on \( X \). Then \( \text{Ab}(\text{Shv}(X)^{\text{Gop}}) \) is the category of abelian group-valued sheaves on \( X \) equipped with a left action of \( G \) compatible with the abelian group structure. So \( \text{Ab}(\text{Shv}(X)^{\text{Gop}}) \) is the category of \( G \)-modules on \( X \), and will be denoted from now on by \( G\text{-Mod}(X) \).
(1.1) SOME REMARKS.

(i) The following diagram of forgetful functors commutes:

\[
\begin{array}{ccc}
G-\text{Mod}(X) & \xrightarrow{V''} & \text{Shv}(X)^{\text{op}} \\
V & \downarrow & \text{Shv}(X) \\
\text{Ab}(\text{Shv}(X)) & \xrightarrow{V'} & \text{Shv}(X)
\end{array}
\]

All these forgetful functors create epimorphisms and monomorphisms, they all have a left adjoint, and they respect injectives \[10\].

(ii) In \(\text{Ab}(\text{Shv}(X))\) the notions of injectivity and internal injectivity coincide \[5\]. For \(A, B\) in \(\text{Ab}(\text{Shv}(X))\) the internal Hom is obtained as follows: for \(U\) open in \(X\), \(\text{Hom}(A, B)(U)\) is defined to be

\[\text{Hom}_{\text{Ab}(\text{Shv}(U))}(A/U, B/U)\].

(iii) \(G-\text{Mod}(X)\) has enough injectives. (This follows immediately from (0.2).)

(iv) In \(G-\text{Mod}(X)\) the internal Hom is obtained as follows: For \(A, B\) \(G\)-modules on \(X\), \(\text{Hom}(A, B)\) is defined to be

\[\text{Hom}(VA, VB)\text{ in } \text{Ab}(\text{Shv}(X))\]

equipped with the following action of \(G\): for \(U\) open in \(X\),

\[
GU \times \text{Hom}_{\text{Ab}(\text{Shv}(U))}(VA/U, VB/U) \to \text{Hom}_{\text{Ab}(\text{Shv}(U))}(VA/U, VB/U)
\]

\((s, h) \mapsto s \circ h\)

is defined by:

\[
(s \circ h)_W(x) = (s/W).h_W((s^{-1}/W).x),
\]

where \(W \subseteq U\), \(W\) open in \(X\) and \(x \in A/W\).

(1.2) PROPOSITION. Let \(B\) be a \(G\)-module on \(X\).

(i) \(B\) is internally injective iff \(VB\) is injective in \(\text{Ab}(\text{Shv}(X))\).

(ii) If \(B\) is internally injective, then \(B^G\) is injective.

PROOF. (i): Suppose \(B\) to be internally injective. To show that \(B\) is injective in \(\text{Ab}(\text{Shv}(X))\), it is sufficient to show that \(B\) is internally injective in \(\text{Ab}(\text{Shv}(X))\) (cf. (1.1) (ii)). So let \(m: A \to C\) be a mono-
morphism in $Ab(\text{Shv}(X))$. Let $G$ operate trivially on $A$ and $C$; then $m: A \to C$ becomes a monomorphism in $G\text{-Mod}(X)$. Since $B$ is internally injective it follows that $\text{Hom}(C, B) \to \text{Hom}(A, B)$ is an epimorphism in $\text{Shv}(X)^{G^{op}}$, and hence an epimorphism in $\text{Shv}(X)$ (cf. (1.1) (i) and (iv)). So $B$ is injective in $Ab(\text{Shv}(X))$.

The other implication is equally easy.

(ii): Let

\[
\begin{array}{ccc}
A & \xrightarrow{m} & C \\
\downarrow f & & \downarrow \\
B^G & & \\
\end{array}
\]

be a diagram in $G\text{-Mod}(X)$ and suppose $B$ to be internally injective. We have a sequence of natural bijections between the following sets:

\[
\begin{array}{rcl}
A & \to & B^G \text{ in } G\text{-Mod}(X) \\
G & \to & V^*(\text{Hom}(A, B)) \text{ in } \text{Shv}(X)^{G^{op}} \\
1 & \to & \overline{V}^*(\text{Hom}(A, B)) \text{ in } \text{Shv}(X) \\
1 & \to & \overline{V}^*(\text{Hom}(VA, VB)) \text{ in } \text{Shv}(X) \\
VA & \to & VB \text{ in } Ab(\text{Shv}(X)).
\end{array}
\]

So $f: A \to B^G$ determines, and is determined by, a morphism $\overline{f}: VA \to VB$ in $Ab(\text{Shv}(X))$. $B$ is supposed to be internally injective, so, by (i), $VB$ is injective in $Ab(\text{Shv}(X))$. Hence there is a morphism $h$ in $Ab(\text{Shv}(X))$ such that

\[
\begin{array}{ccc}
VA & \xrightarrow{V_m} & VC \\
\downarrow \overline{f} & & \downarrow h \\
VB & & \\
\end{array}
\]

commutes. As above $h$ determines a morphism $\hat{h}: C \to B^G$ in $G\text{-Mod}(X)$, and it is easy to verify that $\hat{h}m = f$. So $B^G$ is injective in $G\text{-Mod}(X)$.

(1.3) REMARK. Let $\Delta Z$ be the ring-valued sheaf on $X$ associated to the
constant presheaf with value $Z$. Then $U \mapsto \Delta Z(U)[GU]$ defines an abelian group-valued presheaf with the usual left action of $G$, where we denote by $\Delta Z(U)[GU]$ the group-ring over $GU$. The associated sheaf is a $G$-module on $X$ and is denoted by $Z[G]$. Some obvious calculations show that there is a natural isomorphism

$$\text{Hom}(Z[G], A) \cong A^G.$$

In the following the composite

$$A \xrightarrow{\eta_A} A^G \xrightarrow{=} \text{Hom}(Z[G], A)$$

is again denoted by $\eta_A$ (cf. (0.4)(iii)).

(1.4) PROPOSITION. Let $B$ be a $G$-module on $X$. The following conditions on $B$ are equivalent:

(i) $\text{Hom}(Z[G], B)$ is an injective $G$-module on $X$.

(ii) There exists an epimorphism $D \twoheadrightarrow 1$ in $\text{Shv}(X)^{\text{op}}$ such that $B^D$ is an injective $G$-module on $X$.

(iii) $\eta_B : B \Rightarrow \text{Hom}(Z[G], B)$ is an injective effacement in $G\text{-Mod}(X)$.

(iv) There exists an epimorphism $D \twoheadrightarrow 1$ in $\text{Shv}(X)^{\text{op}}$ such that $\eta_B : B \Rightarrow B^D$ is an injective effacement in $G\text{-Mod}(X)$.

(v) $B$ is injective in $\text{Ab}(\text{Shv}(X))$.

(vi) $B$ is internally injective in $G\text{-Mod}(X)$.

(vii) There exists an open cover of $X$, $X = \bigcup_{i \in I} U_i$, such that $B/U_i$ is injective in $\text{Ab}(\text{Shv}(U_i))$ for every $i \in I$.

PROOF. (iii) $\Rightarrow$ (iv) $\Rightarrow$ (1.1)(iii), (0.4)(iii), (0.6) $\Rightarrow$ (vi) $\Rightarrow$ (1.2)(ii)

$\Rightarrow$ (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii).

(vi) $\Rightarrow$ (1.2)(i) $\Rightarrow$ (v) $\Rightarrow$ (vi).

So they are all equivalent.

(1.5) REMARK. If $X$ is reduced to a point, then $\text{Shv}(X)^{\text{op}}$ is the topos
of $G$-sets, $Ab(Shv(X))^{G^{op}}$ is the category of left $\mathbb{Z}[G]$-modules, and $Ab(Shv(X))$ is the category of abelian groups. For two $\mathbb{Z}[G]$-modules $A, B$ the internal Hom, $Hom(A, B)$, is the abelian group of all $\mathbb{Z}$-linear maps from $A$ to $B$ equipped with the following $G$-action:

$$(s \circ f)(x) = sf(s^{-1}x) \quad \text{for} \quad s \in G, \quad f \in Hom_{\mathbb{Z}}(A, B), \quad x \in A.$$

$\eta_B : B \rightarrow Hom(\mathbb{Z}[G], B)$ is here defined by

$\eta_B(b)(s) = b \quad \text{for every} \quad s \in G.$

We remark that in the above case the Proposition (1.4) remains true if $G$ is replaced by a monoid $M$ and consequently $Hom(\mathbb{Z}[G], B)$ by $B^M$ (cf. [3]).
REFERENCES.


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