CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

JIŘÍ ADÁMEK
VÁCLAV KOUBEK

Completions of concrete categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 22, n° 2 (1981), p. 209-228
<http://www.numdam.org/item?id=CTGDC_1981__22_2_209_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE
Vol. XXII-2 (1981)

3^e COLLOQUE SUR LES CATEGORIES
DEDIE A CHARLES EHRESMANN
Amiens, Juillet 1980

COMPLETIONS OF CONCRETE CATEGORIES

by Jiří ADÁMEK and Václav KOUBEK

INTRODUCTION

Given a complete base-category $\mathfrak{X}$, we study completions of concrete categories, i.e., categories $\mathfrak{K}$ endowed with a faithful (forgetful) functor $U: \mathfrak{K} \to \mathfrak{X}$. We prove that each concrete category $\mathfrak{K}$ has a universal concrete completion $U^*: \mathfrak{K}^* \to \mathfrak{X}$. This means that:

(i) $\mathfrak{K}^*$ is a complete category and its limits are concrete (i.e., preserved by U^*),

(ii) $\mathfrak{K}$ is a full, concrete subcategory of $\mathfrak{K}^*$ closed under all the existing concrete limits, and

(iii) each concrete functor on $\mathfrak{K}$, which preserves concrete limits, has a unique such extension to $\mathfrak{K}^*$.

It turns out that, moreover, $\mathfrak{K}$ is codense in $\mathfrak{K}^*$, i.e., each object of $\mathfrak{K}^*$ is a limit of some diagram in $\mathfrak{K}$.

The category $\mathfrak{K}^*$ is constructed by adding formal limits to the objects of $\mathfrak{K}$. The same method has already been used by C. Ehresmann [3]. New in our approach is the fact that the addition of limits need not be iterated - hence the codensity. The morphisms of $\mathfrak{K}^*$ are defined by a natural transfinite induction. A direct construction of the universal completion will be presented by H. Herrlich in [5].

The completion of concrete categories yields much more satisfactory results than that of «abstract» categories, see for example [6,7,8]. V. Trnková even exhibits in [8] a category $\mathfrak{K}$ which cannot be fully embedded into any finitely productive category with all the finite products of $\mathfrak{K}$ preserved.

1. *Concrete categories* over a base category $\mathfrak{X}$ (assumed to be complete throughout the paper) are categories $\mathcal{K}$ together with a functor $U: \mathcal{K} \to \mathfrak{X}$ (denoted by $UA = |A|$ on objects, $Uf = f$ on morphisms) which is faithful and amnestic, i.e., for each isomorphism $f: A \to B$ in $\mathcal{K}$ with Uf a unit morphism in $\mathfrak{X}$ we have $A = B$. Given concrete categories $\mathcal{K}$ and $\mathcal{L}$ a *concrete functor* is a functor $F: \mathcal{K} \to \mathcal{L}$ commuting with the forgetful functors (i.e., on objects $|FA| = |A|$; on morphisms $Ff = f$).

A concrete category $\mathcal{K}$ is *concretely complete* if the forgetful functor «detects» limits in the following sense. Let D be a diagram in $\mathcal{K}$. (In the present paper this will always mean a small collection of objects

$$D^o = \{ A_i \}_{i \in I(D)}$$

and sets of morphisms

$$D[i,j] \subset hom(A_i, A_j) \text{ for } i, j \in I.)$$

The forgetful functor *detects* the limit of D if for each limiting cone $\pi_i: X \to |A_i|$, $i \in I$ of the underlying diagram $|D|$ in $\mathfrak{X}$ (with objects $|A_i|$, $i \in I(D)$, and morphisms $|D[i,j]| = D[i,j]$) there exists an *initial lift* A in $\mathcal{K}$. Recall that an object A is an initial lift of a cone $\pi_i: X \to A_i$ if:

(i) $|A| = X$ and each $\pi_i: A \to A_i$ is a morphism in $\mathcal{K}$;

(ii) given an object B and a map $h: |B| \to X$ such that each $\pi_i . h: B \to A_i$ is a morphism in $\mathcal{K}$, then so is $h: B \to A$.

Now, an initial lift of a limiting cone of $|D|$ is clearly a limit of D. Note that we can speak about *the* initial lift since, due to amnesticity, it is unique. Note also that a concretely complete category is *transportable*, i.e., for each isomorphism $f: X \to Y$ in $\mathfrak{X}$ and for each object A in $\mathcal{K}$ with $|A| = X$ there exists an object B in $\mathcal{K}$ such that $|B| = Y$ and $f: A \to B$ is an isomorphism, too. In fact, a concrete category is concretely complete iff it is complete and the forgetful functor

(i) preserves limits and (ii) is transportable.

Fortunately neither «amnestic» nor «transportable» are severe restrictions:

2. LEMMA. *For each faithful functor $U: \mathcal{K} \to \mathfrak{X}$ there exists a transportable concrete category $U': \mathcal{K}' \to \mathfrak{X}$ and a concrete equivalence $E: \mathcal{K} \to \mathcal{K}'$*

with $U = U'.E$.

PROOF. Let $(\mathcal{K}'', U'')$ denote the following category and functor: objects of $\mathcal{K}''$ are triples (X, f, A) with X an object in $\mathcal{X}$, A an object in $\mathcal{K}$ and $f: X \to |A|$ an isomorphism in $\mathcal{X}$; morphisms $p: (X, f, A) \to (Y, g, B)$ of $\mathcal{K}''$ are maps $p: X \to Y$ such that $g.p.f^{-1}: A \to B$ is a morphism in $\mathcal{K}$;

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[d, "p"'] & A \arrow[l, shift left, "f^{-1}"] \arrow[d, dashed] \\ Y \arrow[r, "g"] & B \end{tikzcd}$$

the functor $U'': \mathcal{K}'' \to \mathcal{X}$ sends (X, f, A) to X and p to p.
Then U'' is transportable but not amnestic. Therefore, we define an equivalence $\approx$ on objects by:

$$(X, f, A) \approx (Y, g, B) \quad \text{iff} \quad X = Y \text{ and } id_X: (X, f, A) \to (Y, g, B) \text{ is an isomorphism in } \mathcal{K}''.$$

Denote by $\mathcal{K}'$ any choice class of this equivalence, as a full subcategory of $\mathcal{K}''$, and let $U' = U''/\mathcal{K}'$. Then $(\mathcal{K}', U')$ is clearly a transportable concrete category and the functor $E: \mathcal{K} \to \mathcal{K}'$, where $E(A)$ is the representant of (A, id_A, A), is an equivalence functor with $U = U'.E$.

3. DEFINITION. A *universal concrete completion* of a category $\mathcal{K}$ is a concretely complete category $\mathcal{K}^*$, in which $\mathcal{K}$ is a full and concrete subcategory (i.e., the forgetful functor of $\mathcal{K}$ is inherited from $\mathcal{K}^*$) closed to concrete limits and with the following universal property:

Let $\mathcal{L}$ be a concretely complete category. Then each concrete functor $F: \mathcal{K} \to \mathcal{L}$ preserving concrete limits has a concrete continuous extension $F^*: \mathcal{K}^* \to \mathcal{L}$, unique up to natural equivalence.

4. MAIN THEOREM. *Every concrete category $\mathcal{K}$ has a universal concrete completion in which $\mathcal{K}$ is codense.*

5. REMARK. «Codense» means that each object of the extension $\mathcal{K}^*$ is a limit of some diagram in $\mathcal{K}$. It then follows that $\mathcal{K}$ is closed under arbitrary colimits in $\mathcal{K}^*$ (see [4]).

6. PROOF OF THE MAIN THEOREM. Let $\mathcal{K}$ be a concrete category. We shall define its concrete completion $\mathcal{K}^*$ of which we shall verify the properties of a universal concrete completion, except transportability. Then we use Lemma 2: there exists a transportable concrete category, say $\mathcal{K}^{*\prime}$, concretely equivalent to $\mathcal{K}^*$, and this is the universal concrete completion of $\mathcal{K}$.

Denote by $\mathfrak{D}$ the class of all diagrams in $\mathcal{K}$ which have a concrete limit in $\mathcal{K}$. For each diagram D in $\mathcal{K}$ with $D \notin \mathfrak{D}$ choose a limiting cone (in $\mathfrak{X}$) of the underlying diagram $|D|$, where $D^o = \{Q_i\}_{i \in I(D)}$, say

$$\pi_i^D \colon X^D \to |Q_i| \quad \text{for } i \in I(D).$$

Define a concrete category $\mathcal{K}^*$. Its objects are:

1) all objects in $\mathcal{K}$, and

2) objects P^D, indexed by all diagrams D in $\mathcal{K}$ with $D \notin \mathfrak{D}$ (we assume $P^D \notin \mathcal{K}_o$ and $P^D \neq P^{D'}$ whenever $D \neq D'$).

The forgetful functor of $\mathcal{K}^*$ agrees with that of $\mathcal{K}$ on $\mathcal{K}$-objects and it sends P^D to X^D. The morphisms of $\mathcal{K}^*$ will be defined by a transfinite induction: for each ordinal k and each pair Q, R of objects in $\mathcal{K}^*$ we define a set of maps $H_k(Q, R) \subset hom(|Q|, |R|)$ and then a map is a morphism $f \colon Q \to R$ in $\mathcal{K}^*$ iff there exists an ordinal k with $f \in H_k(Q, R)$.

H_0-morphisms are

(i) all $\mathcal{K}$-morphisms between $\mathcal{K}$-objects,

(ii) all the connection maps $\pi_i^D \colon P^D \to Q_i$ (for a diagram $D \notin \mathfrak{D}$ and $i \in I(D)$), and

(iii) the identity maps $id_{X^D} \colon P^D \to P^D$ (for a diagram $D \notin \mathfrak{D}$).

CONVENTION. A collection of H_0-morphisms is said to be *distinguished* if either:

(a) it forms a limiting cone (in $\mathcal{K}$) of a diagram $D \in \mathfrak{D}$; or

(b) it is the collection $\{\pi_i^D \mid i \in I(D)\}$ for some diagram $D \notin \mathfrak{D}$; or

(c) it is the singleton collection $\{id_Q\}$ for an object Q of $\mathcal{K}^*$.

H_{k+1}-morphisms are «basic» morphisms and their composition. A map $f \colon |Q| \to |R|$ (where Q, R are objects in $\mathcal{K}^*$) is a basic H_{k+1}-morphism if there exists a distinguished collection $r_j \colon R \to R_j$, $j \in J$, in H_0 such that

$r_j . f \in H_k(Q, R_j)$ for each $j \in J$.

$H_\gamma = \bigcup_{k<\gamma} H_k$ for each limit ordinal γ.

It is clear that the above defines correctly a concrete category $\mathcal{K}^*$ except that $\mathcal{K}^*$ fails to be amnestic.

(1) *$\mathcal{K}$ is a full subcategory of $\mathcal{K}^*$.*

PROOF. We shall prove, by induction in k, that for each morphism f in $H_k(Q, R)$ such that Q is an object of $\mathcal{K}$, there exists a distinguished collection $p_i : R \to R_i$ such that each $p_i . f : Q \to R_i$ is a morphism in $\mathcal{K}$. It then follows that $\mathcal{K}$ is full in $\mathcal{K}^*$: if also $R \in \mathcal{K}^o$, then the distinguished collection must be a concrete limiting cone of a diagram $D \in \mathfrak{D}$. The compatible collection $\{p_i . f\}$ (in $\mathcal{K}$!) factorizes through the collection $\{p_i\}$- the factorization is necessarily f, thus f is a $\mathcal{K}$-morphism.

For $k = 0, 1$ the proposition can be proved by a simple inspection.

Assuming the proposition holds for $k \geq 1$, we shall prove it for $k+1$. This is clear for basic H_{k+1}-morphisms: there exists a distinguished collection $p_i : R \to P_i$ such that each $p_i . f : Q \to P_i$ is in H_k and, by induction hypothesis, each $p_i . f$ is a $\mathcal{K}$-morphism.

Let f be a composite of $n+1$ basic morphisms, $f = f_{n+1} . f_n . \dots . f_1$ (with $f_i : R_{i-1} \to R_i$, where $R = R_0$ and $Q = R_{n+1}$) and assume the proposition holds for compositions of n basic morphisms.

$$\begin{tikzcd} R \arrow[r, "f_1"] & R_1 \arrow[r, "f_2"] & R_2 \arrow[r, "f_3"] & \cdots \arrow[r, "f_n"] & R_n \arrow[r, "f_{n+1}"] \arrow[d, "p_i"'] & Q \arrow[d, "q_j"] \\ & & & & P_i & Q_j \end{tikzcd}$$

Particularly, the proposition holds for $g = f_n . \dots . f_1$: there exists a distinguished collection $p_i : R_n \to P_i$ such that each $p_i . g : R \to P_i$ is a $\mathcal{K}$-morphism. There follows $g \in H_1$. Moreover, since f_{n+1} is basic, there exists a distinguished collection $q_j : Q \to Q_j$ with each $q_j . f_{n+1}$ in H_k. Then also each

$$(q_j . f_{n+1}) . g = q_j . f : R \to Q_j$$

is in H_k, hence (by the inductive hypothesis) in $\mathcal{K}$.

(2) *$\mathcal{K}$ is closed to limits of $\mathfrak{D}$-diagrams in $\mathcal{K}^*$.*

PROOF. Let D be a diagram in $\mathfrak{D}$; denote its limiting cone by $\pi_i : P \to Q_i$, $i \in I$. We are to show that for each compatible cone $\pi_i' : P' \to Q_i$, $i \in I$ in $\mathcal{K}^*$, there is a unique morphism

$$p : P' \to P \quad \text{with} \quad \pi_i' = \pi_i . p \quad \text{for each } i.$$

Since the limit of D is concrete, we have a limiting cone $\pi_i : |P| \to |Q_i|$ for the diagram $|D|$ in $\mathfrak{X}$. And the cone $\pi_i' : |P'| \to |Q_i|$ is compatible for $|D|$, hence there exists a unique map $p : |P'| \to |P|$ with the required property. It remains to show that $p : P' \to P$ is a morphism in $\mathcal{K}^*$. Since each π_i' is a morphism in $\mathcal{K}^*$, there exists an ordinal γ such that

$$\pi_i' \in H_\gamma(P', Q_i) \quad \text{for each} \quad i \in I.$$

Now, the collection $\{\pi_i\}_{i \in I}$ is distinguished (it is of the first type of distinguished collections), thus $\pi_i' = \pi_i . p \in H_\gamma$ (for each $i \in I$) implies $p \in H_{\gamma+1}$.

(3) *Each diagram $D \notin \mathfrak{D}$ in $\mathcal{K}$ has a concrete limit in $\mathcal{K}^*$, viz,*

$$\pi_i^D : P^D \to Q_i, \quad i \in I(D).$$

The proof is analogous to (2) above: the collection $\{\pi_i^D\}$ is distinguished and it forms a limit of the underlying diagram.

COROLLARIES. *Every diagram in $\mathcal{K}$ has a concrete limit in $\mathcal{K}^*$;*
$\mathcal{K}$ is codense in $\mathcal{K}^*$.

(4) *$\mathcal{K}^*$ has limits, preserved by U.*

We shall prove it in two steps: first, with each diagram D in H_0 (more precisely, each diagram in $\mathcal{K}^*$, all morphisms of which belong to H_0) we associate a diagram D^+ in $\mathcal{K}$ (which has a concrete limit by (2) and (3)) such that $\lim D^+ = \lim D$. Second, with each diagram D in $\mathcal{K}^*$ we associate a diagram $\hat{D}$ in H_0 with $\lim \hat{D} = \lim D$.

(4A) Let D be a diagram in H_0, say on objects Q_j, $j \in J$. Put

$$J' = \{ j \in J \mid Q_j \text{ is not an object of } \mathcal{K} \} ;$$

thus, for each $j \in J'$ we have a diagram $D_j \notin \mathfrak{D}$ in $\mathcal{K}$ (say, on objects Q_{ji} for $i \in I_j$) with $Q_j = P^{D_j}$. Assuming the index sets I_j are pairwise disjoint

and disjoint from J (by which we do not lose generality, of course) we define a diagram D^+ in $\mathcal{K}$ as follows: Its objects are

$$\{Q_j\}_{j \in J - J'} \cup \{Q_{ji}\}_{j \in J\,,\, i \in I_j}.$$

Its morphisms are:

(i) all D-morphisms in $\mathcal{K}$:

$$D^+[j_1, j_2] = D[j_1, j_2] \quad \text{for each } j_1, j_2 \in J - J';$$

(ii) for $j \in J'$, all morphisms in D_j:

$$D^+[ji_1, ji_2] = D_j[ji_1, ji_2] \quad \text{for each } j \in J' \text{ and } i_1, i_2 \in I_j;$$

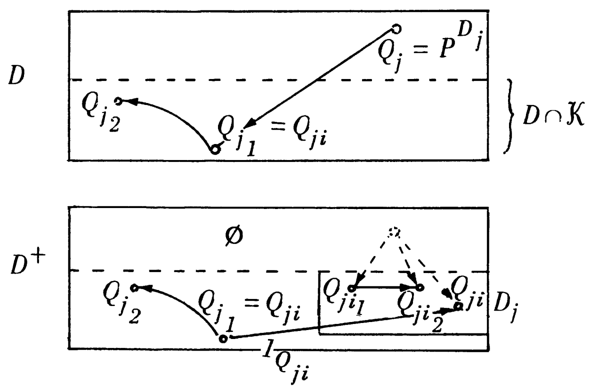
(iii) for each limiting-cone morphism in D

$$\pi_i^{D_j}: P^{D_j} \to Q_{j_1} = Q_{ji} \quad \text{with } j_1 \in J - J' \text{ and } j \in J',\ i \in I_j,$$

(thus $P^{D_j} = Q_j$), we add the unit morphism to D^+:

$$D^+[j_1, ji] = \{1_{Q_{j_1}}\} \quad \text{if } \pi_i^{D_j} \in D[j, j_1], \text{ where } j_1 \in J - J',\ j \in J',\ i \in I_j \text{ with } Q_{ji} = Q_{j_1},$$

$$D^+[j_1, ji] = \emptyset \quad \text{else.}$$



We shall prove that $\lim D^+ = \lim D$. More precisely, given the (concrete) limit $S = \lim D^+$ with the limiting cone

$$\phi^j: S \to Q_j \text{ for } j \in J - J' \quad \text{and} \quad \phi_{ji}: S \to Q_{ji} \text{ for } j \in J' \text{ and } i \in I_j,$$

define for $j \in J'$ a morphism $\phi^j: S \to Q_j$ by

$$\pi_i^{D_j} \cdot \phi^j = \phi_{ji} \quad \text{for each } i \in I_j.$$

(This is correct: $\{\pi_i^{D_j}\}_{i\in I_j}$ is a limit of D_j and, by (ii) above, $\{\phi_{ji}\}_{i\in I_j}$ is a compatible cone.) Then *the cone* $\{\phi^j\}_{j\in J}$ *is a concrete limit of* D.

PROOF. (a) The cone $\{\phi^j\}$ is compatible for D, i.e., for each morphism $f\in D[j,j_1]$ we have $\phi^{j_1}=f.\phi^j$. This is clear if f is a morphism of $\mathcal{K}$ (then it belongs to D^+). If f is in H_0-$\mathcal{K}^m$ then either f is a unit morphism (and the compatibility is clear) or $f=\pi_i^{D_j}$ for some $j\in J'$ and $i\in I_j$ such that $Q_j=P^{D_j}$ and $Q_{j_1}=Q_{ji}$. In that case

$$1_{Q_{j_1}}\in D^+[j_1,ji], \text{ hence } \phi^{j_1}=\phi_{ji}.$$

Since

$$f.\phi^j=\pi_i^{D_j}.\phi^j=\phi_{ji}=\phi^{j_1},$$

compatibility is proved.

(b) The cone $\{\phi^j\}$ is universal. Let $\psi^j: T\to Q_j$, $j\in J$, be a compatible cone for D. Define a compatible cone for D^+ by putting

$$\psi_{ji}=\pi_i^{D_j}.\psi^j: T\to Q_{ji} \quad \text{for each } j\in J' \text{ and } i\in I_j .$$

Then there exists a unique morphism $\psi: T\to S$ with

I) $\psi^j=\phi^j.\psi$ for $j\in J-J'$,

II) $\psi_{ji}=\phi_{ji}.\psi$ for $j\in J'$, $i\in I_j$.

For each $j\in J'$, the condition II is equivalent to $\psi^j=\phi^j.\psi$ (because $\{\pi_i^{D_j}\}$ is a limiting cone for D_j and we have

$$\pi_i^{D_j}.\psi^j=\psi_{ji}=\phi_{ji}.\psi=\pi_i^{D_j}.(\phi^j.\psi)).$$

Thus, the cone $\{\psi^j\}$ factorizes uniquely through the cone $\{\phi^j\}$.

(c) This limit of D is concrete. More generally: given an arbitrary functor $F:\mathcal{K}^*\to\mathcal{L}$ (e.g., the forgetful functor) which preserves limits of all diagrams in $\mathcal{K}$, then F preserves the limit of D. The proof is analogous to (b): Given a compatible cone $\psi^j: T\to FQ_j$ for $F(D)$ in $\mathcal{L}$, put

$$\psi_{ji}=(F\pi_i^{D_j}).\psi^j \quad \text{for } j\in J' \text{ and } i\in I_j.$$

This yields a compatible cone for $F(D^+)$. By hypothesis, F preserves the limit of D^+, hence there exists a unique $\psi: T\to FS$ with

I) $\psi^j=F\phi^j.\psi$ for $j\in J-J'$,

II) $\psi_{ji}=F\phi_{ji}.\psi$ for $j\in J'$, $i\in I_j$.

For each $j \in J'$, the condition II is equivalent to $\psi^j = F\phi^j . \psi$.

(4B) For each diagram D in $\mathcal{K}^*$ we shall construct a diagram $\hat{D}$ in H_0 such that each D-object is a $\hat{D}$-object, and we shall prove:

(i) $\lim D = \lim \hat{D}$ and the restriction of the limiting cone of $\hat{D}$ to the objects of D is the limiting cone of D, and

(ii) each functor, preserving limits of diagrams in H_0, preserves the limit of D.

The method is first to construct $\hat{D}$ in case D consists of a single morphism f (then $\hat{D}$ is denoted by $\hat{D}(f)$) and then, given an arbitrary diagram D, to obtain $\hat{D}$ by merging the diagrams $\hat{D}(f)$ with f ranging through the morphisms of D.

Thus, we first define a diagram $\hat{D}(f)$ for each morphism $f: P \to Q$ in $\mathcal{K}^*$. We shall proceed by induction in k where $f \in H_k(P, Q)$. The objects of $\hat{D}(f)$ will form a collection R_{fi}, $i \in I(f)$, with two distinguished ones: R_{fd_f} (the domain object) and R_{fc_f} (the codomain object) for

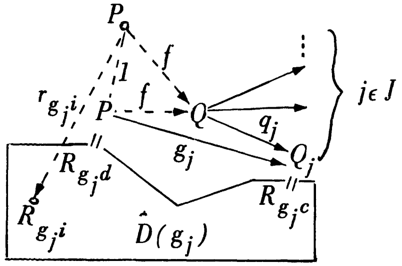
$$d_f, c_f \in I(f) \text{ such that } P = R_{fd_f} \text{ and } Q = R_{fc_f}$$

(we write also just R_{fd} and R_{fc}). And we shall also observe that there exist morphisms $r_{fi}: P \to R_{fi}$, $i \in I(f)$, forming a limiting cone of $\hat{D}(f)$ such that $r_{fd} = id_P$ and $r_{fc} = f$.

I. For $k = 0$ we let $\hat{D}(f)$ have just two objects $P = R_{fd}$, $Q = R_{fc}$, and just one morphism f. The limit is $id_P: P \to R_{fd}$ and $f: P \to R_{fc}$, of course.

II. Let $f \in H_{k+1}$ be a basic morphism. We fix a distinguished collection $q_j: Q \to Q_j$, $j \in J$, such that $g_j \overset{\text{def}}{=} q_j . f$ is in H_k for each $j \in J$. Thus we have diagrams $\hat{D}(g_j)$. The diagram $\hat{D}(f)$ is obtained as follows:

(i) Form the disjoint union of the diagrams $\hat{D}(g_j)$, $j \in J$;



(ii) Merge all their domain objects $R_{g_j d}$ $(= P)$: the merged object will be the domain object R_{fd} of $\hat{D}(f)$;

(iii) Add Q as a new object; this is the codomain object R_{fc} of $\hat{D}(f)$;

(iv) Add a new morphism $q_j : Q \to R_{g_j c}$ for each $j \in J$.

Thus, we obtain a diagram $\hat{D}(f)$ in H_0. We claim that its limiting cone is

$$1_P : P \to R_{fd},\ f : P \to R_{fc} \ \text{ and } \ r_{g_j i} : P \to R_{g_j i} \ \text{ for } j \in J,\ i \in I(g\ ).$$

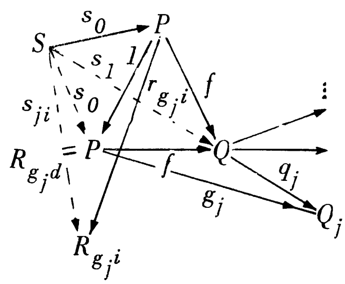
First, this cone is compatible for $\hat{D}(f)$: for each $j \in J$ we have

$$g_j \cdot 1_P = g_j = q_j \cdot f\,;$$

and the compatibility with each morphism inside $\hat{D}(g_j)$ is clear. Second, given another compatible cone

$$s_0 : S \to R_{fd},\ s_1 : S \to R_{fc} \ \text{ and } \ s_{ji} : S \to R_{g_j i} \ (\, j \in J \ \text{ and } \ i \in I(g_j)\,),$$

we shall show that its unique factorization through the given cone is s_0. The uniqueness is evident, since $1_P : P \to R_{fd}$ is in the given cone. Further, for each $j \in J$ we have a compatible cone $\{ s_{ji} \}_{i \in I(g_j)}$ for $\hat{D}(g_j)$,



which factorizes through the limiting cone $\{ r_{g_j i} \}$ of $\hat{D}(g_j)$ - and the factorizing morphism must be s_0 again, thus

$$s_{ji} = r_{g_j} \cdot s_0 \quad \text{for} \quad j \in J' \ \text{ and } \ i \in I_j\,.$$

Finally, there follows $s_1 = f \cdot s_0$ because, for each $j \in J$, we have

$$r_{g_j c} = g_j = q_j \cdot f$$

(by the inductive hypothesis), hence

$$q_j \cdot s_1 = s_{jc} = r_{g_j c} \cdot s_0 = q_j \cdot (f \cdot s_0) \quad \text{for} \quad j \in J\,.$$

Now, $\{ q_j \}_{j \in J}$ is a distinguished family, hence a limiting cone for some diagram (see (2) and (3) above), thus $s_1 = f \cdot s_0$.

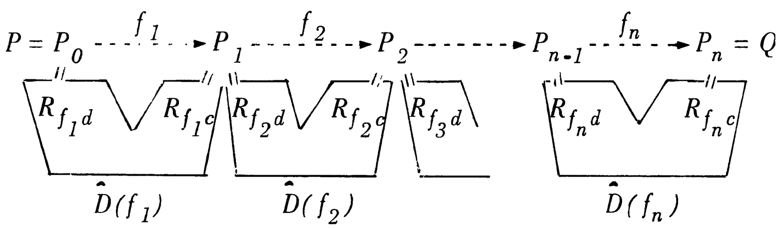
III. Let $f = f_n \cdot \ldots \cdot f_1$ be a composite of basic morphisms in H_{k+1}. We have diagrams $\hat{D}(f_1), \ldots, \hat{D}(f_n)$ and we define $\hat{D}(f)$ as follows:

(i) Form the disjoint union of the diagrams $\hat{D}(f_1), \ldots, \hat{D}(f_n)$;

(ii) Merge the codomain object of $\hat{D}(f_t)$ with the domain object of $\hat{D}(f_{t+1})$; the domain object of $\hat{D}(f_1)$ will be R_{fd} and the codomain object of $\hat{D}(f_n)$ will be R_{fc}.

We claim that the limiting cone of $\hat{D}(f)$ is:

$$r_{f_1 i}: P \to R_{f_1 i} \quad \text{for } i \in I(f_1),$$
$$r_{f_2 i} \cdot f_1 : P \to R_{f_2 i} \quad \text{for } i \in I(f_2),$$
$$\vdots$$
$$r_{f_n i} \cdot (f_{n-1} \cdot \ldots \cdot f_1): P \to R_{f_n i} \quad \text{for } i \in I(f_n).$$



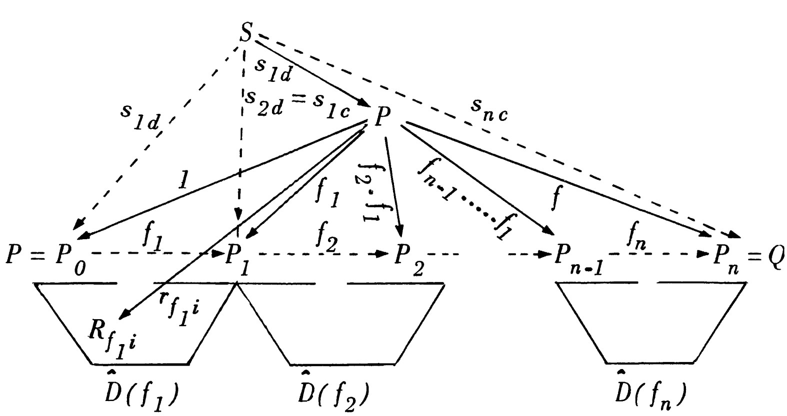
(In particular,

$$r_{f_1 d} = id_P : P \to R_{f_1 d} \quad \text{and} \quad r_{f_n c} = f_n \cdot (f_{n-1} \cdot \ldots \cdot f_1) = f : P \to R_{fc}.)$$

The compatibility of this cone is evident. Given another cone

$$s_{ti}: S \to R_{f_t i}\ , \quad \text{for } t = 1, \ldots, n \text{ and } i \in I(f_t)$$

compatible with $\hat{D}(f)$, for each t we have a cone $\{s_{ti}\}$ compatible with



$\hat{D}(f)$. Thus, there exists a unique $s^t\colon S \to P_t$ with $s_{ti} = r_{f_t i}\cdot s^t$ - in particular $s_{td} = s^t$ (because $r_{f_t d} = id_{P_{t-1}}$ by the inductive hypothesis); further $s_{tc} = f_t \cdot s^t$ (because $r_{f_t c} = f_t$), hence

$$s^{t+1} = s_{(t+1)d} = s_{tc} = f_t \cdot s^t .$$

Thus, the cone $\{ s_{ti} \}$ factorizes (uniquely) through the above cone

$$\{ r_{f_t i} \cdot (f_{t-1} \cdot \ldots \cdot f_1) \} ;$$

viz., by : $s_{1d}\colon S \to P$,

$$s_{1i} = r_{f_1 i}\cdot s^1 = r_{f_1 i}\cdot s_{1d} \quad \text{for } i \in I(f_1),$$

$$s_{2i} = r_{f_2 i}\cdot s^2 = (r_{f_2 i}\cdot f_1)\cdot s_{1d} \quad \text{for } i \in I(f_2),$$

$$\vdots$$

$$s_{ni} = r_{f_n i}\cdot s^n = (r_{f_n i}\cdot f_{n-1}\cdot \ldots \cdot f_1)\cdot s \quad \text{for } i \in I(f_n).$$

IV. Let γ be a limit ordinal. If $\hat{D}(f)$ is constructed for all $f \in H_k$ with $k < \gamma$, then $\hat{D}(f)$ is constructed for all $f \in H_\gamma$.

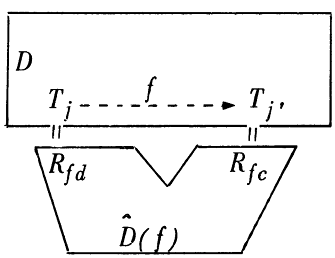
Thus we have constructed $\hat{D}(f)$ for each morphism in $\mathcal{K}^*$.

V. Given an arbitrary diagram D in $\mathcal{K}^*$ on objects T_j, $j \in J$, define a diagram $\hat{D}$ as follows:

(i) Form the disjoint union of diagrams $\hat{D}(f)$, with f ranging over all morphisms of the diagram D ;

(ii) Add the objects of D as new objects ;

(iii) For each $f \in D[j, j']$ merge T_j with the domain object of $\hat{D}(f)$ and merge $T_{j'}$ with the codomain object of $D(f)$.



The diagram $\hat{D}$ lies in H_0, hence it has a concrete limit (by (4A)), say

$$t_j\colon T \to T_j \quad (j \in J),$$

$$t_{fi}\colon T \to R_{fi} \quad (f \text{ a morphism of } D,\ i \in I(f)).$$

We claim that the former part $\{t_j\}_{j\in J}$ is a concrete limiting cone for $\hat{D}$.

(a) The cone $\{t_j\}$ is compatible for D, i.e.,

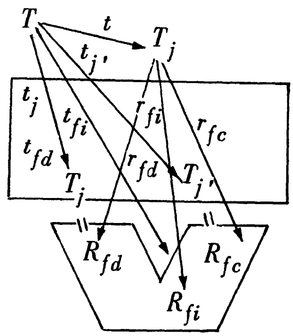
$$t_{j'} = f.t_j \quad \text{for each} \quad f \in D[j,j'] .$$

Morphisms t_{fi}, $i \in I(f)$, form a compatible cone for $\hat{D}(f)$. This cone factorizes through the limiting cone $\{r_{fi}\}$: there is a

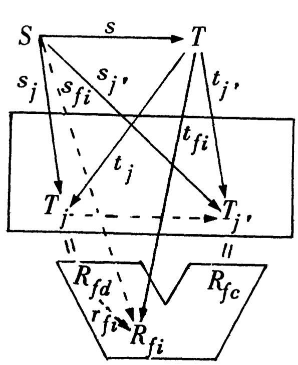
$$t: T \to T_j \quad \text{with} \quad t_{fi} = r_{fi}.t .$$

Necessarily $t = t_{fd} = t_j$ (since $r_{fd} = id_{T_j}$), hence (since $r_{fc} = f$):

$$t_{j'} = t_{fc} = r_{fc}.t_j = f.t_j .$$



(b) The cone $\{t_j\}$ is universal. Proof: Given another compatible cone $\{s_j\}$ with $s_j: S \to T_j$, define $s_{fi} = r_{fi}.s_j$ for each morphism $f: T_j \to T_{j'}$ in D and each $i \in I(f)$. This clearly yields a compatible cone for $\hat{D}$ (the compatibility of $\{s_j\}$ guarantees that the definition of s_{fi} is correct, i.e., $s_j = s_{fd}$ and $s_{j'} = s_{fc}$: recall $r_{fd} = id$ and $r_{fc} = f$). Thus, there exists a unique morphism $s: S \to T$ with



$$s_j = t_j \,.\, s \text{ and } s_{fi} = t_{fi} \,.\, s \text{ for all } j,\, f \text{ and } i.$$

Since the latter follows from the former ($s_{fi} = r_{fi} . s_j$ and $t_{fi} = r_{fi} . t_j$ imply

$$s_{fi} = r_{fi} . s_j = r_{fi} . t_j . s = t_{fi} . s\ )$$

the unicity holds also for D.

(c) This limit of D is concrete. More generally: given an arbitrary functor $F: \mathfrak{K}^* \to \mathfrak{L}$ which preserves limits of all diagrams in H_0, then F preserves the limit of D. (E.g., the forgetful functor can be taken as F, see (4 A c).) The proof is analogous to (b): Given a compatible cone $s_j : S \to F\,T_j$ for $F(D)$ in $\mathfrak{L}$, define $s_{fi} = F\,r_{fi} . s_j$ to obtain a compatible cone for $F(\hat{D})$. Since $\{F t_j\} \cup \{F t_{fi}\}$ is a limiting cone for $F(\hat{D})$, the cone $\{s_j\}$ factorizes uniquely through $\{F t_j\}$.

(5) *The conclusion of the proof.* Let $\mathfrak{K}^{*\prime}$ be a transportable category, concretely equivalent to $\mathfrak{K}^*$. We shall verify that $\mathfrak{K}^{*\prime}$ is a universal concrete completion of $\mathfrak{K}$. Without loss of generality we assume that $\mathfrak{K}$ is a full concrete subcategory of $\mathfrak{K}^{*\prime}$.

Since $\mathfrak{K}^*$ has limits preserved by the forgetful functor, so does $\mathfrak{K}^{*\prime}$ - recall that equivalences preserve limits. This implies that $\mathfrak{K}^{*\prime}$ is concretely complete; also, since $\mathfrak{K}$ is closed to concrete limits in $\mathfrak{K}^*$, so it is in $\mathfrak{K}^{*\prime}$.

Given a concretely complete category $\mathfrak{L}$ and a concrete functor $F: \mathfrak{K} \to \mathfrak{L}$ preserving concrete limits, we are to find a concrete continuous extension of F to $\mathfrak{K}^{*\prime}$. We shall verify that F has a unique concrete, continuous extension to $\mathfrak{K}^*$; then it has such an extension to $\mathfrak{K}^{*\prime}$, unique up to a natural equivalence. For each diagram D in $\mathfrak{K}$, $D \notin \mathfrak{D}$, we have a diagram $F(D)$ in $\mathfrak{L}$ such that $|D| = |FD|$ (since F is a concrete functor). We have choosen a limit $\pi_i^D : X^D \to |Q_i|$ in $\mathfrak{X}$ for the diagram $|F(D)|$. Since $\mathfrak{L}$ is a transportable concretely complete category, there exists an object R^D in $\mathfrak{L}$ with $|R^D| = X^D$ and such that $\pi_i^D : R^D \to F\,Q_i$ is a limiting cone for $F(D)$ (since $\mathfrak{L}$ is amnestic, R^D is unique). There is no other choice of a concrete, continuous extension F^* of F than

$$F^*(P^D) = R^D \text{ on objects, } \quad F^*f = Ff \text{ on morphisms.}$$

We must verify that, on the other hand, this defines a concrete continuous functor $F^*: \mathfrak{K}^* \to \mathfrak{L}$. First, F^* is indeed a functor, i.e., given a morphism $f: P^D \to P^{D'}$ in $\mathfrak{K}^*$ then also $f: R^D \to R^{D'}$ is a morphism in $\mathfrak{L}$. This is easy to see (by induction in i with $f \in H_i$). Second, F^* is concrete by its very definition:

$$|F^*(P^D)| = X^D = |P^D|.$$

Finally, given a diagram D in $\mathfrak{K}^*$, we shall verify that F^* preserves its limit. This is clear if D is a diagram in $\mathfrak{K}$: either $D \in \mathfrak{D}$ and then F^* ($= F$ on $\mathfrak{K}$) preserves its limit by hypothesis; or $D \notin \mathfrak{D}$, in which case the limiting cone is $\pi_i^D: P^D \to Q_i$ (see (3)). This is mapped by F^* to the cone $\pi_i^D: R^D \to FQ_i$, which has been chosen as the limiting cone for $F(D)$. Further, if D is a diagram in H_0 then, by (4Ac) above, F^* preserves its limit, too. Hence, if D is an arbitrary diagram in $\mathfrak{K}^*$, then, by (4Bc), F^* preserves its limit again.

This concludes the proof of the theorem.

7. Without any change in the proof, the completion theorem can be generalized to $\mathfrak{D}$-universal completions. Let $\mathfrak{K}$ be a concrete category and let $\mathfrak{D}$ be a class of diagrams in $\mathfrak{K}$, each having a concrete limit in $\mathfrak{K}$. Then a *$\mathfrak{D}$-universal concrete completion of* $\mathfrak{K}$ is its concrete completion $\mathfrak{K}^*$, in which $\mathfrak{K}$ is closed to limits of diagrams in $\mathfrak{D}$, and which has the following universal property:

Let $\mathfrak{L}$ be a concretely complete category; then each concrete functor $F: \mathfrak{K} \to \mathfrak{L}$, preserving limits of $\mathfrak{D}$-diagrams, has a unique concrete, continuous extension $F^*: \mathfrak{K}^* \to \mathfrak{L}$.

In the proof of the Main Theorem, let $\mathfrak{D}$ denote the given class (and not, as before, the class of *all* diagrams with concrete limits). Then the proof of the following theorem is obtained:

8. THEOREM. *Let $\mathfrak{D}$ be a class of diagrams in a concrete category $\mathfrak{K}$, each having a concrete limit in $\mathfrak{K}$. Then $\mathfrak{K}$ has a $\mathfrak{D}$-universal completion, in which $\mathfrak{K}$ is codense.*

9. We shall use this theorem to prove the existence of universal bicompletions. First, we observe that the completion theorems above can be dualized: if $\mathcal{K}$ is concrete over $\mathfrak{X}$, then $\mathcal{K}^{op}$ is concrete over $\mathfrak{X}^{op}$. Hence for a cocomplete base-category, we see that each concrete category has a universal concrete cocompletion. (The generalization to $\mathfrak{D}$-universality is obvious.)

Now, let *bicomplete* stand for complete plus cocomplete. Let $\mathfrak{X}$ be a bicomplete base-category. Then a *universal concrete bicompletion* of a concrete category $\mathcal{K}$ is a full, concrete and concretely bicomplete extension $\mathcal{K}^*$ of $\mathcal{K}$ in which $\mathcal{K}$ is closed to concrete limits and concrete colimits with the following universal property:

Let $\mathfrak{L}$ be a concretely bicomplete category; then each functor $F: \mathcal{K} \to \mathfrak{L}$ preserving concrete limits and concrete colimits has a unique bicontinuous extension $F^*: \mathcal{K}^* \to \mathfrak{L}$, unique up to natural equivalence.

10. THEOREM. *Every concrete category over a bicomplete base-category has a universal concrete bicompletion.*

PROOF. We shall define a transfinite sequence $\mathcal{K}^{(i)}$ of concrete categories the union of which will be the universal concrete bicompletion [1].

First, $\mathcal{K}^{(0)} = \mathcal{K}$ and $\mathcal{K}^{(1)}$ is the universal (concrete) completion of $\mathcal{K}$ (we omit the word concrete for shortness); second, $\mathcal{K}^{(2)}$ is the $\mathfrak{D}^{(2)}$-universal cocompletion of $\mathcal{K}^{(1)}$, where $\mathfrak{D}^{(2)}$ is the class of all diagrams in $\mathcal{K}^{(1)}$ which lie in $\mathcal{K}^{(0)}$ and have a concrete colimit in $\mathcal{K}^{(0)}$.

Generally, given a limit ordinal γ, then:

$\mathcal{K}^{(\gamma)}$ is the $\mathfrak{D}^{(\gamma)}$-universal completion of $\bigcup_{i<\gamma} \mathcal{K}^{(i)}$, where

$$\mathfrak{D}^{(\gamma)} = \bigcup_{i<\gamma} \mathfrak{D}^{(i)};$$

$\mathcal{K}^{(\gamma+1)}$ is the $\mathfrak{D}^{(\gamma+1)}$-universal cocompletion of $\mathcal{K}^{(\gamma)}$, where

$$\mathfrak{D}^{(\gamma+1)} = \mathfrak{D}^{(\gamma)};$$

$\mathcal{K}^{(\gamma+2)}$ is the $\mathfrak{D}^{(\gamma+2)}$-universal completion of $\mathcal{K}^{(\gamma+1)}$, where $\mathfrak{D}^{(\gamma+2)}$

1) This union is set-theoretically legitimate: the transfinite induction defines a relation ρ of all pairs (x, i) where i is an ordinal and $x \in \mathcal{K}^{(i)}$; the domain of ρ is then the union.

is the class of all diagrams in $\mathcal{K}^{(\gamma)}$ with a concrete limit;

$\mathcal{K}^{(\gamma+3)}$ is the $\mathcal{D}^{(\gamma+3)}$-universal cocompletion of $\mathcal{K}^{(\gamma+2)}$, where $\mathcal{D}^{(\gamma+3)}$ is the class of all diagrams in $\mathcal{K}^{(\gamma+1)}$ with a concrete colimit,

etc...

Then the concrete category $\mathcal{K}^* = \bigcup_{i \in Ord} \mathcal{K}^{(i)}$ is concretely bicomplete and it has $\mathcal{K}$ as its full, concrete subcategory, closed by concrete limits and concrete colimits. All this easily follows from the fact that every concrete category is closed to concrete $\mathcal{D}$-limits as well as concrete (in fact, all) colimits in its $\mathcal{D}$-universal completion; analogously for cocompletions. And every diagram in $\mathcal{K}^*$, being small, it lies in some $\mathcal{K}^{(i)}$ and so it has a concrete limit and a concrete colimit in $\mathcal{K}^{(i+1)}$.

What remains to verify is the universality. Let $\mathcal{L}$ be a concretely bicomplete category and let $F: \mathcal{K} \to \mathcal{L}$ be a concrete functor, preserving concrete limits and colimits. Then F can be uniquely extended into a concrete functor $F^{(1)}: \mathcal{K}^{(1)} \to \mathcal{L}$, and $F^{(1)}$ preserves concrete colimits of diagrams in $\mathcal{K}$ (i.e., of $\mathcal{D}^{(2)}$-diagrams), hence it has a unique cocontinuous concrete extension $F^{(2)}: \mathcal{K}^{(2)} \to \mathcal{L}$, preserving concrete limits of diagrams in $\mathcal{K}^{(1)}$ (i.e., of $\mathcal{D}^{(3)}$-diagrams), etc... Given functors $F^{(i)}$ for all $i < \gamma$, where γ is a limit ordinal, then their joint extension to $\mathcal{K}^{(\gamma)} = \bigcup_{i<\gamma} \mathcal{K}^{(i)}$ preserves concrete colimits and concrete limits of the diagrams lying in some $\mathcal{K}^{(i_0)}$ (e.g., of $\mathcal{D}^{(\gamma)}$-diagrams). Then there is a unique continuous concrete extension to $F^{(\gamma)}: \mathcal{K}^{(\gamma)} \to \mathcal{L}$. Etc. This concludes the proof.

11. REMARK. A closely related problem to concrete completions is that of initial completions. Let $\mathcal{C}$ be a conglomerate of cones in $\mathcal{X}$, i.e., of (possibly large) collections

$$< f_i : X \to X_i \mid i \in I >$$

of maps with a joint domain. A concrete category $\mathcal{K}$ is *initially $\mathcal{C}$-complete* if for each cone $< f_i : X \to X_i >$ in $\mathcal{C}$ and each collection $\{A_i\}$ of objects of $\mathcal{K}$ with $X_i = |A_i|$ there exists an initial lift (see Introduction). A concrete functor $F: \mathcal{K} \to \mathcal{L}$ *preserves $\mathcal{C}$-initial lifts* if, given an initial lift A of a cone $< f_i : X \to |A_i| >$ in $\mathcal{C}$, then FA is an initial lift of the cone

$< f_i : X \to | FA_i | >$ in $\mathcal{L}$.

A *universal initial $\mathcal{C}$-completion* of a concrete category $\mathcal{K}$ is its full, initially $\mathcal{C}$-complete extension $\mathcal{K}^*$, in which $\mathcal{K}$ is closed to $\mathcal{C}$-initial lifts (i.e., the embedding $\mathcal{K} \to \mathcal{K}^*$ preserves $\mathcal{C}$-initial lifts) and which has the obvious universal property with respect to functors preserving $\mathcal{C}$-initial lifts. The existence of a universal initial completion is investigated in [1] for $\mathcal{C}$ = all cones in $\mathcal{X}$: a possibly non-legitimate concrete category $\hat{\mathcal{K}}$ is constructed such that either $\hat{\mathcal{K}}$ is legitimate and then it is the universal initial completion, or $\hat{\mathcal{K}}$ fails to be legitimate, in which case the universal completion fails to exist.

In case $\mathcal{C}$ is a class of *small* cones in $\mathcal{X}$, the universal initial $\mathcal{C}$-completion always exists: we have $\mathcal{K} = \mathcal{K}_0$ as a subcategory of the (possibly non-legitimate) category $\hat{\mathcal{K}}$ and we denote by

$\mathcal{K}_1$ the closure of $\mathcal{K}_0$ for initial lifts of $\mathcal{C}$-sources in $\hat{\mathcal{K}}$,

$\mathcal{K}_2$ the closure of $\mathcal{K}_1$, etc...

$\mathcal{K}_\omega = \bigcup_{i < \omega} \mathcal{K}_i$,

$\mathcal{K}_{\omega+1}$ the closure of $\mathcal{K}_\omega$ for initial lifts of $\mathcal{C}$-sources, etc...

Then the category $\mathcal{K}^* = \bigcup_{i \in Ord} \mathcal{K}_i$ is always legitimate and it is evidently the universal $\mathcal{C}$-initial completion of $\mathcal{K}$.

Starting with $\mathcal{C}$ = all limiting cones for diagrams in $\mathcal{X}$, we obtain the universal concrete completions. But in this way we cannot verify that a concrete category is codense in its universal concrete completion. This is why we had to prove our theorem in a much more complicated manner.

The proof of the Main Theorem above can also be modified for this situation of initial $\mathcal{C}$-completion but, again, an iteration would be used generally. This would lose the codensity, but not the closedness for colimits.

12. EXAMPLE. Let $\mathcal{X}$ be a finitely productive base-category. For each concrete category $\mathcal{K}$ there exists a universal CFP-extension $\mathcal{K}^*$. This is a CFP-category (= concrete category with Concrete Finite Products) in which $\mathcal{K}$ is a full CFP-subcategory such that, given a CFP-category $\mathcal{L}$, then each CFP-functor $F : \mathcal{K} \to \mathcal{L}$ (= concrete functor preserving concrete

finite products) has a «unique» CFP-extension $F^*: \mathcal{K}^* \to \mathcal{L}$. Proof: let $\mathcal{C}$ be the class of all limiting cones for finite discrete diagrams, then a universal $\mathcal{C}$-initial completion is precisely a universal CFP-cxtension.

13. REMARK. In a subsequent paper [2] on cartesian closed extensions we shall need a generalization of the previous example: Given a concrete category $\mathcal{K}$ and a class $\mathfrak{D}$ of finite collections of its objects, there exists a $\mathfrak{D}$-universal CFP-extension of $\mathcal{K}$. (This is a CFP-category $\mathcal{K}^*$, in which $\mathcal{K}$ is closed to concrete products of $\mathfrak{D}$-collections, which has the obvious universal property.) The proof of this statement is an easy modification of the proof of the Main Theorem above: the objects of $\mathcal{K}^*$ will be the objects of $\mathcal{K}$ and objects P^D, where D is a finite collection of objects of $\mathcal{K}$ with $D \notin \mathfrak{D}$; morphisms are defined transfinitely in a natural way.

REFERENCES

1. J. ADÁMEK, H. HERRLICH & G.S. STRECKER, Least and largest initial completion, *Comment. Math. Univ. Carolinæ* 20 (1979), 43-77.
2. J. ADÁMEK & V. KOUBEK, *Cartesian closed concrete categories*, to appear.
3. C. EHRESMANN, Prolongements universels d'un foncteur par adjonction de limites, *Dissertationes Math.* 64 (1969).
4. A. & C. EHRESMANN, Multiple functors II, *Cahiers Topo. et Géom. Diff.* XIX-3 (1978), 295-334.
5. H. HERRLICH, Universal completions of concrete categories, *Proc. «Categorical aspects of Topology and Analysis*, Ottawa 1980.
6. J. ISBELL, Structure of categories, *Bull. A.M.S.* 72 (1966), 619-655.
7. J. LAMBEK, Completions of categories, *Lecture Notes in Math.* 24, Springer (1966).
8. V. TRNKOVÁ, Limits in categories and limit-preserving functors, *Commen. Math. Univ. Carolinæ* 7 (1966), 1-73.

Faculty of Electrical Engineering
Technical University Prague
Suchbatarova 2,
166 27 PRAGUE 6

and

Faculty of Mathematics and Physics
Charles University Prague
Malostranské nám. 25
PRAGUE 1. TCHECOSLOVAQUIE