R. F. C. Walters

Sheaves and Cauchy-complete categories


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I want to consider the point of view (see [2, 4]) that sheaves are sets with a generalized equality, in the context of enriched category theory (see [3]), where such structures as metric spaces and additive categories are regarded as categories with a generalized hom-functor. In this context sheaves on a locale \( H \) turn out to be precisely symmetric Cauchy-complete \( B \)-categories for a suitable bicategory \( B \) constructed out of \( H \).

This idea arose in conversations with Stefano Kasangian and Renato Betti in Milan. The necessary \( B \)-category theory was developed with Betti. I present here only the basic idea; developments will appear elsewhere.

1. CATEGORIES BASED ON A BICATEGORY (see [1])

The theory of categories with hom taking values in a bicategory, rather than a monoidal category (= bicategory with one object) seems to be very little developed. I have only some unpublished notes of R. Betti. However, most of what we need for this lecture is a simple translation of [3]. For our application we need only consider the case where the base bicategory \( B \) is locally partially-ordered; i.e., \( B(a, b) \) is a poset for all \( a, b \) in \( B \). We need also to assume that all these posets are co-complete and that suprema are preserved by composition in \( B \).

DEFINITIONS. A \( B \)-category \( X \) is a set \( X \) with a function \( e: X \to \text{obj. } B \) and a function \( d: X \times X \to \text{morph. } B \) satisfying:

(i) \( d(x_1, x_2): e(x_1) \to e(x_2) \),

(ii) \( l_e(x) \leq d(x, x) \),

(iii) \( d(x_2, x_3) \cdot d(x_1, x_2) \leq d(x_1, x_3) \).

(Draw a picture: \( X \) is a space lying over \( B \).)
A B-functor $f$ from $X$ to $Y$ is a function $f: X \to Y$ satisfying:

(i) $e(f(x)) = e(x)$,
(ii) $d(x_1, x_2) \leq d(f x_1, f x_2)$.

**EXAMPLE.** Let $H$ be a locale. Form a bicategory $B$ from $H$ as follows:
- objects of $B$: opens $u$ in $H$,
- arrows from $u$ to $v$: elements $w \leq u \wedge v$,
- 2-cells: order in $H$,
- composition of arrows: intersection.

Notice that $B = \text{Relations}(H)$.

From a sheaf $F$ on $H$ we can form a $B$-category $L(F)$ as follows:

$$L(F) = \text{set of partial sections of } F,$$

$$e: L(F) \to \text{obj. } B: s \mapsto \text{domain of } s,$$

$$d: L(F) \times L(F) \to \text{morph. } B: (s, t) \mapsto \bigvee \{ u; s \mid u = t \mid u \}.$$

Notice that $L(F)$ has the property that if

$$s, t \in L(F) \text{ and } d(s, t) = e(s) = e(t),$$

then $s = t$. Call such a $B$-category **skeletal**.

Notice that the bicategory $B = \text{Span}(H)$ of this example has the property that $B^{\text{op}}$ (arrows reversed) = $B$. This property allows us to say that a $B$-category $X$ is symmetric if

$$d(x_1, x_2) = d(x_2, x_1) \text{ for all } x_1, x_2 \in X.$$

Clearly $L(F)$ is symmetric and in fact $L$ is a fully-faithful functor

$$L: \text{Sheaves}(H) \to \text{skeletal symmetric } B\text{-categories}.$$

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2. **CAUCHY-COMPLETENESS**

To express Lawvere's notion of Cauchy-completeness we need to define bimodules. A bimodule $\phi$ from $X$ to $Y$ (denoted $\phi: X \star Y$) is a function $\phi: X \times Y \to \text{morph. } B$ satisfying (for all $x, x' \in X, y, y' \in Y$)

(i) $\phi(x, y): e(x) \to e(y)$,
(ii) $\phi(x, y). d(x', x) \leq \phi(x', y)$,
(iii) $d(y, y'). \phi(x, y) \leq \phi(x, y')$.

As usual a $B$-functor $f: X \to Y$ yields a pair of bimodules
\[ f^* : X \rightarrow Y \text{ and } f_* : Y \rightarrow X \]
defined by
\[ f^*(x, y) = d(fx, y) \quad \text{and} \quad f_*(y, x) = d(y, fx). \]
Further \( f^* \) and \( f_* \) are adjoint in the sense that
\[
( i ) \; d(x, x') \leq \exists y \left[ f_*(y, x') \cdot f^*(x, y) \right] \\
( \text{where we write } \exists y \text{ for the supremum (over } y \text{) in } B(x, x') \text{) and} \\
(ii) \; \exists x \left[ f^*(x, y') \cdot f_*(y, x) \right] \leq d(y, y').
\]
Then a \( B \)-category \( Y \) is Cauchy-complete if every adjoint pair of bimodules \( \phi, \psi : X \leftrightarrow Y \) arises from a functor \( X \rightarrow Y \).

3. SHEAVES

We now have the definitions required to state the result.

**THEOREM.** If \( H \) is a locale, then \( \text{Sheaves}(H) \) is equivalent to the category of skeletal symmetric Cauchy-complete \( \text{Rel}(H) \)-categories.

**PROOF.** We want to see

(a) that \( L \) lands in Cauchy-complete \( B \)-categories, and

(b) that every skeletal Cauchy-complete symmetric \( B \)-category is isomorphic to \( L(F) \) for some sheaf \( F \).

For each element \( u \in H \) we can define a \( B \)-category \( \hat{u} \) with one element * and with \( e(*) = u \), \( d(*, *) = u \). Then, in testing Cauchy-completeness of \( Y \), we need only consider adjoint pairs of bimodules from \( \hat{u} \) to \( Y \) for each \( u \in H \).

To prove (a) consider an adjoint pair of bimodules \( \phi(s), \psi(s) \) \(( s \in L(F) \)) from \( \hat{u} \) to \( F \). Then condition (i) of adjointness says that:
\[ u_s = \phi(s) \wedge \psi(s) \quad (s \in L(F)) \]
is a cover of \( u \). Condition (ii) says that
\[ s \upharpoonright u_s \quad (s \in L(F)) \]
is a compatible family of sections, and so there is a section \( s_\theta \in F(u) \) such that
\[ s_\theta \upharpoonright u_s = s \upharpoonright u_s \quad \text{for all } s \in L(F). \]
Now it is clear that for a general \( s \),
\[ d(s_\theta, s) = \bigvee_t d(s, t \upharpoonright u_t). \]
From property (ii) of adjunction:

\[ \phi(s) \wedge \psi(t) \wedge \phi(t) \leq d(s, t) \leq d(s, t | u_t) \]

and so by (i)

\[ \phi(s) \leq \bigvee_t d(s, t | u_t) = d(s_0, s). \]

From property (iii) of bimodules

\[ \phi(s) \geq d(s, t) \wedge \phi(t) \geq d(s, t | u_t), \text{ and so } \phi(s) \geq d(s_0, s). \]

Hence,

\[ \phi(s) = \psi(s) = d(s_0, s). \]

That is, the pair of bimodules arises from a functor.

To prove (b) consider a skeletal Cauchy-complete symmetric B-category \( Y \). We need to be able to define the restriction of an element \( y \) over \( u \) to \( v \leq u \). But this restriction comes from the fact that the adjoint pair of bimodules

\[ \phi(y') = \psi(y') = v \wedge d(y, y') : \hat{v} \rightarrow Y \]

is given by a functor. We need also to have the glueing together of a compatible family of elements \( (y_a) \) with \( \bigvee_a e(y_a) = u \). In this case the required section comes from the representation of the bimodules

\[ \phi(y') = \psi(y') = \bigvee_a d(y_a, y') : \hat{a} \rightarrow Y \]

as a functor.

REFERENCES