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Sheaves and Cauchy-complete categories


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I want to consider the point of view (see [2, 4]) that sheaves are sets with a generalized equality, in the context of enriched category theory (see [3]), where such structures as metric spaces and additive categories are regarded as categories with a generalized hom-functor. In this context sheaves on a locale $H$ turn out to be precisely symmetric Cauchy-complete $B$-categories for a suitable bicategory $B$ constructed out of $H$.

This idea arose in conversations with Stefano Kasangian and Renato Betti in Milan. The necessary $B$-category theory was developed with Betti. I present here only the basic idea; developments will appear elsewhere.

1. Categories Based on a Bicategory (see [1])

The theory of categories with hom taking values in a bicategory, rather than a monoidal category (= bicategory with one object) seems to be very little developed. I have only some unpublished notes of R. Betti. However, most of what we need for this lecture is a simple translation of [3]. For our application we need only consider the case where the base bicategory $B$ is locally partially-ordered; i.e., $B(a, b)$ is a poset for all $a, b$ in $B$. We need also to assume that all these posets are co-complete and that suprema are preserved by composition in $B$.

Definitions. A $B$-category $X$ is a set $X$ with a function $e: X \to \text{obj. } B$ and a function $d: X \times X \to \text{morph. } B$ satisfying:

(i) $d(x_1, x_2): e(x_1) \to e(x_2)$,

(ii) $I = (x) \leq d(x, x)$,

(iii) $d(x_2, x_3). d(x_1, x_2) \leq d(x_1, x_3)$.

(Draw a picture: $X$ is a space lying over $B$.)
A \textit{B-functor} $f$ from $X$ to $Y$ is a function $f: X \to Y$ satisfying:

(i) $e(f(x)) = e(x),$

(ii) $d(x_1, x_2) \leq f(x_1, f(x_2)).$

\textbf{EXAMPLE.} Let $H$ be a locale. Form a bicategory $B$ from $H$ as follows:

- objects of $B$: opens $u$ in $H$,
- arrows from $u$ to $v$: elements $w \leq u \land v$,
- 2-cells: order in $H$,
- composition of arrows: intersection.

Notice that $B = \text{Relations}(H)$.

From a sheaf $F$ on $H$ we can form a $B$-category $L(F)$ as follows:

$L(F) = \text{set of partial sections of } F,$

$e: L(F) \to \text{obj. } B: s \mapsto \text{domain of } s,$

$d: L(F) \times L(F) \to \text{morph. } B: (s, t) \mapsto \forall u; s \upharpoonright u = t \upharpoonright u.$

Notice that $L(F)$ has the property that if

$s, t \in L(F) \quad \text{and} \quad d(s, t) = e(s) = e(t),$

then $s = t$. Call such a $B$-category \textit{skeletal}.

Notice that the bicategory $B = \text{Span}(H)$ of this example has the property that $B^{\text{op}}$ (arrows reversed) = $B$. This property allows us to say that a $B$-category $X$ is \textit{symmetric} if

$d(x_1, x_2) = d(x_2, x_1) \quad \text{for all } x_1, x_2 \in X.$

Clearly $L(F)$ is symmetric and in fact $L$ is a fully-faithful functor

$L: \text{Sheaves}(H) \to \text{skeletal symmetric } B\text{-categories}.$

\section*{2. CAUCHY-COMPLETENESS}

To express Lawvere's notion of Cauchy-completeness we need to define bimodules. A \textit{bimodule} $\phi$ from $X$ to $Y$ (denoted $\phi: X \longrightarrow Y$) is a function $\phi: X \times Y \to \text{morph. } B$ satisfying (for all $x, x' \in X, y, y' \in Y$)

(i) $\phi(x, y): e(x) \to e(y),$

(ii) $\phi(x, y). d(x', x) \leq \phi(x', y),$

(iii) $d(y, y'). \phi(x, y) \leq \phi(x, y').$

As usual a $B$-functor $f: X \to Y$ yields a pair of bimodules
defined by
\[ f^*(x, y) = d(fx, y) \quad \text{and} \quad f_*(y, x) = d(y, fx). \]

Further \( f^* \) and \( f_* \) are adjoint in the sense that

(i) \( d(x, x') \leq \exists y [f_*(y, x') \cdot f^*(x, y)] \)
(where we write \( \exists y \) for the supremum (over \( y \)) in \( B(x, x') \)) and

(ii) \( \exists x [f^*(x, y') \cdot f_*(y, x)] \leq d(y, y') \).

Then a \( B \)-category \( Y \) is Cauchy-complete if every adjoint pair of bimodules \( \phi, \psi : X \rightleftarrows Y \) arises from a functor \( X \to Y \).

3. SHEAVES

We now have the definitions required to state the result.

**Theorem.** If \( H \) is a locale, then \( \text{Sheaves}(H) \) is equivalent to the category of skeletal symmetric Cauchy-complete \( \text{Rel}(H) \)-categories.

**Proof.** We want to see

(a) that \( L \) lands in Cauchy-complete \( B \)-categories, and

(b) that every skeletal Cauchy-complete symmetric \( B \)-category is isomorphic to \( L(F) \) for some sheaf \( F \).

For each element \( u \in H \) we can define a \( B \)-category \( \hat{u} \) with one element \(*\) and with \( e(*) = u, \quad d(*, *) = u \). Then, in testing Cauchy-completeness of \( Y \), we need only consider adjoint pairs of bimodules from \( \hat{u} \) to \( Y \) for each \( u \in H \).

To prove (a) consider an adjoint pair of bimodules \( \phi(s), \psi(s) \) ( \( s \in L(F) \) ) from \( \hat{u} \) to \( F \). Then condition (i) of adjointness says that:
\[ u_s = \phi(s) \land \psi(s) \quad (s \in L(F)) \]
is a cover of \( u \). Condition (ii) says that
\[ s \mid u_s \quad (s \in L(F)) \]
is a compatible family of sections, and so there is a section \( s_\theta \in F(u) \) such that
\[ s_\theta \mid u_s = s \mid u_s \quad \text{for all} \quad s \in L(F). \]

Now it is clear that for a general \( s \),
\[ d(s_\theta, s) = \bigvee_t d(s, t \mid u_t). \]
From property (ii) of adjunction:
\[ \phi(s) \wedge \psi(t) \wedge \phi(t) \leq d(s, t) \leq d(s, t | u_t) \]
and so by (i)
\[ \phi(s) \leq \bigvee_t d(s, t | u_t) = d(s_0, s). \]

From property (iii) of bimodules
\[ \phi(s) \geq d(s, t) \wedge \phi(t) \geq d(s, t | u_t), \text{ and so } \phi(s) \geq d(s_0, s). \]
Hence,
\[ \phi(s) = \psi(s) = d(s_0, s). \]

That is, the pair of bimodules arises from a functor.

To prove (b) consider a skeletal Cauchy-complete symmetric B-category \( Y \). We need to be able to define the restriction of an element \( y \) over \( u \) to \( v \leq u \). But this restriction comes from the fact that the adjoint pair of bimodules
\[ \phi(y') = \psi(y') = v \wedge d(y, y') : \begin{array}{c} \hat{v} \rightarrow \rightarrow Y \end{array} \]
is given by a functor. We need also to have the glueing together of a compatible family of elements \( (y_\alpha)_\alpha \) with \( \bigvee_{\alpha} e(y_\alpha) = v \). In this case the required section comes from the representation of the bimodules
\[ \phi(y') = \psi(y') = \bigvee_{\alpha} d(y_\alpha, y') : \begin{array}{c} \hat{v} \rightarrow \rightarrow Y \end{array} \]
as a functor.

REFERENCES

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