EDUARDO J. DUBUC

Open covers and infinitary operations in $C^\infty$-rings

*Cahiers de topologie et géométrie différentielle catégoriques*, tome 22, n° 3 (1981), p. 287-300

<http://www.numdam.org/item?id=CTGDC_1981__22_3_287_0>


L’accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l’accord avec les conditions générales d’utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d’une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

*Numdam*

Article numérisé dans le cadre du programme

*Numérisation de documents anciens mathématiques*

http://www.numdam.org/
OPEN COVERS AND INFINITARY OPERATIONS IN $C^\infty$-RINGS

by Eduardo I. DUBUC

INFINITE ADDITIONS.

In the $C^\infty$-ring $C^\infty(\mathbb{R}^n)$ of smooth real valued functions, one can define elements (functions) by adding certain infinite families. A typical application of this method is the following: Suppose $M \to \mathbb{R}^n$ is a closed smooth submanifold, and let $h : M \to \mathbb{R}$ be a smooth function defined on $M$. By definition this means that there are open sets $U_a \subset \mathbb{R}^n$, $a \in \Gamma$, such that they cover $M$, and smooth functions $h_a : U_a \to \mathbb{R}$ such that

$$\forall p \in U_a \cap M, \quad h_a(p) = h(p).$$

Let $U_0$ be $\mathbb{R}^n - M$, and take any function (e.g. the zero function) $h_0 : U_0 \to \mathbb{R}$. We then have an open covering $U_a$, $a \in \Gamma + \{0\}$, of the whole space $\mathbb{R}^n$, and functions $h_a : U_a \to \mathbb{R}$ such that

$$\forall p \in U_a \cap M, \quad h_a(p) = h(p).$$

(since $U_0 \cap M = \emptyset$). This family does not agree in the intersections $U_a \cap U_\beta$; thus it does not define a global function $f : \mathbb{R}^n \to \mathbb{R}$ which extends $h$. However, we can construct an extension of $h$. Let $W_i$ be a locally finite refinement of $U_a$, for each $i$ take $a$ such that $W_i \subset U_a$, and let $g_i : W_i \to \mathbb{R}$ be equal to $h_a | W_i$. Let $\phi_i$ be an associated partition of unity. The functions $f_i = \phi_i g_i$ are defined globally, $f_i : \mathbb{R}^n \to \mathbb{R}$, and have support contained in $W_i$. (This is so since $\text{support}(\phi_i) \subset W_i$ by definition.) It follows that the equality $f = \sum f_i$ defines a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^n, \quad f(x) = \sum f_i(x)$$

(which is a finite sum on an open neighborhood of $x$). If we compute $f$ at

1) Partially supported by the Danish Natural Science Research Council.
a point \( p \in M \), we have:
\[
f(p) = \sum_{i \in I} f_i(p) = \sum_{i \in I} \sum_{p \in W_i} \phi_i(p) g_i(p) = \sum_{i \in I} \phi_i(p) f_i(p) = h(p) \sum_{i \in I} \phi_i(p) = h(p) .
\]
Thus \( f \) is an extension of \( h \).

Two questions arise:

1. Which is the algebraic meaning of these infinitary additions?

2. To which extent can they be defined and performed in arbitrary \( \mathcal{C}^\infty \)-rings of finite type \( A = \mathcal{C}^\infty(\mathbb{R}^n)/I \)?

In particular, can we give a meaning to expressions of the form \( a = \sum a_{a} \), \( a_{a} \in A \)? In a way such that if \( a_{a} = [f_{a}] \), \( f_{a} \in \mathcal{C}^\infty(\mathbb{R}^n) \), and the \( f_{a} \) can be added in \( \mathcal{C}^\infty(\mathbb{R}^n) \) defining a function \( f = \sum f_{a} \) (as before), then \( a = [f] \) (where brackets indicate equivalence class modulo \( I \)). That this is not always possible is seen as follows: Let
\[
l = \{ h \mid h \text{ is of compact support} \}
\]
and let \( \phi_{a} \) be a partition of unity such that \( \phi_{a} \in l \) for all \( a \). Let
\[
a_{a} = [\phi_{a}] \quad \text{and} \quad a = \sum a_{a} .
\]
Then \( a = l \) since \( l = \sum a_{a} \) in \( \mathcal{C}^\infty(\mathbb{R}^n) \). But also \( a_{a} = [0] \) and then \( a = 0 \) since \( 0 = \sum a_{a} \) in \( \mathcal{C}^\infty(\mathbb{R}^n) \). Thus \( l = 0 \) in \( A \), which is impossible since \( l \notin I \). We see that a condition is needed in the ideal that presents \( A \). Namely, that if \( l_{a} \in l \) and the \( l_{a} \) can be added in \( \mathcal{C}^\infty(\mathbb{R}^n) \) defining a function \( l = \sum l_{a} \) (as before), then \( l \in l \). This leads to the notion of ideal of local character (cf. Definition 6, iv).

**Answer to Question II.**

The first step is to define the open cover topology in the category \( \mathcal{O}_{f, \text{open}} \), dual of the category of \( \mathcal{C}^\infty \)-rings of finite type. Given a \( \mathcal{C}^\infty \)-ring \( A \) and an element \( a \in A \), we denote \( A \to A \{ a^{-1} \} \) the solution in \( \mathcal{O} \) to the universal problem of making \( a \) invertible (cf. [1,2]). The following is straightforward:
1. PROPOSITION. Given any morphism \( \phi: A \to B \) and element \( a \in A \), the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & A \setminus \{a^{-1}\} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\phi} & C
\end{array}
\]

is a pushout diagram iff \( C = B \setminus \{\phi(a)^{-1}\} \).

We recall the following

2. PROPOSITION. Let \( U \subset \mathbb{R}^n \) be an open set, and let \( f \) be such that \( U = \{ x \mid f(x) \neq 0 \} \). Then \( C^\infty(\mathbb{R}^n) \to C^\infty(U) = C^\infty(\mathbb{R}^n) \setminus \{f^{-1}\} \).

PROOF. The basic idea is to consider the map

\( U \to \mathbb{R}^{n+1}: p \mapsto (p, f(p)^{-1}) \)

which makes \( U \) the closed sub-manifold of \( \mathbb{R}^{n+1} \) defined by the equation \( 1 - x_{n+1}f(p) = 0 \). Then use the fact that this equation is independent to deduce that the equalizer is preserved when taking the \( C^\infty \)-rings of smooth functions (cf. [1, 2]). A different proof of the preservation of this equalizer is given in [4].

3. DEFINITION. The open cover topology in \( \mathbb{R}^n \) is the topology generated by the empty family covering \( \{0\} \), and families of the form

\( C^\infty(\mathbb{R}^n) \to C^\infty(U_a), \) for all \( n \) and all open coverings \( U_a \) of \( \mathbb{R}^n \).

It follows from Propositions 1 and 2 that basic (generating) covers of an arbitrary \( C^\infty \)-ring of finite type \( A \) are families \( A \to A \setminus \{a_a^{-1}\} \) which can be completed into pushout diagrams

\[
\begin{array}{ccc}
C^\infty(\mathbb{R}^s) & \xrightarrow{\phi} & C^\infty(V_a) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi} & A \setminus \{a_a^{-1}\}
\end{array}
\]

where \( V_a = \{ x \mid g_a(x) \neq 0 \} \) is an open cover of \( \mathbb{R}^s \) and \( \phi(g_a) = a_a \). The morphism \( \phi \) can be chosen to be a quotient map. Let \( A = C^\infty(\mathbb{R}^n)/I; \) then, since \( C^\infty(\mathbb{R}^s) \) is free, there is a smooth function \( f: \mathbb{R}^n \to \mathbb{R}^s \) making
the following triangle commutative:

\[
\begin{array}{c}
\xymatrix{ C^\infty(R^s) \ar[r]^{f^*} & C^\infty(R^n) \ar[d]^{\phi} \\
A & A\{a^{-1}_a\} }
\end{array}
\]

One checks then that the following diagrams are pushouts:

\[
\begin{array}{c}
\xymatrix{ C^\infty(R^n) \ar[r] & C^\infty(U_a) \ar[d] \\
A & A\{a^{-1}_a\} }
\end{array}
\]

where

\[ U_a = f^{-1}(V_a) = \{ x | f_a(x) \neq 0 \}, \quad f_a = g_a f, \]

is an open cover of \( R^n \) and \( a_a = \lfloor f_a \rfloor \).

Open covers \( C^\infty(R^n) \to C^\infty(U_a) \) are effective epimorphic families. This just means that given a compatible family \( f_a \in C^\infty(U_a) \) (meaning that they agree in the intersections), there exists a unique \( f \in C^\infty(R^n) \) such that \( f|U_a = f \). (Remark that \( C^\infty(U_a \cap U_\beta) \) is the pushout of \( C^\infty(U_a) \) with \( C^\infty(U_\beta) \) over \( C^\infty(U_a) \).) However, they are not universal. Open covers of an arbitrary \( A \) will not, in general, be effective epimorphic families. For example, let as before \( I \) be the ideal

\[ I = ( h | h \text{ is of compact support}) , \]

and let \( \phi_a \) be a partition of unity such that \( \phi_a \in I \) for all \( a \). Then the diagrams

\[
\begin{array}{c}
\xymatrix{ C^\infty(R^n) \ar[r] & C^\infty(U_a) \ar[d] \\
A & \{0\} = A\{a^{-1}_a\} }
\end{array}
\]

are pushout diagrams for all \( a \), where \( U_a \) is the open covering

\[ U_a = \{ x | \phi_a(x) \neq 0 \}, \quad A = C^\infty(R^n)/I, \quad \text{and} \quad a_a = \lfloor \phi_a \rfloor = 0 . \]

On the way we see that the empty family covers \( A \) since it covers \( \{0\} \)
Consider now an open covering of an arbitrary $\mathcal{C}^\infty$-ring $A = \mathcal{C}^\infty(\mathbb{R}^n)/I$

\[
\begin{array}{ccc}
\mathcal{C}^\infty(\mathbb{R}^n) & \longrightarrow & \mathcal{C}^\infty(U_\alpha) \\
& & \downarrow \\
A & \longrightarrow & A\{a^{-1}\}
\end{array}
\]

$U_\alpha = \{ x \mid f_\alpha(x) \neq 0 \}$ and $a_\alpha = [f_\alpha]$.

For $b \in A$, we denote $b|_\alpha$ its image in $A\{a^{-1}\}$. Suppose $b, b' \in A$ are such that $b|_\alpha = b'|_\alpha$ for all $\alpha$. Let $b = [f]$ and $b' = [f']$. Then

\[
[f|U_\alpha] = b|_\alpha \quad \text{and} \quad [f'|U_\alpha] = b'|_\alpha.
\]

Thus $[f|U_\alpha] = [f'|U_\alpha]$, which means that $(f-f')|U_\alpha \in I\{U_\alpha\}$. If the covering is going to be effective epimorphic, it should follow that $(f-f') \in I$.

This leads to the notion of ideal of local character (cf. Definition 6, ii).

We recall the following elementary facts in order to fix the notation.

4. PROPOSITION. Let $A$ be a $\mathcal{C}^\infty$-ring and $p: A \rightarrow R$ a morphism into $R$, which we will also call a point of $A$. We denote $A \rightarrow A_p$ the solution in the category of $\mathcal{C}^\infty$-rings to the universal problem of making invertible all the elements $a \in A$ such that $p(a) \neq 0$. There is a factorization of $p$, $A \rightarrow A_p \rightarrow R$, and $A_p$ is a $\mathcal{C}^\infty$-local ring. If $a \in A$, we denote $a|_p \in A_p$ its image in $A_p$. Suppose $A = \mathcal{C}^\infty(\mathbb{R}^n)/I$, let $U \subset \mathbb{R}^n$ open, $f$ such that $U = \{ x \mid f(x) \neq 0 \}$, and $a = [f]$. Consider the following diagram (where the upper square is a pushout):

\[
\begin{array}{ccc}
\mathcal{C}^\infty(\mathbb{R}^n) & \longrightarrow & \mathcal{C}^\infty(U) \\
& & \downarrow \\
A & \longrightarrow & A\{a^{-1}\} \\
& \downarrow p & \Downarrow \quad \Downarrow \\
A_p & \longrightarrow & R
\end{array}
\]

Given a point $p$ of $A$, $p: A \rightarrow R$, since $\mathcal{C}^\infty(\mathbb{R}^n)$ is free, $p$ can be identified with a point of $\text{Zeros}(I) \subset \mathbb{R}^n$ which we will also denote $p$. When
there exist factorizations (necessarily unique) as shown in the dotted arrows, \( p \) can be identified with a point of \( A\{a^{-1}\} \), which we will also denote by \( p \). Then

\[
p \text{ is a point of } A\{a^{-1}\} \iff p(a) \neq 0 \iff p \in U.
\]

It follows that if \( A \to A\{a^{-1}\} \) is an open cover and \( p \) is a point of \( A \) then there exists \( a \) such that \( p \) is a point of \( A\{a^{-1}\} \) (since there exist \( a \) such that \( p \in U_a \)).

**Proof.** This is all rather straightforward. For a proof, cf. [2, Exposé 11].

Suppose now \( b, b' \in A \) are such that \( b|_p = b'|_p \) for all points \( p \) of \( A \). Let \( b = [f] \) and \( b' = [f'] \). Then

\[
b|_p = [f|_p] \quad \text{and} \quad b'|_p = [f'|_p].
\]

Thus \( [f|_p] = [f'|_p] \), which means that \( (f-f')|_p \in I|_p \). If we want to deduce that \( b = b' \), it should follows that \( (f-f') \in I \). This leads to the notion of *ideal of local character* (cf. Definition 6, i).

5. **Definition.** A family \( l_i \in C^\infty (R^n) \) (indexed by an arbitrary set) is *locally finite* if there is an open covering \( U_a \) such that, for every \( a \), \( l_i|_{U_a} = 0 \) except for a finite number of \( i \). Equivalently, if each point \( p \) of \( R^n \) has an open neighborhood \( U \) such that \( l_i|_U = 0 \) except for a finite number of \( i \).

Given a locally finite family \( l_i \), the finite sums \( \sum l_i|_{U_a} \in C^\infty (U_a) \) form a compatible family. Thus there exists a unique

\[
l \in C^\infty (R^n) \quad \text{such that} \quad l|_{U_a} = \sum l_i|_{U_a}.
\]

We denote this \( l \) by the formula \( l = \sum l_i \).

6. **Definition.** An ideal \( I \in C^\infty (R^n) \) is of *local character* if it satisfies any one of the following equivalent conditions:

i) \( (f|_p \in I|_p \quad \text{for all} \quad p \in \text{Zeros}(I)) \Rightarrow f \in I \).

ii) \( (f|_{U_a} \in I|_{U_a} \quad \text{for some open cover} \ U_a) \Rightarrow f \in I \).

iii) \( \phi_a f \in I \quad \text{for some partition of unity} \ \phi_a \Rightarrow f \in I \).

iv) \( l_i \in I \quad \text{for all} \quad i \Rightarrow (\sum l_i) \in I \quad \text{for every locally finite family} \ l_i \).
PROOF OF THE EQUIVALENCE. The antecedents of the first three implications are themselves equivalent, cf. [1], Lemma 10, and for detailed proof, [2] Théorème 1.5; thus the equivalence of the first three conditions. That \( iv \Rightarrow iii \) is immediate since \( \phi_a f \) is a locally finite family and \( f = \sum \phi_a f \). Finally, it is evident that \( ii \Rightarrow iv \). Notice that the implications in the first three conditions always hold in the other sense.

It is clear that any ideal \( I \) has a «closure» of local character; namely
\[
\hat{I} = \{ f \mid f\mid_p \in I \mid_p \ \forall p \in \text{Zeros}(I) \}.
\]
\( \hat{I} \) can also be seen as the closure of \( I \) under additions of locally finite families. Let \( \mathcal{B} \) be the category of \( \mathcal{C}^\infty \)-rings presented by an ideal of local character. Let
\[
A = \mathcal{C}^\infty(R^n)/I \quad \text{and} \quad rA = \mathcal{C}^\infty(R^n)/\hat{I}.
\]
Then we have a canonical (quotient) map \( A \to rA \) and the passage \( A \to rA \) is clearly a left adjoint for the inclusion \( \mathcal{B} \to \mathcal{A}_{f,t} \). Thus \( \mathcal{B} \) is closed under all inverse limits and has all colimits. These colimits will not coincide in general with the respective construction in \( \mathcal{A}_{f,t} \). We remark that, since \( \text{Zeros}(I) = \text{Zeros}(\hat{I}) \), \( A \) and \( rA \) have the same points.

7. EXAMPLES. 1° Let \( A = \mathcal{C}^\infty_0(R) \), \( B = \mathcal{C}^\infty(R) \). Then \( A, B \in \mathcal{B} \). By construction, \( A \otimes_\infty B = \mathcal{C}^\infty(R^2)/\hat{I} \), where
\[
\hat{I} = \{ f \mid \exists \eta > 0, f(x, y) = 0 \ \forall x \mid |x| < \eta \}.
\]
This ideal is not of local character:
\[
\hat{I} = \{ f \mid f = 0 \ \text{on an arbitrary neighborhood of the } y\text{-axis} \}.
\]
The coproduct of \( A \) with \( B \) in \( \mathcal{B} \) does not coincide then with the one performed in \( \mathcal{A}_{f,t} \).

2° Let \( A = \mathcal{C}^\infty_0(R) \) and \( a = x_0 \). Then
\[
A\{a^{-1}\} = \mathcal{C}^\infty_0(R)\{x_0^{-1}\} = \mathcal{C}^\infty(R^*)/J,
\]
where \( R^* = R - \{0\} \) and
\[
J = \{ f \mid \exists U \mid 0 \in U, f|_U = 0 \} \subset \mathcal{C}^\infty(R^*).\]
We see this because the following is a pushout diagram:

\[
\begin{array}{ccc}
C^\infty(R) & \longrightarrow & C^\infty(R^*) = C^\infty(R)\{x^{-1}\} \\
\downarrow & & \downarrow \\
C^\infty_0(R) & \longrightarrow & C^\infty(R^*)/I = C^\infty_0(R)\{x|_0^{-1}\}
\end{array}
\]

(cf. Proposition 1). Now, the ring \(A\{a^{-1}\}\) has the presentation

\[A\{a^{-1}\} = C^\infty(R^2)/(l, 1-xy),\]

where \(l\) is the ideal in \(I\) above. Thus the localization \(A\{a^{-1}\}\) in \(\mathcal{B}\) does not coincide with the one performed in \(\mathcal{A}_{f,t}\). Since \(\text{Zeros}(l, 1-xy) = \emptyset\), we have that \(l \in (l, 1-xy)^o\). Thus \(A\{a^{-1}\} = \{0\}\) in \(\mathcal{B}\). This means that if \(\phi: C^\infty_0(R) \to B\) is any morphism, \(B\) is in \(\mathcal{B}\), and \(\phi(x|_0)\) invertible, then \(B = \{0\}\). This is not so if \(B\) is not in \(\mathcal{B}\).

In what follows we will utilize the same notation as in \(\mathcal{A}_{f,t}\) for the constructions performed in \(\mathcal{B}\). Since they are defined by the same universal properties, we have:

8. PROPOSITION. Propositions 1 and 4 remain valid for the category \(\mathcal{B}\). In addition, since all finitely generated ideals are of local character (cf. [1, 2]), if \(A\) is of finite presentation, then for any \(a \in A\), \(A\{a^{-1}\}\) constructed in \(\mathcal{A}_{f,t}\) is already in \(\mathcal{B}\). Thus Proposition 2 remains valid also for the category \(\mathcal{B}\). Furthermore, if \(A\) is in \(\mathcal{B}\), then for any elements \(b, b' \in A\),

\[b = b' \iff b|_p = b'|_p \text{ for all points } p \text{ of } A.\]

9. PROPOSITION. The open coverings are universal effective epimorphic families in the category \(\mathcal{B}^{op}\).

PROOF. The proof is essentially a repetition of the argument given at the beginning of this article. Let \(A = C^\infty(R^n)/I\), \(I\) of local character, \(U_\alpha\) an open cover of \(R^n\), \(f_\alpha\) such that \(U_\alpha = \{ x \mid f_\alpha(x) \neq 0 \}\) and \(a_\alpha \in A, a_\alpha = [f_\alpha]\). We consider the pushout diagram in \(\mathcal{B}\):

\[
\begin{array}{ccc}
C^\infty(R^n) & \longrightarrow & C^\infty(U_\alpha) \\
\downarrow & & \downarrow \\
A & \longrightarrow & A\{a_\alpha^{-1}\}
\end{array}
\]
Let \( b \in A \{ a^{-1}_a \} \) be a compatible family. This implies that for all points of \( A \{ a^{-1}_a \} \), \( b_a \mid_p = b \beta \mid_p \). Thus, for all \( p \) in \( A \) there is a well defined \( b(p) \in A_p \), \( b(p) = b_a \mid_p \) for any \( a \) such that \( p \) is in \( A_a \). We shall construct an element

\[
b \in A \quad \text{such that} \quad b \mid_p = b(p) \quad \text{for all} \quad p \in A.
\]

Let \( h_a \in C^\infty(U_a) \) such that \( b_a = [h_a] \), let \( W_i \) be a locally finite refinement of \( U_a \), for each \( i \) take \( a \) such that \( W_i \subset U_a \), and let \( g_i \in C^\infty(W_i) \) be equal to \( h_a \mid W_i \). Thus, for all \( p \) in \( W_i \), \( [g_i \mid_p] = b(p) \). Let \( \phi_i \) be an associated partition of unity. The functions \( l_i = \phi_i g_i \) are defined globally, \( l_i \in C^\infty(\mathbb{R}^n) \), and have support contained in \( W_i \). Thus, given any \( p \) in \( A \), if \( p \notin W_i \), \( l_i \mid_p = 0 \), and if \( p \in W_i \),

\[
[l_i \mid_p] = [\phi_i \mid_p] b(p).
\]

Since the family \( l_i \) is locally finite (Definition 5), we have a function \( l = \sum_i l_i \). Let \( b = [l] \). Then, for any \( p \) in \( A \),

\[
b \mid_p = [\sum_i l_i \mid_p] = \sum_{i \mid_p \in W_i} [l_i \mid_p] = b(p) \sum [\phi_i \mid_p] = b(p).
\]

Given any \( a \), \( b \mid_a = b_a \) since for all \( p \) in \( A \{ a^{-1}_a \} \), \( b \mid_p = b_a \mid_p \) and \( A \{ a^{-1}_a \} \) is in \( \mathcal{B} \). In the same way one checks the uniqueness of such \( a, b \), since all points of \( A \) are in some \( A \{ a^{-1}_a \} \). This finishes the proof.

10. DEFINITION - PROPOSITION. Given any \( C^\infty \)-ring \( A \) presented by an ideal of local character, a family \( b_i \in A \) (indexed by an arbitrary set) is \textit{locally finite} if there is an open covering \( A \rightarrow A \{ a^{-1}_a \} \) such that for every \( a \), \( b_i \mid_a = 0 \) except for a finite number of \( i \). Given such a family, the finite sums \( \sum_i b_i \mid_a \in A \{ a^{-1}_a \} \) form a compatible family. It follows then from the previous proposition that there exists a unique

\[
b \in A \quad \text{such that} \quad b \mid_a = \sum_i b_i \mid_a.
\]

We denote this \( b \) by the formula \( b = \sum_i b_i \). Given any morphism \( \phi : A \rightarrow B \) in \( \mathcal{B} \), the family \( \phi(b_i) \in B \) is also locally finite, and \( \phi \sum_i b_i = \sum_i \phi(b_i) \).

This finishes the answer to Question II. Infinite additions of locally finite families make sense and can be performed in certain \( C^\infty \)-rings. The
C∞-rings which have this extra structure are precisely those presented by an ideal of local character. Before passing to Question I, we make a final remark on the open covering topology.

11. **Proposition.** Given any C∞-ring \( A \in \mathcal{F}_f, \lambda \) and a family \( a_\alpha \in A, A \to A\{a_\alpha^{-1}\} \) covers in the open covering topology iff for all points \( p \) in \( A \) there exists \( \alpha \) such that \( p \) is in \( A\{a_\alpha^{-1}\} \). Thus any C∞-ring without points is covered by the empty family.

**Proof.** One of the implications has already been seen (Proposition 4). Suppose then that for all \( p \) in \( A \) there is \( \alpha \) such that \( p \) is in \( A\{a_\alpha^{-1}\} \). Let \( A = C^\infty(\mathbb{R}^n)/l, f_\alpha \in C^\infty(\mathbb{R}^n) \) such that

\[
a_\alpha = [f_\alpha] \quad \text{and} \quad U_\alpha = \{x \mid f_\alpha(x) \neq 0\}.
\]

The hypothesis means that for all \( p \in \text{Zeros}(l) \subset \mathbb{R}^n \) there is \( \alpha \) such that \( p \in U_\alpha \). If \( p \notin \text{Zeros}(l) \), there is

\[
l \in l \quad \text{such that} \quad p \notin U_l = \{c \mid l(c) \neq 0\}.
\]

Thus the \( U_\alpha \) together with the \( U_l \) form an open cover of \( \mathbb{R}^n \). Let \( b_l \in A, b_l = [l] \). Then \( A \to A\{a_\alpha^{-1}\} \) together with \( A \to A\{b_l^{-1}\} \) form an open cover of \( A \). But \( A\{b_l^{-1}\} = \{0\} \); thus it is covered by the empty family. By composition of coverings, it follows that \( A \to A\{a_\alpha^{-1}\} \) is an open covering of \( A \).

**Answer to Question I.**

We consider the free C∞-ring in \( \lambda \) generators, \( \lambda \in \Gamma \), cf. [2]. This ring, which we denote \( C^\infty(\mathbb{R}^\Gamma) \), is the ring of functions \( \mathbb{R}^\Gamma \to \mathbb{R} \) which depend only on a finite number of variables, and which are smooth on these variables. Clearly, this is a ring of continuous functions (for the product topology in \( \mathbb{R}^\Gamma \)). If we take a locally finite family of functions \( \mathbb{R}^\Gamma \to \mathbb{R} \) in \( C^\infty(\mathbb{R}^\Gamma) \), and add it up, we get a continuous function which will not be in general in \( C^\infty(\mathbb{R}^\Gamma) \). However, every point will have a neighborhood in which this function will coincide with the restriction of a function in \( C^\infty(\mathbb{R}^\Gamma) \). We consider all functions \( \phi \) which locally depend on a finite number of variables. More precisely, functions for which there exists an
open cover $U_a \subset \mathbb{R}^\Gamma$, finite sets $I_a$, smooth functions $h_a$, and factorizations as indicated in the following diagram (where $\rho$ is the projection):

$$
\begin{array}{c}
U_a \\
\downarrow \rho \\
I_a \\
\downarrow h_a \\
R_a \\
\end{array} \\
\begin{array}{c}
\mathbb{R}^\Gamma \\
\phi \\
\mathbb{R} \\
\end{array}
$$

for all $a$. Thus we add to $C^\infty(\mathbb{R}^\Gamma)$ all the functions needed to render the open covers of $\mathbb{R}^\Gamma$ effective epimorphic families. We will, for the lack of a better notation, denote this ring by $\infty C^\infty(\mathbb{R}^\Gamma)$. Thus a function $\phi$ is in $\infty C^\infty(\mathbb{R}^\Gamma)$ iff for each point $p \in \mathbb{R}^\Gamma$, there is an open neighborhood $U$ of $p$, a finite set of indices $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and a smooth function of $n$ variables $h$ such that for all $(x_{\lambda}) \in U$, $\lambda \in \Gamma$,

$$
\phi(x_{\lambda}) = h(x_{\lambda_1}, \ldots, x_{\lambda_n}).
$$

We leave to the reader the proof of the following (where $\Gamma$ and $\Lambda$ are any sets, including finite):

12. **Proposition.** Given any $\phi \in \infty C^\infty(\mathbb{R}^\Gamma)$ and a $\Gamma$-tuple $\phi_\Lambda \in \infty C^\infty(\mathbb{R}^\Lambda)$, $(\phi_\Lambda) : \mathbb{R}^\Lambda \to \mathbb{R}^\Gamma$,

the composite $\phi(\phi_\Lambda) \in \infty C^\infty(\mathbb{R}^\Lambda)$.

It follows then that there is an infinitary algebraic theory in the sense of Linton (cf. [3]) which has as $\Gamma$-ary operations the rings $\infty C^\infty(\mathbb{R}^\Gamma)$. We shall call this theory the *infinitary theory of $C^\infty$-rings*. Its finitary part is the theory of $C^\infty$-rings, since $\infty C^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. Thus, this ring is still free on $n$ generators for the infinitary theory. (The action of a $\Gamma$-ary operation on $C^\infty(\mathbb{R}^n)$ is given in Proposition 12, with $\Lambda = n$.) All $\infty C^\infty$-rings of finite type are then quotients of $C^\infty(\mathbb{R}^n)$ presented by $\infty C^\infty$-ideals, that is, ideals $I$ such that the congruence

$$
f \sim g \Rightarrow f - g \in I
$$

is an $\infty C^\infty$-congruence. Recall that this congruence is always a $C^\infty$-congruence (cf. [1, 2]).
13. PROPOSITION. Let $A = C^\infty(\mathbb{R}^n)/I$ be a $C^\infty$-ring presented by an ideal $I$ of local character. Then, $A$ is an $\infty C^\infty$-ring. Or, equivalently, $I$ is an $\infty C^\infty$-ideal.

PROOF. Let $\phi \in \infty C^\infty(\mathbb{R}^\Gamma)$, and let

$$g_\lambda, f_\lambda \in C^\infty(\mathbb{R}^n), \lambda \in \Gamma,$$

such that $g_\lambda - f_\lambda \in I$.

Let $p$ be any point of $\mathbb{R}^n$ and let $U \subset \mathbb{R}^\Gamma$ be an open neighborhood of the two points $(g_\lambda(p))$, $(f_\lambda(p)) \in \mathbb{R}^\Gamma$ where $\phi$ depends on a finite number of variables:

$$\phi(x_\lambda) = h(x_{\lambda_1}, \ldots, x_{\lambda_k}) \text{ over } U,$$

for some $h \in C^\infty(\mathbb{R}^k)$. Let $W \subset \mathbb{R}^n$ be an open neighborhood of $p$ such that $(g_\lambda(W)) \subset U$ and $(f_\lambda(W)) \subset U$.

Then

$$(\phi(g_\lambda) - \phi(f_\lambda))(W) = (h(g_{\lambda_1}, \ldots, g_{\lambda_k}) - h(f_{\lambda_1}, \ldots, f_{\lambda_k}))(W) \in I \mid W,$$

because

$$h(g_{\lambda_1}, \ldots, g_{\lambda_k}) - h(f_{\lambda_1}, \ldots, f_{\lambda_k}) \in I,$$

since all ideals are $C^\infty$-ideals. Thus the term $(\phi(g_\lambda) - \phi(f_\lambda))(W)$ is in $I(p)$. Since $I$ is of local character, this implies $\phi(g_\lambda) - \phi(f_\lambda) \in I$.

The converse of this proposition says, in a way, that there are enough operations in the infinitary theory of $C^\infty$-rings to force any $\infty C^\infty$-ideal to be of local character. This is actually the case, and we prove it by showing that given any locally finite family $l_\lambda \in C^\infty(\mathbb{R}^n)$, there is an infinitary operation that adds it up.

14. PROPOSITION. Let $l_\lambda \in C^\infty(\mathbb{R}^n), \lambda \in \Gamma$, be any locally finite family. Then, the following function:

$$L: \mathbb{R}^n \times \mathbb{R}^\Gamma \to \mathbb{R}, L(x_1, \ldots, x_n, x_\lambda) = \sum_\lambda l_\lambda(x_1, \ldots, x_n)x_\lambda$$

is in $\infty C^\infty(\mathbb{R}^n + \Gamma)$.

PROOF. Given any point $p \in \mathbb{R}^n + \Gamma$, $p = (a_1, \ldots, a_n, a_\lambda)$, take an open neighborhood $U \subset \mathbb{R}^n$ of $(a_1, \ldots, a_n)$, where all but a finite number of $l_\alpha$
are zero. Then the open set $U \times \mathbb{R}^\Gamma$ is a neighborhood of $p$ where $L$ depends on only a finite number of variables.

Let $h_1, h_2, \ldots, h_n$ and $f_\lambda$ be any $(n + \Gamma)$-tuplet of smooth functions in $k$ variables. The family (indexed by $\Gamma$) $l_\lambda(h_1, \ldots, h_n)f_\lambda$ is a locally finite family in $C^\infty(\mathbb{R}^k)$, and it follows from Proposition 12 (with $k = \Gamma$) that the action of $L$ for the $\infty C^\infty$-ring structure of $C^\infty(\mathbb{R}^k)$ is given by the formula

$$L(h_1, \ldots, h_n, f_\lambda) = \sum_\lambda l_\lambda(h_1, \ldots, h_n)f_\lambda.$$

We have

15. **PROPOSITION.** Let $l_\lambda$ and $L$ be as in Proposition 14.

   i) Given any $n$-tuple $h_1, \ldots, h_n \in C^\infty(\mathbb{R}^k)$, $L(h_1, \ldots, h_n, 0) = 0$.

   ii) Given any $\infty C^\infty$-ideal $I \subset C^\infty(\mathbb{R}^k)$ and any $n + \Gamma$-tuple $h_1, \ldots, h_n, f_\lambda \in C^\infty(\mathbb{R}^n)$, if $f_\lambda \in I$ for all $\lambda$, then $\sum_\lambda l_\lambda(h_1, \ldots, h_n)f_\lambda \in I$.

   iii) Given any $\infty C^\infty$-ideal $I \subset C^\infty(\mathbb{R}^n)$ and any $\Gamma$-tuple $f_\lambda \in C^\infty(\mathbb{R}^n)$ if $f_\lambda \in I$ for all $\lambda$, then $\sum_\lambda f_\lambda \in I$.

   iv) Given any $\infty C^\infty$-ideal $I \subset C^\infty(\mathbb{R}^n)$ and any locally finite family $f_\lambda \in C^\infty(\mathbb{R}^n)$, if $f_\lambda \in I$ for all $\lambda$, then $\sum_\lambda f_\lambda \in I$.

**PROOF.** i is clear, and ii follows clearly from i. We get iii by putting $k = n$ and $h_i = \pi_i$ the projections.

Finally, observe that given any open set $U \subset \mathbb{R}^n$ and any smooth function $f$ with $\text{Supp}(f) \subset U$, then there exists a function $l$ with $\text{supp}(l) \subset U$ and such that $f = fl$. To prove iv, we apply this observation to each of the functions $f_\lambda$. The family $l_\lambda$ so obtained is also locally finite; thus it has an associated operation $L$. iv follows then from iii.

16. **COROLLARY.** Let $A$ be a $\infty C^\infty$-ring of finite type. Then $A$ is a $C^\infty$-ring presented by an ideal of local character.

**PROOF.** Immediate from iv in the previous proposition and Definition 6, iv.

Thus, the ideals of local character are exactly the congruences for
the infinitary theory of $C^\infty$-rings. Remark that we have also proved that
given any locally finite family $f_\lambda$ in $C^\infty(\mathbb{R}^n)$, $\lambda \in \Gamma$, there is a $(n + \Gamma)$-
ary operation $L$ such that

$$L(\pi_1, \ldots, \pi_n, f_\lambda) = \sum_\lambda f_\lambda,$$

where $\pi_i$ are the projections. Suppose now $I$ is an ideal of local character,
let $A = C^\infty(\mathbb{R}^n)/I$, let $e_i = [\pi_i]$ be the generators of $A$, and let

$$b_\lambda = f_\lambda(e_1, \ldots, e_n) = [f_\lambda].$$

Then, by Definition-Proposition 10, $b_\lambda$ is a locally finite family, and

$$[\sum_\lambda f_\lambda] = \sum_\lambda b_\lambda.$$

But since $I$ is a $\infty C^\infty$-ideal, we also have

$$[L(\pi_1, \ldots, \pi_n, f_\lambda)] = L(e_1, \ldots, e_n, b_\lambda).$$

Thus, given any locally finite family $b_\lambda = [f_\lambda]$ in a $C^\infty$-ring $A$ presented
by an ideal of local character, $A = C^\infty(\mathbb{R}^n)/I$, if $f_\lambda$ is locally finite in
$C^\infty(\mathbb{R}^n)$, there is an infinitary operation $L$ such that

$$L(e_1, \ldots, e_n, b_\lambda) = \sum_\lambda b_\lambda,$$

where $e_1, \ldots, e_n$ are generators of $A$. With this we finish the answer to
Question I.

REFERENCES.

1. E.J. DUBUC, $C^\infty$-schemes, Aarhus Univ. Preprint Series 1979/80, n°3. A revis-

2. E.J. DUBUC, Schémas $C^\infty$, Exposés 3 et 11 in Géom. Diff. Synth., Rapports

3. F.J. LINTON, Some aspects of equational categories, Proc. Conf. on categorical