PARTIAL COMPLETIONS OF CONCRETE FUNCTORS

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INTRODUCTION.

If $f$ and $g$ are differentiable maps between manifolds $M$ and $M'$, the equation $f(m) = g(m)$ may not define a submanifold of $M$; two differentiable maps toward $M$ may not have a pullback unless they are transversal. Such difficulties have hindered a categorical study of Differential Geometry; e.g., differentiable categories /50/ are only those internal categories in the category $\mathcal{D}$ of manifolds whose domain and codomain maps are submersions. Is it not possible to embed $\mathcal{D}$ into an «adequate» category? Charles was mainly motivated by this question and inspired by his many works on completions of posets and of local categories /47, 55, 76, 85, 86/ when he wrote his paper /107/ in the mid-sixties: here he constructs «optimal» extensions of a concrete functor $P : H \to E$ into a concrete functor which initially lifts a given class of singleton sources («spreading» functors) or of limit cones («completions») in $E$. For example, the smallest spreading extension of $\mathcal{D} \to \text{Ens}$ equips each subset of a manifold $M$ with a structure which has been independently worked out (without categorical aims!) by Ngo Van Qué [6], Aronszajn and Marshall [5].

Later on, several authors tackled analogous problems, often with a view to embedding the category of topological spaces in an initially complete cartesian closed category (Antoine, Chartrelle, Day, Wyler,...); generalizing results of Banaschewski and Bums on completions of posets, Herrlich describes in [2] the smallest or Mac Neille and the largest or universal (preserving initial lifts) initial completions of $P$; for the bibliography, we refer to [3] where most papers on initial completions (Adamek, Herrlich, Strecker; Börger; Hoffmann; Tholen...) are summarized. Recently
universal completions of $P$ have been constructed by Adamek-Koubek [1] and, without transfinite induction, by Herrlich [4].

Here all these results are unified: Given a class $\Gamma$ of cones in $E$, and a class $\Delta$ of initial cones in $H$, the concrete functor $P : H \to E$ is extended into a concrete functor with initial lifts of cones of $\Gamma$, and for which the cones of $\Delta$ remain initial; two «optimal» solutions, the Mac Neille $\Gamma$-completion and the universal $(\Delta, \Gamma')$-completion of $P$, are constructed by methods making the most out the ideas of Charles /55, 107/ and Herrlich [2, 4]. If $\Gamma$ is «not too large» these solutions live in the same universe as $P$.

HYPOTHESES. There is given a category $E$ (the «base category») and a class $\Gamma$ of cones in $E$; let $\text{Ind} \Gamma$ be the class formed by the indexing categories of the cones $\gamma \in \Gamma$. Cone always means projective cone.

We denote by $P : H \to E$ a concrete (i.e., faithful and amnestic) functor, by $S, S', ...$ the objects of $H$, by $E, E', ...$ the objects of $E$. These notations come from the primitive case where $P$ is the forgetful (Projection) functor from the category of Homomorphisms between Structures of some kind to $\text{Ens}$.

$\mathcal{U}_0$ and $\mathcal{U}$ are two universes such that $\mathcal{U}_0 \subset \mathcal{U}$; the elements of $\mathcal{U}$ are large sets, or classes, those of $\mathcal{U}_0$ are small sets. We suppose $P$ lives in $\mathcal{U}$ (as does for instance the forgetful functor $\text{Hom} \to \text{Ens}$).

1. PARTIAL COMPLETIONS.

In this section, we recall definitions, and state some results on cones and initial lifts which are used in the sequel.

A. Commutative hull of $\Gamma$.

Let $\gamma$ be a cone in $E$ with vertex $E$ and basis $\phi : I \to E$, abbreviated in $\gamma : E \Rightarrow \phi$. If the indexing category $I$ is discrete, $\gamma$ is called a source, and also denoted by $(\gamma(l) \mid l \in I)$; for instance $E$ defines the the singleton source $E^\bullet = (1d_E)$.

To the cone $\gamma$ is associated the source $(\gamma(l) \mid l \in I_0)$ indexed by the class $I_0$ of objects of $I$; this source is written $\gamma_0 : E \Rightarrow \phi_0$. We
denote by $\Gamma_0$ the class of all sources $\gamma_0$, for $\gamma \in \Gamma$.

If $\gamma_I: \phi(I) \Rightarrow \phi_I$ is a cone indexed by $I_I$ for each $I \in I_0$, the source

$$(\gamma_I(I), \gamma(I): E \rightarrow \phi_I(I) \mid (I, I) \in \Sigma_{I_0})$$

is called the composite source of $((\gamma_I)_I, \gamma)$, denoted $(\gamma_I)_I \circ \gamma$. For instance: $(\phi(I)^* I_0 \circ \gamma = \gamma_0$; if $I = 1$ and $\gamma(I) = f$, then $(\gamma_I)_1 \circ \gamma$ is the source $\gamma_I \circ f = (\gamma_I(J), f \mid J \in I_{I_0})$.

**DEFINITION.** A class $\Sigma$ of sources in $E$ is said commutative if

1. $E^{*} \epsilon \Sigma$ for each object $E$ of $E$,
2. $\gamma \in \Sigma$ and $\gamma_I \in \Sigma$ for each $I \in I_0$ imply $(\gamma_I)_I \circ \gamma \epsilon \Sigma$.

**PROPOSITION 1.** Let $\Sigma$ be a class of sources in $E$; the smallest commutative class of sources $\Sigma^0$ containing $\Sigma$ is constructed by transfinite induction. If $\text{Ind} \Sigma$ and each $I \in \text{Ind} \Sigma$ belong to the universe $U$, so does $\text{Ind} \Sigma^0$.

$\Sigma^0$ is the union of the transfinite increasing sequence $(\Sigma_\lambda)_\lambda$ defined by induction as follows:

$$\Sigma_0 = \{ E^* \mid E \epsilon E_0 \} \cup \Sigma; \quad \Sigma_a = \cup_{\lambda < a} \Sigma_\lambda \text{ for each limit ordinal } a,$$

$$\Sigma_{\lambda + 1} = \{(\gamma_I)_I \circ \gamma \mid \gamma \epsilon \Sigma_\lambda, \gamma_I \epsilon \Sigma_\lambda \forall I \epsilon I \}.$$  

The construction stops at the limit ordinal larger than the ordinal of $I$, for each $I \in \text{Ind} \Sigma$.

**DEFINITION.** The smallest commutative class of sources containing $\Gamma_0$ is called the commutative hull of $\Gamma$, denoted by $\Gamma^0$.

**EXAMPLES.** The class $\text{Sour} E$ of all sources in $E$ is commutative and it is the commutative hull of the class $\text{Cone} E$ of all cones in $E$. If $A$ is a class of morphisms of $E$, the class $/A/$ of singleton sources $\{a\}$, $a \epsilon A$ has for its commutative hull the class $/A'/$, corresponding to the sub-category of $E$ generated by $A \cup E_0$.

**B. Initial lifts.**

Let $P: H \rightarrow E$ be a concrete functor. A morphism $h$ from $S$ to $S'$ in $H$ is written $g: S \rightarrow S'$, where $g = P(h)$. If $\theta$ is a cone in $H$ with vertex $S$ and basis $\Phi$ and if $\gamma = P\theta$, we also denote $\theta$ by $\gamma: S \Rightarrow \Phi$. 

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A \( P \)-cone indexed by \( I \) is a pair \((P, y)\), where \( P \colon I \to H \) is a functor and \( y : E \to P \Phi \) is a cone. An initial lift of \((\Phi, \gamma)\) is an object \( S \) of \( H \) such that:

1. \( y(I) : S \to \Phi(I) \) is in \( H \) for each \( I \in I_o \),
2. If \( f : E' \to E \) in \( E \) and if \( y \circ f : S' \to \Phi \) is a cone in \( H \), then \( f \) lifts into \( f : S' \to S \) in \( H \).

As \( P \) is concrete, such an \( S \) (if it exists) is unique; it is then denoted by \( \text{il}(\Phi, \gamma) \); so \( y : S \to \Phi \) is an initial cone for \( P \).

A \( P \)-source is a \( P \)-cone \((\Phi, \gamma)\) indexed by a discrete category \( I_o \); it is often identified to the family of \( P \)-morphisms (singleton \( P \)-sources) \( (\Phi(I), \gamma(I)) \mid I \in I_o \).

The dual notion is a \( P \)-sink.

To the \( P \)-cone \((\Phi, \gamma)\) indexed by \( I \) is associated the \( P \)-source \((\Phi_0, \gamma_0)\), where \( \Phi_0 : I_o \to H \) is the restriction of \( \Phi \) to the objects of \( I \).

**Proposition 2.** Let \((\Phi, \gamma)\) be a \( P \)-cone. It has an initial lift iff the \( P \)-source \((\Phi_0, \gamma_0)\) has one; in this case, \( \text{il}(\Phi, \gamma) = \text{il}(\Phi_0, \gamma_0) \).

The following proposition (whose proof is straightforward) is important for the sequel. Let \((\Phi, \gamma)\) be a \( P \)-cone indexed by \( I \) and \((\Phi_I, \gamma_I)\) be a \( P \)-cone indexed by \( I_I \) such that \( \gamma_I : \Phi(I) \to \Phi_I \) is a cone in \( H \), for \( I \) in \( I_o \). We denote by \((\Phi_I, \gamma_I)_{I_o} \circ (\Phi, \gamma)\) the \( P \)-source \((\Psi, (\gamma_I)_{I_o} \circ \gamma)\) where \( \Psi : \sum_{I \in I_o} I \to H : (J, I) \mapsto \Phi_J(I) \).

**Proposition 3 (Commutativity of initial lifts).** If \( \Phi(I) = \text{il}(\Phi_I, \gamma_I) \) for each object \( I \) of \( I \), we have

\[
\text{il}(\Phi, \gamma) = \text{il}(\Phi_I, \gamma_I)_{I_o} \circ (\Phi, \gamma)
\]

as soon as one of these terms is defined.

**Definition.** \( P \) is called \( \Gamma \)-complete if each \( P \)-cone \((\Phi, \gamma)\) with \( \gamma \in \Gamma \) admits an initial lift.

From Propositions 1, 2, 3, it follows by transfinite induction:

**Corollary.** If \( P \) is \( \Gamma_o \)-complete, then it is \( \Gamma \)-complete and \( \Gamma^o \)-complete, where \( \Gamma^o \) is the commutative hull of \( \Gamma \).
We consider the category of concrete functors over $E$, whose objects are the concrete functors $Q: K \to E$ and whose morphisms $F: Q \to Q'$, where $Q': K' \to E$, are the functors $F: K \to K'$ such that $Q' F = Q$. It has a non-full subcategory formed by the $\Gamma$-morphisms, which are the morphisms $F: Q \to Q'$ such that $F(il(\Phi, \gamma)) = il(F \Phi, \gamma)$ whenever $(\Phi, \gamma)$ is a $Q$-cone with $\gamma \in \Gamma$ which has an initial lift.

**Corollary.** If $F: Q \to Q'$ is a $\Gamma_0$-morphism, it is a $\Gamma$-morphism and if $Q$ is $\Gamma_0$-complete, a $\Gamma^0$-morphism.

### C. $\Gamma$-density et $\Gamma$-generation.

Here $Q: K \to E$ is a concrete functor, $H$ a full subcategory of $K$ and $P: H \to E$ is the restriction of $Q$.

**Definition.** $H$ is called $\Gamma$-dense for $Q$ if each object $K$ of $K$ is the initial lift of a $P$-cone $(\Psi, \gamma)$ with $\gamma \in \Gamma$ and $\Psi$ valued in $H$. If $H$ is $\text{Sour } E$-dense, it is said initially dense.

**Proposition 4.** The following conditions are equivalent:

1° $H$ is initially dense for $Q$.

2° For each $K \in K_0$ the source $(k: K \to S \mid S \in H_0)$ is initial for $Q$.

3° Let $K, K'$ be objects of $K$ and $g: Q(K) \to Q(K')$ a $E$-morphism; then we have $g: K \to K'$ in $K$ iff

$$f: K' \to S \text{ in } K \text{ and } S \in H_0 \implies f, g: K \to S \text{ in } K.$$  

*If they are satisfied, the insertion $H \subseteq K$ preserves final lifts.*

The dual notion is «finally dense». It will be used in Section 2 through the third characterization above (introduced in /107/ under the name «$Q$ is $P$-generated»).

**Definition.** We call $\Gamma$-hull (resp. strict $\Gamma$-hull) of $H$ for $Q$ the smallest full subcategory (resp. smallest subcategory) $H'$ of $K$ containing $H$ and $il(\Psi, \gamma)$ for each $Q$-cone $(\Psi, \gamma)$ with $\gamma \in \Gamma$ and $\Psi$ valued in $H'$. If $H' = K$, we say that $Q$ is $\Gamma$-generated (resp. strictly $\Gamma$-generated) by $H$.

If $Q$ is $\Gamma$-complete, so is its restriction to $H'$. If $H$ is $\Gamma$-dense for $Q$, then $Q$ is $\Gamma$-generated by $H$ (but not conversely).
PROPOSITION 5. The $\Gamma$-hull $C$ and the strict $\Gamma$-hull $B$ of $H$ for $Q$ are constructed by transfinite induction; they are in the same universe $U$ as $H$ if so are $\text{Ind } \Gamma$ and each $I \in \text{Ind } \Gamma$.

$\Delta$. $C$ and $B$ are respectively the union of the transfinite increasing sequence $(C_\lambda)_\lambda$ and $(B_\lambda)_\lambda$ defined as follows:

$C_0 = H = B_0$,

$C_\lambda = \bigcup_{\lambda < \alpha} C_\alpha$ and $B_\lambda = \bigcup_{\lambda < \alpha} B_\alpha$ for each limit ordinal $\alpha$,

$C_{\lambda + 1}$ is the full subcategory of $K$ with objects $il(\Psi, \gamma)$ where $(\Psi, \gamma)$ is a $Q$-cone, $\gamma \in \Gamma$ and $\Psi$ valued in $C_\lambda$, 

$B_{\lambda + 1}$ is the subcategory of $K$ generated by all the morphisms $\gamma(l): il(\Psi, \gamma) \to \Psi(l)$ for each $l \in I_\alpha$, 

$g: K_\lambda \to il(\Psi, \gamma)$ whenever $\gamma \circ g: K_\lambda \Rightarrow \Psi_\alpha$ is a cone in $K$, 

where $(\Psi, \gamma)$ is any $Q$-cone with $\gamma \in \Gamma$ and $\Psi: I \to K$ valued in $B_\lambda$.

The construction stops at the first limit ordinal greater than the ordinals of $I$ for each $I \in \text{Ind } \Gamma$.

COROLLARY. If $Q$ is $\Gamma$-generated by $H$, then $H$ is $\Gamma^\circ$-dense for $Q$.

Proof by induction on $C_\lambda$ using the commutativity of initial lifts.

D. $\Gamma$-completions.

DEFINITION. A $\Gamma$-completion of $P: H \to E$ is defined as a concrete $\Gamma$-complete functor $Q: K \to E$ of which $P$ is a full restriction. The $\Gamma$-completion is $\Gamma$-dense (resp. initially dense) if so is $H$ for $Q$; it is (strictly) $\Gamma$-generated if $K$ is the (strict) $\Gamma$-hull of $H$ for $Q$.

An order is defined on the $\Gamma$-completions of $P$ as follows:

$Q \preceq Q'$ (say $Q$ is $\Gamma$-smaller than $Q'$) iff there exists one unique $\Gamma$-morphism $F: Q \to Q'$ extending the identity on $H$.

We are going to construct completions which are optimal for this order.

EXAMPLES. 1. Sour $E$-completions have been considered by several authors, e.g. Herrlich [2,4] under the name: initial completions.

2. If $A$ is a class of morphisms of $E$, the $/A/$-completions of $P$ are called $A$-completions; they are the $A$-spreading functors extending $P$ which are dealt with in /107/.
3. If \( E \) is a complete category and \( \text{Lim } E \) is the class of all its small limit-cones, \( \text{Lim } E \)-completions, just called completions of \( P \), are constructed in Adamek-Koubek [1] and Herrlich [3]. More generally, if \( \mu \) is a partial (multiple) choice of limit-cones on \( E \) and \( \Gamma \) the class of limit-cones distinguished by \( \mu \), we find the \( \mu \)-completions studied in [107].

2. MAC NEILLE COMPLETIONS.

In this section, we construct a \( \Gamma \)-completion of \( P: H \rightarrow E \) which is both finally dense and \( \Gamma \)-generated; such a \( \Gamma \)-completion is called a Mac Neille \( \Gamma \)-completion of \( P \) (by analogy with Herrlich's Mac Neille initial completions, named after the Mac Neille completions of posets).

In [107], Charles constructs the Mac Neille \( A \)-completion of \( P \) for \( A \) a subcategory of \( E \) (Theorem 2, 3) and, using it, the Mac Neille \( \mu \)-completion of \( P \) (Theorem 5, 6), which he calls «smallest prolongations of \( P \».

His method, which generalizes for any class \( \Gamma \) of cones, may be sketched as follows: To \( H \) he adds «formal initial lifts» of cones of \( \Gamma \) and as many morphisms as possible for getting a faithful (non-amnestic) functor in which these formal initial lifts become initial lifts; so \( H \) is \( \Gamma \)-dense and finally dense for the associated concrete functor \( Q: K \rightarrow E \) (this condition entirely characterizes \( Q \)). In the case \( \Gamma = /A/ \), this functor is the Mac Neille \( A \)-completion of \( P \). For \( \mu \)-completions or more generally, the construction has to be transfinitely reiterated, because \( Q \) is not \( \Gamma \)-complete. (In fact, Charles gets \( Q \) as a Mac Neille \( A \)-completion of a certain extension of \( P \).)

Now we remark that \( Q \) is \( \Gamma \)-complete whenever \( \Gamma \) is equal to its commutative hull \( \Gamma^0 \) thanks to the commutativity of initial lifts; in this case, an object of \( K \), which is an equivalence class of \( P \)-sources, may be identified to the union of these \( P \)-sources, hence to a closed source in Herrlich's sense [3]; so, for \( \Gamma = \text{Sour } E \), \( Q \) is exactly the Mac Neille initial completion \( P_4 \) as constructed by Herrlich in [2].

Whence the idea of the following proof: we first construct the Mac Neille \( \Gamma^0 \)-completion of \( P \); the \( \Gamma \)-hull of \( H \) in it then gives the Mac Neille \( \Gamma \)-completion of \( P \) (constructed by induction via Proposition 5).

**Theorem 1.** \( P \) admits a Mac Neille \( \Gamma \)-completion \( P_\Gamma: H_\Gamma \rightarrow E \), which
lives in the universe $\mathbb{U}$ if so do $P$, $\text{Ind} \Gamma$ and each $I \in \text{Ind} \Gamma$.

\[ \Delta. \] Construction of the Mac Neille $\Gamma^{\circ}$-completion $V : M \to E$ of $P$:
If $(\Phi, \gamma)$ is a $P$-source, we denote by $(\Phi, \gamma)^*$ the opposite $P$-sink [3]:

\[(S, f : P(S) \to E \mid S \in H_0, \gamma \circ f : S \Rightarrow \Phi),\]

where $E$ is the vertex of $\gamma$. Let $M_\circ$ be the class of the $P$-sinks of the form $(\Phi, \gamma)^*$ for some $P$-source with $\gamma \in \Gamma^{\circ}$; the vertex of $\gamma$ is denoted by $V(M)$. If $M$ and $M'$ are in $M_\circ$, there'll be a morphism $g : M \to M'$ in $M$ mapped by $V$ on $g$ iff

\[(S, f) \text{ in } M \text{ implies } (S, g.f) \text{ in } M'.\]

This defines the concrete functor $V : M \to E$. We identify $H$ to a full subcategory of $M$ by identifying $S \in H_0$ to the $P$-sink

\[(S, id_{P(S)})^* = (S', h \mid h : S' \to S \text{ in } H).\]

So $H$ becomes finally dense for $Q$; it is also $\Gamma^{\circ}$-dense, because the object $M$ is the initial lift of each $P$-source (considered as a $V$-source!) $(\Phi, \gamma)$ such that $M = (\Phi, \gamma)^*$. Hence $V$ is a Mac Neille $\Gamma^{\circ}$-completion of $P$ if it is $\Gamma^{\circ}$-complete; this is true: let $(\Psi, \theta)$ be a $V$-source with $\theta \in \Gamma^{\circ}$, indexed by $I$; for each $I$ in $I$, we have

\[\Psi(I) = (\Phi_I, \gamma_I)^* = \text{il}(\Phi_I, \gamma_I)\text{ for some } \gamma_I \in \Gamma^{\circ};\]

as $\Gamma^{\circ}$ is commutative $(\Phi_I, \gamma_I)_I \circ (\Psi, \theta)$ is a $P$-source $\sigma$ whose dual $P$-sink $\sigma^*$ is in $M_\circ$, so that $\sigma^* = \text{il}(\sigma) = il(\Psi, \theta)$ (by Proposition 3).

- Let $H^{\Gamma}$ be the $\Gamma$-hull of $H$ for $V$, and $P^{\Gamma} : H^{\Gamma} \to E$ the restriction of $V$. It is $\Gamma$-complete (Corollary, 1), and $H$ is still finally dense, $H^{\Gamma}$ being a full subcategory of $M$. Hence $P^{\Gamma}$ is a Mac Neille $\Gamma$-completion of $P$. If $\text{Ind} \Gamma$ and all $I \in \text{Ind} \Gamma$ are in the universe $\mathbb{U}$, then so does $\text{Ind} \Gamma^{\circ}$ (Proposition 1), which implies $M$ and $H^{\Gamma}$ are also in $\mathbb{U}$. \( \Box \)

**Remark.** $M$ is a full subcategory of the Mac Neille initial completion $P_4$ so that $H^{\Gamma}$ may also be defined as the $\Gamma$-hull of $H$ for $P_4$. However $M$ is in $\mathbb{U}$ while $P_4$ may not; conditions for it to be in $\mathbb{U}$ are given in [3].

The «optimality» of $P^{\Gamma}$ will be deduced from the following proposition, which generalizes Theorems 3 and 6 of /107/ and has a similar proof.
THEOREM 2. Let $Q: K \to E$ be a $\Gamma$-generated $\Gamma$-completion of $P: H \to E$ and $Q': K' \to E$ be a finally dense $\Gamma$-completion of $P': H' \to E$. Let $F: P \to P'$ be a morphism satisfying the «lifting-cones» condition:

If $\Phi: I \to H$ is a functor with $I \in \text{Ind} \Gamma$, each cone in $H'$ with basis $F \Phi$ is the image by $F$ of a cone in $H$ with basis $\Phi$.

Then $F$ extends in a unique $\Gamma$-morphism $F': Q \to Q'$.

COROLLARY 1. The Mac Neille $\Gamma$-completion $P_{\Gamma}$ of $P$ is the $\Gamma'$-smallest finally dense $\Gamma$-completion of $P$ and the $\Gamma$-largest regenerated one; in particular, two Mac Neille completions are isomorphic.

COROLLARY 2. The Mac Neille $\Gamma$-completion $P_{\Gamma}$ of $P$ is fully embedded in any finally dense $\Gamma$-completion $Q'$ of $P$.

Indeed, if $F = \text{Id}_H$ and $Q$ finally dense, $F'$ above is a full embedding.

REMARK. Corollary 2 says that $P_{\Gamma}$ is also the smallest finally dense $\Gamma$-completion of $P$ for the preorder on completions:

$Q < Q'$ iff there exists a full embedding $Q \to Q'$ (not a $\Gamma$-morphism!) extending the identity on $H$.

For this preorder, Herrlich proves that $P_{\Gamma}$ is in fact the smallest initial completion of $P$; this stronger result comes from the duality Theorem for initially complete functors, which has no analogon for a general $\Gamma$.

As in [107], Theorem 2 is easily adapted to characterize the strict $\Gamma$-hull $H_{\Gamma}$ of $H$ for $V$ (or for $P_{\Gamma}$). We say that a $\Gamma$-completion $Q: K \to E$ of $P$ is weakly dense if $K = K'$ whenever:

for each $S \in H_0$, we have: $g: S \to K$ iff $g: S \to K'$ in $K$.

THEOREM 3. The restriction $P_{\Gamma}': H_{\Gamma} \to E$ of $P_{\Gamma}$ is the unique (up to isomorphism) $\Gamma$-completion of $P$ which is both weakly dense and strictly $\Gamma$-generated; it is the $\Gamma$-smallest weakly dense $\Gamma$-completion and its $\Gamma$-
largest strictly $\Gamma$-generated one. \( \nabla \)


In this section $\Delta$ denotes a given class of initial cones in $H$ for $P : H \to E$ such that $P \delta \in \Gamma$ for each $\delta \in \Delta$. Let $\Delta_0 = \{ \delta_0 \mid \delta \in \Delta \}$; by Proposition 1, $\Delta_0$ is a class of initial sources for $P$.

Definition. A $\Gamma$-completion $Q : K \to E$ of $P$ is called a $(\Delta, \Gamma)$-completion of $P$ if the insertion $H \hookrightarrow K$ sends (all $\delta$ in) $\Delta$ on initial cones. It is called a universal $(\Delta, \Gamma)$-completion if it also satisfies:

Let $Q' : K' \to E$ be a $\Gamma$-complete concrete functor and $F : P \to Q'$ be a morphism sending $\Delta$ on initial cones; then there exists one unique $\Gamma$-morphism $F' : Q \to Q'$ extending $F$.

The universal $(\Delta, \Gamma)$-completion is unique (up to isomorphism) if it exists, and it is strictly $\Gamma$-generated.

Examples. If $\Delta = \emptyset$, a universal $(\Delta, \Gamma)$-completion is called a free $\Gamma$-completion. If $\Delta$ is the class of all initial cones $\delta$ in $H$ with $P \delta \in \Gamma$, a universal $(\Delta, \Gamma)$-completion is just called a universal $\Gamma$-completion. The free and universal $\Gamma$-completions are proved to exist in /107/ for $\Gamma$ associated to a subcategory $A$ of $E$ or to a partial choice $\mu$ of limits (Theorem 10), but no explicit construction is given in this last case. Herrlich describes the free initial completion $P_2$ and the universal initial completion $P_3$ in [1] and, in [4] the universal completion $P^*$ (for $\Gamma = \Lim E$), whose objects are the «complete sources». Adapting his method as in Section 2, we'll obtain the universal $(\Delta_0, \Gamma^0)$-completion $U$ of $P$, with objects the $\Delta$-complete $P$-sources; the $\Gamma$-hull of $H$ for $U$ is the universal $(\Delta, \Gamma)$-completion of $P$.

Definition. A $P$-source $\sigma = (S_I, f_I \mid I \in I)$ is said $\Delta$-complete if it contains the $P$-morphisms

- (a) $(S, h, f_I)$ for each $h : S_I \to S$ in $H$,
- (b) $(S', g)$ if there exists $(d_j : S' \to S'_j \mid J \in J)$ in $\Delta_0$ with

$(S'_j, d_j, g)$ in $\sigma$ for each $J \in J$.

(Intuitively, $\sigma$ is closed under left composition by $H$ and factors through
PROPOSITION 6. Each $P$-source $\sigma = (\Phi, \gamma)$ is included in a smallest $\Delta$-complete $P$-source, denoted by $\Delta\sigma = (\Delta\Phi, \Delta\gamma)$, which is constructed by transfinite induction. We have $il\sigma = il\Delta\sigma$ as soon as one of them is defined.

The $P$-source $\Delta\sigma$ is the union of the transfinite sequence $\sigma_{\lambda}$ where $\sigma_0 = \sigma$, $\sigma_\alpha = \bigcup_{\lambda < \alpha} \sigma_\lambda$ for a limit ordinal $\alpha$, and $\sigma_{\lambda+1}$ is deduced from $\sigma_\lambda$ by adding elements of the form (a) and (b) above. For the last assertion, we prove by induction on $\sigma_\lambda$ that, if the $P$-source $\sigma \circ g = (\Phi, \gamma \circ g)$ lifts into a source $\gamma \circ g : S \Rightarrow \Phi$ in $H$, then the $P$-source $\Delta \sigma \circ g$ lifts into a source with basis $\Delta\Phi$ in $H$.

THEOREM 4. $P$ has a universal $(\Delta, \Gamma)$-completion $U_{\Gamma} : L_{\Gamma} \to E$ which is $\Gamma$-generated, hence $\Gamma^\circ$-dense. It lives in the universe $\mathcal{U}$ if so do $\text{Ind}_{\Gamma}$ and each of its elements.

1. Construction of a $(\Delta_{\circ}, \Gamma^\circ)$-completion $U : L \to E$ of $P$. Let $L_{\circ}$ be the class formed by the $\Delta$-complete $P$-sources $L$ of the form (Proposition 6) $\Delta(\Phi, \gamma)$ for some $\gamma \in \Gamma^\circ$; let $U(L)$ be the vertex of $\gamma$. If $L'$ is also in $L_{\circ}$, then $g : L' \to L$ is a morphism in $L$ mapped by $U$ on $g$ iff $(\Phi, \gamma \circ g)$ is included in $L'$ (which implies $L \circ g \subset L'$). We identify $H$ to a full subcategory of $L$ by identifying the object $S$ to

$$\Delta(S, id_{P(S)}) = (S', f \mid f : S \to S' \text{ in } H).$$

As we have $h : L \to S$ in $L$ iff $(S, h)$ is in $L$, it follows that $L$ is the initial lift of $(\Phi, \gamma)$ considered as a $U$-source, and that $\delta_{\circ} \in \Delta_{\circ}$ remains an initial source for $U$. The fact that $U$ is $\Gamma^\circ$-complete is proved as in Theorem 1, thanks to the commutativity of initial lifts and of $\Gamma^\circ$.

2. Universality of $U$. Let $Q : \mathcal{K} \to E$ be a $\Gamma^\circ$-complete functor, and $F : P \to Q$ a morphism sending $\Delta_{\circ}$ on initial sources. If there is a $\Gamma^\circ$-morphism $F' : U \to Q$ extending $F$, it maps $L$ on $il(F\Phi, \gamma)$ and $g : L' \to L$ on $g : F'(L') \to F'(L)$. So we have just to prove that this $F'$ is well-defined i.e., that

$$il(F\Phi', \gamma') = il(F\Phi, \gamma) \quad \text{if} \quad \Delta(\Phi', \gamma') = L = \Delta(\Phi, \gamma).$$
Indeed \((F\Phi, y)\) generates a \(F\Delta\)-complete \(Q\)-source \(\sigma\) and \(il\sigma = il(F\Phi, y)\) (Proposition 6). From the construction of \(\Delta(\Phi, y) = (\Delta\Phi, \Delta y)\), we deduce by induction that \((F\Delta\Phi, \Delta y) \triangleright (F\Phi', y')\) is included in \(\Delta(F\Phi, y) = \sigma\). Therefore

\[\Delta(F\Phi', y') = \sigma \quad \text{and} \quad il(F\Phi', y') = il\sigma = il(F\Phi, y).\]

3. Universal \((\Delta, \Gamma)\)-completion of \(P\). Let \(U_\Gamma : L_\Gamma \to E\) be the restriction of \(U\) to the \(\Gamma\)-hull of \(H\) for \(U\); it is a \((\Delta, \Gamma)\)-completion of \(P\). To prove the universality, let \(Q' : K' \to E\) be a \(\Gamma\)-complete functor, and \(G : P \to Q'\) a morphism sending \(\Delta\) on initial cones. We consider the universal \(\Gamma^o\)-completion \(U' : L' \to E\) of \(Q'\); as \(Q' \subseteq U'\) is a \(\Gamma^o\)-morphism, hence a \(\Gamma\)-morphism, \(G : P \to U'\) still sends \(\Delta\) on initial cones and, by Part 2, it extends into a unique \(\Gamma^o\)-morphism \(G' : U \to U'\). If \(G'\) maps \(L_\Gamma\) into the full subcategory \(K'\) of \(L'\) its restriction \(G'' : U_\Gamma \to Q'\) will be the unique \(\Gamma\)-morphism extending \(G\). This is proved by induction on \(\lambda_\lambda\), where \(L_\Gamma = \bigcup \lambda \subseteq \lambda\) (Proposition 5): suppose \(G'\) maps \(C_\lambda\) into \(K'\); if \(L\) is an object of \(C_{\lambda+1}\), we have \(L = il(\Psi, y')\), where \(y' \in \Gamma\) and \(\Psi\) valued in \(C_\lambda\); as \(G'\Psi\) is valued in \(K'\) and \(Q'\) is \(\Gamma\)-complete, the \(Q'\)-cone \((G'\Psi, y')\) has an initial lift for \(Q'\), which remains an initial lift for the universal \(\Gamma^o\)-completion \(U'\), hence is equal to \(G'(L) = il(G'\Psi, y')\). It follows that \(G'\) maps \(C_{\lambda+1}\) into the full subcategory \(K'\). \(\Box\)

REMARKS. 1. Suppose \(\Gamma = \text{Lim} E\). Then the universal completion \(U_\Gamma\) of \(P\) is \((\Gamma^o)\)-dense (not only \(\Gamma\)-generated); this has been proved by Adamek-Koubek [1] via a construction which is transfinite only for morphisms, and by Herrlich [4] thanks to his one-step construction. His proof rests on the two facts:

A functor is complete as soon as it lifts products and equalizers;

Let \(\sigma = \Delta(\Phi, y)\) with \(y \in \text{Lim} E\); any \(P\)-source included in \(\sigma\) is also included in a \(P\)-source \((\Phi, y) \subseteq \sigma\) with \(y\) a limit-cone [4]; it is easily adapted, whatever be \(\Delta\), to prove that \(L_\Gamma\) reduces to the full subcategory \(C_1\) of \(L\) (Part 3 above), whence:

COROLLARY. The universal \((\Delta, \text{Lim} E)\)-completion of \(P\) is \((\text{Lim} E^o)\)-dense.
2. The free initial completion of $P$ is the largest initial completion, while its universal one is the largest preserving initial lifts initial completion [2]. This maximality property is no more valid for a general class $\Gamma$; we only prove as in Theorem 4 the

**Proposition 7.** Let $Q: K \to E$ and $Q': K' \to E$ be $(\Delta_0, \Gamma^o)$-completions of $P$. If $Q$ satisfies

1) For each $K$ in $K_0$ there exists a $P$-source $(\Phi, \gamma)$ with $\gamma \in \Gamma^o$ such that $\Delta(\Phi, \gamma) = (S', h \mid h: K \to S' \text{ in } K)$ and $K = il(\Phi, \gamma)$, then there exists a morphism $Q \to Q'$ extending the identity on $H$.

**Corollary.** The universal $(\Delta_0, \Gamma^o)$-completion $U$ of $P$ is the largest $(\Delta_0, \Gamma^o)$-completion which satisfies (1).

Another maximality property of the universal $(\Delta, \Gamma)$-completion is given in Comment 91-1 [0], Part IV-1.

3. The problem of «lifting singleton sources» may be translated in the world of internal functors in a category, leading to universal internal $(\Delta, / A/)$-completions; cf. /95, 96/ and Synopsis no 5 [0], Part III-2.

4. Many authors consider only concrete functors which are transportable. The preceding results are easily adapted to this case.
REFERENCES.


3. H. HERRLICH, Initial and final completions, Lecture Notes in Math., Springer 719 (1979), 137-149. This paper contains an important bibliography.


In particular, the often cited

// 107// Prolongements universels d'un foncteur par adjonction de limites, Dissertations Math. LXIV, Varsovie (1969), 1-72; reprinted in [0], Part IV-1.