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On the preservation of homotopy invariance by Kan extensions


<http://www.numdam.org/item?id=CTGDC_1981__22_3_329_0>
1. Introduction.

In the literature (e.g. in [1, 2, 3, 4, 6, 7]) one considers Kan extensions of functors (usually homology or cohomology functors) defined on a category $\mathcal{F}_0$ of spaces and homotopy classes of maps to another such category $\mathcal{F}_1$, containing $\mathcal{F}_0$ as a full subcategory. The following question arises: (in rough terms) does it matter whether one takes the Kan extension at the level of homotopy classes or at the level of maps? In order to lay out the situation we consider the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{E} & \mathcal{F}_1 \\
H_0 \downarrow & & \downarrow H_1 \\
\mathcal{E}_0 & \xrightarrow{\mathcal{E}} & \mathcal{E}_1
\end{array}
\]

where $\mathcal{F}_0$ and $\mathcal{F}_1$ are full subcategories of the category of spaces and continuous maps, $E$ is a full embedding and $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}$ are their quotients modulo homotopy. $H_1$ and $H_0$ are the quotient functors and $F$ is any homotopy invariant functor. Throughout the paper we use the notations $G^K$, $KG$ for the right respectively the left Kan extension of $G$ along $K$. As we take Kan extensions along full embeddings we may, and do, always choose them so that the universal transformation is the identity. Thus, in more precise terms, our question is:

When is $\mathcal{E}^E H_1 = F^E$?

It turns out that this is the case precisely when $F^E$ exists and is homotopy invariant (Theorem 3.4). By duality the same holds when $F^E$, $\mathcal{E}^E$ are replaced by the left Kan extensions $E^F$, $\mathcal{E}^F$, or when $F$ is a cofunctor.
view of this we investigate when the Kan extension of a homotopy invariant functor is homotopy invariant.

We shall always refer to the situation of diagram (1.1). We assume that $\mathcal{K}$ is sufficiently complete or cocomplete in order to admit the limits or colimits defining the pointwise right or left Kan extensions which we consider.

In Section 2 we treat the case where $\mathcal{J}_0$ contains the cofibration $X \to Zf$ for any map $f: X \to EY$ in $\mathcal{J}_1$ ($Zf$ denotes the mapping cylinder of $f$) and its dual, that is, the case where $\mathcal{J}_0$ contains the mapping path fibration $Y^I \to X$ for any map $EY \to X$. In the first case $F^E$ is homotopy invariant and in the second case $E^F$ is. Both conditions are clearly satisfied when $\mathcal{J}_0$ contains entire homotopy types, as is the case for categories which are admissible in the sense of [1, 6]. Section 3 contains some categorical facts about Kan extensions, which shall be used in the sequel. Although what is said in this section applies to more general situations, for the sake of shortness and ease of reading, we adhere to the situation of diagram (1.1). In Section 4 we show that if $\mathcal{J}_0$ contains $Y^I$ whenever it contains $Y$, then $F^E$ is homotopy invariant, and, dually, if it contains $Y \times I$ together with $Y$, then $E^F$ is homotopy invariant. The condition that $\mathcal{J}_0$ contain $Y \times I$ along with $Y$ is rather harmless when $\mathcal{J}_0$ is a category of polyhedra or of CW-complexes as $\times I$ preserves them and also preserves finiteness, local finiteness and finite dimension, thus left Kan extensions of homotopy invariant functors from any reasonable category of polyhedra or CW-complexes are homotopy invariant. Unfortunately the dual is not adequate for such categories; we shall consider right Kan extensions from categories of polyhedra in another paper.

2. THE CASE WHERE $\mathcal{J}_0$ CONTAINS $Zf$ FOR ALL $f: X \to EY$.

**Theorem 2.1.** If the category $\mathcal{J}_0$ contains the mapping cylinder $Zf$ for any map $f: X \to EY$ in $\mathcal{J}_1$, then:

(i) $F^E$ exists iff $E^F$ exists.

(ii) $\overline{F}^E H_1 = F^E$, in particular $F^E$ is homotopy invariant.
PROOF. Let $H_X : (X/E) \rightarrow (X/\tilde{E})$ be the functor which takes the object $(f, Y)$ of $(X/E)$, that is a map $f : X \rightarrow EY$, to its homotopy class $H_1 f$ and a map $g$ of $(X/E)$ to its homotopy class $H_1 g$. Then the diagram

\[
\begin{array}{ccc}
(X/E) & \xrightarrow{Q} & \mathcal{F}_0 \\
\downarrow H_X & & \downarrow H_0 \\
(X/\tilde{E}) & \xrightarrow{Q} & \tilde{\mathcal{F}}_0 \\
\end{array}
\]

where $Q$ denotes the usual forgetful functor, is commutative. Hence it suffices to show that the functor $H_X$ is initial for all $X$ in $\mathcal{F}_1$. Now, every object in $(X/\tilde{E})$ is clearly of the form $H_X(f, Y)$ and the category $\mathcal{X} = (H_X : H_X(f, Y))$ is nonempty. In order to show that it is connected we consider two objects

\[
H_X(f_1, Y_1) \quad H_X(f_2, Y_2)
\]

in $\mathcal{X}$, i.e., a homotopy commutative diagram

\[
\begin{array}{ccc}
EY_1 & \xrightarrow{v_1} & EY \\
\downarrow f_1 & & \downarrow f \\
X & \xrightarrow{v_2} & EY_2
\end{array}
\]

in $\mathcal{F}_1$. We replace $f_1, f_2$ by cofibrations $c_1, c_2$ respectively and obtain a diagram

\[
\begin{array}{ccc}
EY_1 & \xrightarrow{r_1} & E\text{Z}f_1 \\
\downarrow f_1 & & \downarrow c_1 \\
X & \xrightarrow{v_1} & EY \\
\downarrow f_2 & & \downarrow c_2 \\
& \xrightarrow{r_2} & E\text{Z}f_2 \\
& \downarrow f_2 & \\
& EY_2 & \\
\end{array}
\]

where $r_1$ and $r_2$ are homotopy equivalences and where $v_i, r_i, c_i - f$. By the homotopy extension property there exist maps $u_i : Zf_i \rightarrow Y_i$ with $u_i c_i = f$. Thus we obtain a strictly commutative diagram in $\mathcal{F}_1$.
A dual procedure, replacing the appropriate maps by their mapping path fibrations, gives the proof of:

**Theorem 2.2.** If the category $\mathcal{C}_0$ contains the mapping path fibration for any map $EY \to X$ in $\mathcal{C}_1$, then:

- (i) $EF$ exists iff $E'F$ exists.
- (ii) $EF \simeq E \simeq F$, in particular $E \simeq F$ is homotopy invariant.

When $\mathcal{C}_0$ contains entire homotopy types, then the hypotheses of both theorems are satisfied and we immediately have

**Corollary 2.3.** If $\mathcal{C}_0$ is closed under homotopy types, then the conclusions of both Theorems, 2.1 and 2.2, hold.

### 3. Some Categorical Facts.

As the Kan extension of $F$ along a fully faithful functor $E$ can be chosen so that $F^E E = F$, there is a very explicit form of the limiting cone defining $F^E X$.

**Proposition 3.1.** For any object $X$ in $\mathcal{C}_1$ the family

$$\{ F^E f \mid (f, Y) \in (X \ast E) \}$$

is a universal cone.
PROOF. Let $X$ be an object in $\mathcal{J}$ and let
$$\{ \eta(g, Z) : FEX \to FQ(g, Z) \},$$
where $Q : (X \downarrow E) \to \mathcal{J}$ is the forgetful functor, be a universal cone. For a map $f : X \to EY$, the map $Ff$ renders the diagram
$$\begin{array}{ccc}
FEX & \xrightarrow{Ff} & FEY = FY \\
\eta(hf, Z) & \nearrow & \\
FZ & & \lambda(h, Z)
\end{array}$$
commutative for any $(h, Z)$ in $(EY \downarrow E)$, where $\lambda$ denotes the universal cone defining $FEY$. In particular for $(1, Y)$ we have that $\lambda(1, Y) = 1$, thus $\eta(f, Y) = Ff$.

The proposition above leads to a criterion for $FE$ to be homotopy invariant. Thus

**PROPOSITION 3.2.** If $FE$ takes homotopic maps $f, g : X \to EY$ to the same map, then $FE$ is homotopy invariant.

**PROOF.** Let $h, i : X_1 \to X_2$ be homotopic maps in $\mathcal{J}$. Using the representation given in Proposition 3.1 of the universal cone defining $FE$, the map $FEh$ is the unique one for which $FEx \cdot FEh = FEth$, while $FEi$ is the unique one for which $FEx \cdot FEi = FEti$, for all objects $(t, Y)$ in $(X_2 \downarrow E)$. As $th, ti : X_1 \to EY$ are homotopic, one has, by hypothesis, that $FEx \cdot FEh = FEti$, hence $FEh = FEi$ by universality.

The next proposition points out a nice feature of quotient functors.

**PROPOSITION 3.3.** If $G : \mathcal{J} \to K$ is any functor, then $G$ is the right and the left pointwise Kan extension of $GH_1$ along $H_1$, with the identity as universal transformation. The same holds, of course, for $H_0$.

**PROOF.** As $H_1$ is full and onto objects it induces a bijection
$$\text{Nat}[G, G'] \to \text{Nat}[GH_1, G'H_1]$$
for all functors $G, G' : \mathcal{J} \to K$. Hence $G$ is not only a pointwise, but even
an absolute Kan extension.

We now proceed to show that precisely when $F^E$ is homotopy invariant, Kan extending at the map or at the homotopy class level give the same. More precisely:

**Theorem 3.4.** In the situation of diagram (1.1) the following are equivalent:

(i) $F^E$ exists and is homotopy invariant.

(ii) $F^E$ exists and $F^E H_1 = F^E$.

**Proof.** (ii) $\Rightarrow$ (i) is trivial. (i) $\Rightarrow$ (ii): As $F^E$ is homotopy invariant there is a unique functor $\bar{F} : \mathcal{T}_1 \to K$ with $\bar{F} H_1 = F^E$. Then by Proposition 3.3,

$$\bar{F} = (F^E)^{H_1}$$

by Lemma 1.2 (i) of [5],

$$\bar{F} = F^{H_1 E}$$

as (1.1) is commutative.

By Proposition 3.3, $\bar{F} = F^{H_0}$ and by Lemma 1.2 (ii) of [5] there is a unique $\mu : F^E \to \bar{F}$ with $\mu H_0 = 1$, thus $\mu = 1$ as $H_0$ is onto objects, and $\bar{F} = F^E$.

4. **The Case Where $\mathcal{T}_0$ is Closed Under (-)$^I$ or $\times I$.

Suppose that for every $Y$ in $\mathcal{T}_0$ also $Y^I$ lies in $\mathcal{T}_0$. Let $f, g : X \to E Y$ be homotopic maps. There is a map $k : X \to Y^I$ such that

$$d_0 k = f \quad \text{and} \quad d_1 k = g,$$

where $d_0$ and $d_1$ are the evaluations at 0 and 1 respectively. As $d_0$ and $d_1$ are homotopic and lie in $\mathcal{T}_0$ one has, for any functor $\bar{F} : \mathcal{T}_1 \to K$ with $\bar{F} E$ homotopy invariant, that $\bar{F} f = \bar{F} g$. This holds in particular for the right Kan extension $F^E$ of $F$. Invoking Proposition 3.2 we then have:

**Theorem 4.1.** If $\mathcal{T}_0$ contains $X^I$ whenever it contains $X$, then $F^E$ is homotopy invariant.

Dually we have

**Theorem 4.2.** If $\mathcal{T}_0$ contains $X \times I$ whenever it contains $X$, then $E F$ is homotopy invariant.
We conclude by observing that, if $\mathcal{F}_0$ is closed under homotopy types, one obtains the conclusion (ii) of the theorems in Section 2 via the theorems in this section and Theorem 3.4 (and its dual). Nevertheless the theorems in Section 2 lead a separate life from those in Section 4 as one sees by taking $\mathcal{F}_0$ to be the category of polyhedra, or CW-complexes or of compact spaces, which are closed under $\times I$ but not under homotopy types.
REFERENCES.


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