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Structures defined by finite limits in the enriched context, I

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INTRODUCTION

Ehresmann's notion of the category of algebras defined by a (projective) sketch, along with the elementary properties of such categories and the basic relations between such categories, are generalized to the case of enriched categories in Chapter 6 of the author's forthcoming book [*]. That generalization, more precisely, is to the case of \( \mathcal{V} \)-categories where \( \mathcal{V} \) is a symmetric monoidal closed category whose underlying ordinary category \( \mathcal{V}_0 \) is locally small, complete, and cocomplete, and where, moreover, \( \mathcal{V} \) is locally bounded in the sense of [*] - as most base-categories of interest seem to be. In the present article we set about giving more precise results when the sketch, or the corresponding theory, is finitary: generalizing the results of Gabriel-Ulmer [7] on locally finitely presentable (ordinary) categories.

It turns out that finitariness has a good definition in the enriched case, leading to results analogous to the classical ones, only when the base category \( \mathcal{V} \) itself is suitably special. The analogy works most perfectly when the ordinary category \( \mathcal{V}_0 \) is locally finitely presentable in the classical sense, and when the finitely-presentable objects of \( \mathcal{V}_0 \) are closed under the monoidal structure of \( \mathcal{V} \); that is, when the unit-object \( I \) of \( \mathcal{V} \) is finitely presentable in \( \mathcal{V}_0 \), and when the object \( x \otimes y \) is so whenever \( x \) and \( y \) are; in which case we say that \( \mathcal{V} \) is locally finitely presentable as a closed category.

For such a \( \mathcal{V} \), a \( \mathcal{V} \)-category \( \mathcal{A} \) is the category of algebras for a
finitary $\mathcal{C}$-sketch precisely when $\mathcal{A}$ is *locally finitely presentable as a $\mathcal{C}$-category*; and the corresponding finitary $\mathcal{C}$-theory $\mathcal{T}$ is $(\mathcal{A}^\text{op})^\text{op}$, where $\mathcal{A}$ is the full subcategory of $\mathcal{A}$ determined by the finitely-presentable objects - whereupon $\mathcal{A} = \mathcal{T} \text{-Alg}$. A functor $T : \mathcal{T}' \text{-Alg} \to \mathcal{T} \text{-Alg}$ is induced by a morphism $M : \mathcal{T} \to \mathcal{T}'$ of theories precisely when $T$ is finitary and admits a left adjoint $S$; whereupon $M^\text{op}$ is the restriction of $S$ to the finitely-presentable objects. Here a *finitary $\mathcal{C}$-theory* is a small $\mathcal{C}$-category $\mathcal{T}$ that is finitely-complete in the appropriate $\mathcal{C}$-enriched sense; a morphism of theories $\mathcal{T} \to \mathcal{T}'$ and a $\mathcal{T}$-algebra $\mathcal{T} \to \mathcal{O}$ are $\mathcal{C}$-functors which are left exact, in that they preserve the appropriate finite $\mathcal{C}$-limits; and a $\mathcal{C}$-functor is finitary if it preserves (classical, conical) filtered colimits.

The underlying ordinary category $\mathcal{I}_0$ of a finitary $\mathcal{C}$-theory $\mathcal{I}$ is itself finitely complete in the classical sense; and $\mathcal{I}_0 \text{-Alg}$ is precisely the underlying category $(\mathcal{I} \text{-Alg})_0$ of $\mathcal{I} \text{-Alg}$. A given classical finitary theory may or may not be of the form $\mathcal{I}_0$ for some finitary $\mathcal{C}$-theory $\mathcal{I}$; and when it is so, $\mathcal{I}$ need not be unique.

Examples of such base-categories $\mathcal{O}$, other than $\mathsf{Set}$, include $R$-modules, graded $R$-modules, and differential graded $R$-modules, for any commutative ring $R$; and the categories $\mathsf{Cat}$, $\mathsf{Spd}$, $\mathsf{Ord}$ of (small) categories, groupoids, and ordered sets.

Since many of the arguments below are direct generalizations of what is true when $\mathcal{O} = \mathsf{Set}$, it was very little extra trouble to make the article self-contained, assuming no prior knowledge of that classical case. We have therefore done this, but kept these references to the classical case brief, since so much detail is available in [7] and [4].

A second part of this article, to appear later, will study the special case of those finitary $\mathcal{C}$-theories which arise from finitary $\mathcal{C}$-monads; and will use this study to give a syntactic description of finitary $\mathcal{C}$-theories, as is done for the classical case in the thesis [3] of M. Coste or the book [13] of Makkai- Reyes.
Categories with essentially-algebraic extra structure are algebras for a finitary \( \mathcal{V} \)-theory \( \mathcal{I} \), where \( \mathcal{V} \) is usually \( \mathcal{S} \mathrm{pd} \) but sometimes \( \mathcal{C} \text{at} \). The morphisms in \( \mathcal{I} \text{-Alg} \) are those that preserve the structure strictly; and it is more natural to study a bigger category \( \mathcal{I} \text{-Alg}^* \), with the same objects, but with morphisms that preserve the structure only to within isomorphism. In a forthcoming paper [10], the author will examine the relation between \( \mathcal{I} \text{-Alg} \) and \( \mathcal{I} \text{-Alg}^* \), and combine this with the results of the present article, on the left adjoint of the \( \mathcal{V} \)-functor \( \mathcal{I}' \text{-Alg} \to \mathcal{I} \text{-Alg} \) induced by a map \( \mathcal{I} \to \mathcal{I}' \) of theories, to describe the left bi-adjoint of the corresponding functor \( \mathcal{I}' \text{-Alg}^* \to \mathcal{I} \text{-Alg}^* \).

0. REVISION OF NOMENCLATURE

Our general reference for enriched category theory is [*]. We suppose that our chosen base-category \( \mathcal{V} \) is a (symmetric monoidal) closed category whose underlying category \( \mathcal{V}_0 \) is locally small, complete, and cocomplete. An important special case is that where \( \mathcal{V} \) is the cartesian-closed category \( \mathcal{S} \text{et} \) of small sets; note that a \( \mathcal{S} \text{et} \)-category is a locally-small ordinary category. The tensor product, unit object, and internal hom of \( \mathcal{V} \) are \( X \otimes Y \), \( I \), and \([X,Y] \); and we use \( V: \mathcal{V}_0 \to \mathcal{S} \text{et} \) for the canonical representable functor \( \mathcal{V}_0 (I, -) \).

Recall from Section 1.3 of [*] the careful distinction we make between a \( \mathcal{V} \)-category \( \mathcal{A} \) and the underlying ordinary category \( \mathcal{A}_0 \) which has the same objects but has \( \mathcal{A}_0 (A,B) = V \mathcal{A} (A,B) \); and between a \( \mathcal{V} \)-functor \( T: \mathcal{A} \to \mathcal{B} \) and its underlying functor \( T_0 : \mathcal{A}_0 \to \mathcal{B}_0 \). It is precisely this notational distinction that allows us the simplification of writing "category" for "\( \mathcal{V} \)-category" and "functor" for "\( \mathcal{V} \)-functor" when the context makes clear which base-category is meant.

Recall too that a \( \mathcal{V} \)-category is small if the set of isomorphism classes of its objects is small. Then an ordinary category is small if it is a small \( \mathcal{S} \text{et} \)-category. We often use lower-case letters for the objects of small categories.

Finally, as is appropriate for enriched categories, we use the un-
qualified word limit to mean indexed limit in the sense of [*] Ch. 3, and similarly for colimit - except where 0 = \( S_{\mathsf{et}} \) and the context makes clear that we are referring to classical conical limits.

1. FILTERED COLIMITS AND FINITARY FUNCTORS

(1.1) By a filtered colimit in the \( \mathcal{V} \)-category \( \mathcal{A} \) we mean the conical colimit in \( \mathcal{A} \) (see [*] Section 3.8) of an ordinary functor \( P : \mathcal{L} \to \mathcal{A}_0 \), where the ordinary category \( \mathcal{L} \) is small and filtered. Since \( \operatorname{colim} P \), if it exists in \( \mathcal{A} \), is a fortiori the colimit of \( P \) in \( \mathcal{A}_0 \), it follows that if \( \mathcal{A} \) admits all filtered colimits, so does \( \mathcal{A}_0 \).

(1.2) The notion of a filtered category can be found in [*] Section 4.6, where it is shown that filtered colimits in \( \mathsf{Set} \) commute with finite (conical) limits.

(1.3) We call a \( \mathcal{V} \)-functor \( T : \mathcal{A} \to \mathcal{B} \) finitary if \( \mathcal{A} \) admits filtered colimits and \( T \) preserves them. Clearly \( T \) is finitary if and only if \( \mathcal{T}_0 \) is so, provided that \( \mathcal{A} \) and \( \mathcal{B} \) admit filtered colimits. Of course a composite \( TS \) is finitary if \( T \) and \( S \) are; while, since a conservative (= isomorphism-reflecting) functor reflects such colimits as it preserves ([*] Section 3.6), \( S \) is finitary if \( TS \) is finitary and \( T \) is finitary and conservative.

(1.4) Let the fully-faithful \( T : \mathcal{A} \to \mathcal{B} \) have the left adjoint \( S \), where \( \mathcal{B} \) (and hence \( \mathcal{A} \)) admits filtered colimits. Then \( T \) is finitary if and only if \( TS \) is so.

**Proof.** For the non-trivial part let \( \operatorname{colim} P \) be a filtered colimit in \( \mathcal{A} \). We have isomorphisms

\[
\operatorname{colim} TP = \operatorname{colim} TSP = T \operatorname{colim} TP = T \operatorname{colim} STP = T \operatorname{colim} P,
\]

the first because \( ST = 1 \), the second because \( TS \) is finitary, the third because \( S \) is left adjoint, and the fourth because \( ST = 1 \); and the composite isomorphism is easily verified to be the canonical map \( \operatorname{colim} TP \to T \operatorname{colim} P \).

2. FINITELY-PRESENTABLE OBJECTS

(2.1) We shall call the object \( G \) of the \( \mathcal{V} \)-category \( \mathcal{A} \) finitely-presentable (or f.p.) if the representable \( \mathcal{V} \)-functor \( \mathcal{A}(G, -) : \mathcal{A} \to \mathcal{V} \) is finitary. We
write $\mathcal{A}_f$ for the full subcategory of $\mathcal{A}$ given by the f.p. objects.

(2.2) EXAMPLES. The unit object $I$ is f.p. in the $\mathcal{C}$-category $\mathcal{C}$, since

$$\mathcal{V}(I, -) = [I, -] \cong I: \mathcal{C} \to \mathcal{C}.$$ 

If $x$ and $y$ are f.p. objects in $\mathcal{V}$, so is $x \otimes y$, since

$$[x \otimes y, -] \cong [x, [y, -]].$$

More generally, if $x \in \mathcal{C}_f$ and $G \in \mathcal{A}_f$, the tensor product $x \otimes G$ is f.p. if it exists; for $\mathcal{A}(x \otimes G, -) \cong [x, \mathcal{A}(G, -)]$. For any small $\mathcal{C}$-category $\mathcal{I}$, the representable $Yt = \mathcal{I}(t, -)$ is f.p. in $[\mathcal{I}, \mathcal{C}]$, since $[\mathcal{I}, \mathcal{C}](Yt, -)$ is isomorphic by Yoneda to the evaluation $E_t: [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$, which preserves all small colimits by [*] Section 3.6. More generally, if $G \in \mathcal{A}_f$ and if the tensor product $\mathcal{I}(t, s) \otimes G$ exists in $\mathcal{A}$ for all $t, s \in \mathcal{I}$, the object $Yt \otimes G = \mathcal{I}(t, -) \otimes G$ is f.p. in $[\mathcal{I}, \mathcal{A}]$; for now Yoneda gives

$$[\mathcal{I}, \mathcal{A}](Yt \otimes G, -) \cong \mathcal{A}(G, E_t -).$$

In fact the finite presentability of $Yt \otimes G$ could also be deduced from that of $Yt$ and

(2.4) PROPOSITION. If $S \dashv T: \mathcal{A} \to \mathcal{B}$ where $T$ is finitary, we have $S(\mathcal{B}_f) \subseteq \mathcal{A}_f$.

PROOF. For $G \in \mathcal{B}_f$ we have $\mathcal{A}(SG, -) \cong \mathcal{B}(G, T -)$, the composite of the finitary functors $T$ and $\mathcal{B}(G, -)$.

(2.5) It follows from (1.2) that, when $\mathcal{C} = \text{Set}$ and $\mathcal{A}$ admits filtered colimits, the category $\mathcal{A}_f$ is closed in $\mathcal{A}$ under finite (conical) limits.

Before we can exhibit other "suitable" $\mathcal{C}$ for which an analogue of this is true, we need more examples of finite presentability in the case $\mathcal{C} = \text{Set}$: to which we now restrict ourselves up to the end of (2.9) below.

(2.6) $\text{Set}_f$ is the category of finite sets.

PROOF. $I \in \text{Set}_f$ being f.p. by (2.2), so is every finite set $n$ by (2.5). For the converse, since every set $X$ is the filtered colimit of the diagram $n \to X$ of its finite subsets, we have $\text{Set}_f(X, X) \cong \text{colim} \text{Set}_f(X, n)$, if $X$ is f.p.; but then $I: X \to X$ factorizes as $X \to n \to X$ for some finite $n$, whence $X$ is finite.
(2.7) Let $\mathcal{A}$ be the category of algebras for a one-sorted finitary algebraic theory in the sense of Lawvere [11], and let $U: \mathcal{A} \to \text{Set}$ be the underlying-set functor. It was observed in [*] Section 4.6 that a filtered colimit in $\mathcal{A}$ is formed by taking the colimit of the underlying sets and giving this the algebra structure which it inherits by (1.2). Thus $U$ is finitary: so that, if $F \models U$, the free algebra $F^n$ on a finite set $n$ is f.p. by (2.6) and (2.4). By (2.5), an algebra $A$ is f.p. if it is the coequalizer of two maps $Fm \twoheadrightarrow Fn$ with $m$ and $n$ finite. [It will follow from (8.12) below that every f.p. algebra has this form: so that our definition of finite presentability agrees here with the classical one of universal algebra.]

(2.8) The results are similar when $\mathcal{A}$ is the category of algebras for a many-sorted finitary algebraic theory in the sense of Bénabou [2]. If $X$ is the set of sorts, the forgetful functor $U: \mathcal{A} \to \text{Set}^X$ is finitary, since its component $U_x: \mathcal{A} \to \text{Set}$ is so for each $x \in X$. Hence $F_x n$ is f.p. for finite $n$, where $F_x \models U_x$. A two-sorted example is the category of (small) graphs, where by a graph $A$ we mean a diagram $A_1 \rightarrow A_0$. The graph $A$ is called finite if both $A_0$ and $A_1$ are finite; and it follows much as in (2.6) - essentially because the theory of graphs has no axioms - that the f.p. graphs are exactly the finite ones. Two $\mathcal{H}_3$-sorted examples are the categories of graded and of differential graded $R$-modules; it follows from the above that $A = (A_i)_{i \in \mathbb{Z}}$ is f.p. if each $A_i$ is f.p. and if $A_i = 0$ for all but a finite number of $i$. [It will follow from (8.12) below that these are the only f.p. objects.]

(2.9) The structure of a category or of a groupoid, not being given by operations defined on a finite product, is not algebraic in the sense of (2.8); it is however what Freyd [6] called essentially algebraic, in that it can be described in terms of finite limits; more precisely, as was first pointed out by Ehresmann ([5]; see also [1]), both categories and groupoids are algebras for a finitary essentially-algebraic theory in the sense of [*] Section 6.3 - which theories are the case $\emptyset = \text{Set}$ of our present object of study in this article. It follows easily using (1.2) that the forgetful functor
from the category of (small) categories [resp. groupoids] to that of graphs is finitary; and hence that a category [resp. a groupoid] is f.p. if it is the coequalizer of a diagram $Fg \rightrightarrows FH$, where $Fg$ and $FH$ are the free categories [resp. groupoids] on the finite graphs $g$ and $h$. {Once again it will follow from (8.12) below that these are the only f.p. objects.} In particular the category $2 = (0 \rightarrow 1)$ and the groupoid $1 = (0 \rightrightarrows 1)$ are f.p.; the set-valued functors they represent send a category or a groupoid to its set of morphisms, and are not only finitary but also conservative.

(2.10) We now return to the case of a general $\mathcal{C}$, and consider a $\mathcal{C}$-category $\mathcal{A}$ admitting filtered colimits. For an object $G$ of $\mathcal{A}$, we must carefully distinguish between its finite presentability in $\mathcal{A}$ - the finitariness of $\mathcal{A}(G, -) : \mathcal{A} \rightarrow \mathcal{C}$, or equally of $\mathcal{A}(G, -) : \mathcal{A}_0 \rightarrow \mathcal{C}_0$ - and its finite presentability in $\mathcal{A}_0$ - the finitariness of $\mathcal{A}_0(G, -) : \mathcal{A}_0 \rightarrow \mathcal{S}_{\text{Set}}$, which is the composite of $\mathcal{A}(G, -)$ with $V : \mathcal{C}_0 \rightarrow \mathcal{S}_{\text{Set}}$. Neither implies the other in general, so that neither of $\mathcal{A}_{f_0}$ and $\mathcal{A}_{f_1}$ need contain the other; and this even for $\mathcal{A} = \mathcal{C}$.

(2.11) As an example of this, let $M$ be a group, and let $\mathcal{O}$ be the category $\mathcal{O}_0 = [M, \text{Set}]$ of $M$-sets with its cartesian closed structure. It follows easily from (2.7) that $G \in \mathcal{O}$ is f.p. in $\mathcal{O}_0$ if and only if (i) the set of orbits of $G$ is finite, and (ii) the stabilizer of each $g \in G$ is a finitely-generated group. On the other hand, since the finitary and conservative forgetful functor $U : \mathcal{O}_0 \rightarrow \mathcal{S}_{\text{Set}}$ is represented by the $M$-set $M$, the object $G$ is f.p. in $\mathcal{O}$ precisely when $\mathcal{O}_0(M, [G, -])$ is finitary; that is, when $\mathcal{O}_0(M \times G, -)$ is finitary, or when $M \times G$ is f.p. in $\mathcal{O}_0$; which by (i) and (ii) above is the case exactly when $G$ is finite. Thus the unit object $1 = 1$ is f.p. in $\mathcal{O}$, as it must be by (2.2); but it is f.p. in $\mathcal{O}_0$ only when $M$ is finitely generated as a group. On the other hand the $M$-set $M$, since it represents $U$, is always f.p. in $\mathcal{O}_0$, but it is f.p. in $\mathcal{O}$ only when $M$ is finite.

(2.12) In the case $\mathcal{O} = \mathcal{S}_{\text{Set}}$, let $q_1 : P l \rightarrow C$ be the colimit of $P : L \rightarrow \mathcal{A}$, where $L$ is small and filtered. Then if $C$ is f.p., some $q_1$ is a retraction. If, moreover, every $P l$ is f.p., and if $P \phi$ is epimorphic for every $\phi : l \rightarrow m$
in \( \mathfrak{L} \), then some \( q_l \) is an isomorphism.

**Proof.** Since \( C \) is f.p. the \( \mathfrak{A}(C, q_l) : \mathfrak{A}(C, P_l) \to \mathfrak{A}(C, C) \) constitute a colimit cone in \( \mathfrak{Set} \), so that \( 1_C = q_l i \) for some \( l \) and for some \( i : C \to P_l \). For such an \( i \) we have

\[
q_l i q_l = q_l 1 : P_l \to C;
\]

and since the \( \mathfrak{A}(P_l, q_m) : \mathfrak{A}(P_l, P_m) \to \mathfrak{A}(P_l, C) \) also constitute a colimit in \( \mathfrak{Set} \) if \( P_l \) is f.p., it follows (see [\*] Thm. 4.72) that there is some \( \phi : l \to m \) in \( \mathfrak{L} \) with \( P\phi . i q_l = P\phi \). Since \( q_l = q_m . P\phi \), this gives

\[
P\phi . i q_m . P\phi = P\phi , \text{ whence } P\phi . i q_m = 1
\]

since \( P\phi \) is epimorphic. But \( q_m . P\phi . i = q_l i = 1 \); so that \( q_m \) is an isomorphism.

### 3. Locally Finitely Presentable Categories

**3.1 Proposition.** For a cocomplete \( \mathcal{C} \)-category \( \mathfrak{A} \), the following are equivalent:

(i) \( \mathfrak{A} \) has a (small) strong generator \( \mathfrak{G} \subset \mathfrak{A}_f \);

(ii) there is a small \( \mathfrak{G} \) and a strongly-generating \( K : \mathfrak{G} \to \mathfrak{A} \) with \( K(\mathfrak{G}) \subset \mathfrak{A}_f \);

(iii) there is a small \( \mathfrak{G} \) and a right-adjoint, finitary, conservative functor \( T : \mathfrak{A} \to [\mathfrak{G}^{op}, \mathfrak{V}] \).

**Proof.** (i) and (ii) are equivalent since, by [\*] Section 3.6, the full image of a strongly generating functor is a strong generator. By [\*] Thm 4.51, there is an equivalence between functors \( K : \mathfrak{G} \to \mathfrak{A} \) and right adjoints \( T : \mathfrak{A} \to [\mathfrak{G}^{op}, \mathfrak{V}] \), given by \( T = K \) where \( K A = \mathfrak{A}(K-, A) \). By definition, \( K \) is strongly generating exactly when \( K \) is conservative; moreover, since small colimits in \( [\mathfrak{G}^{op}, \mathfrak{V}] \) are formed pointwise, \( K \) is finitary exactly when \( K(\mathfrak{G}) \subset \mathfrak{A}_f \).

(3.2) We shall call a \( \mathcal{C} \)-category \( \mathfrak{A} \) **locally finitely presentable** (or l.f.p.) when it is cocomplete and satisfies the equivalent conditions of (3.1).

(3.3) It follows from (3.1) (iii) that, if \( I : \mathfrak{B} \to \mathfrak{A} \) is right-adjoint, finitary and conservative, and if \( \mathfrak{A} \) is l.f.p., then \( \mathfrak{B} \) is l.f.p., provided that \( \mathfrak{B} \) is
cocomplete. Since the cocompleteness of \( B \) follows from that of \( \mathcal{A} \) if the right-adjoint \( J \) is fully faithful, we conclude that a reflective full subcategory \( B \) of an l.f.p. \( \mathcal{A} \) is l.f.p. if the inclusion \( J: B \rightarrow \mathcal{A} \) is finitary.

(3.4) EXAMPLES. The \( \mathcal{O} \)-category \( \mathcal{O} \) itself is always l.f.p., since the object \( I \), which is f.p. by (2.2), is (not merely a strong generator but) dense, by (5.17) of [\*]. More generally the functor category \([\mathcal{I}, \mathcal{O}]\) is l.f.p. for a small \( \mathcal{I} \), since the representables \( \mathcal{I}(t, \cdot) \) are f.p. by (2.2) and dense by Proposition 5.16 of [\*]. Still more generally, if \( \mathcal{A} \) is l.f.p. with strong generator \( \mathcal{O} \subset \mathcal{A}_f \), then \([\mathcal{I}, \mathcal{O}]\) is l.f.p.; for the \( \{\mathcal{I}(t, \cdot) \otimes G\}_{t \in \mathcal{I}, G \in \mathcal{O}} \), which are f.p. by (2.2), form a strong generator by (2.3), the \( \mathcal{A}(G, E_t \cdot) \) jointly reflecting isomorphisms.

(3.5) EXAMPLES OF L.F.P. CATEGORIES WHEN \( \mathcal{O} = \mathcal{S}_{\text{set}} \). The category of algebras for a finitary algebraic theory, one-sorted or many-sorted, is l.f.p.; the objects \( \{F_x \mid x \in X\} \) of (2.8) are f.p., and constitute a strong generator - indeed, a regular one. The category of small categories is l.f.p. by (2.9), the f.p. object \( 2 \) being a strong generator; and the same argument applies to the full subcategories of preordered sets and of ordered sets. The category of small groupoids is again l.f.p. by (2.9), the f.p. object \( I \) being a strong generator; and the same argument applies to the full subcategory of sets-with-an-equivalence-relation.

(3.6) By applying the last remark of (1.3) to the finitary and conservative \( \mathcal{A} \rightarrow [\mathcal{O}^\circ, \mathcal{O}] \) of (3.1), we deduce that if \( \mathcal{A} \) is l.f.p. with the strong generator \( \mathcal{O} \subset \mathcal{A}_f \), the functors \( \mathcal{A}(G, \cdot): \mathcal{A} \rightarrow \mathcal{O} \) for \( G \in \mathcal{O} \) jointly reflect filtered colimits; so that \( S: B \rightarrow \mathcal{A} \) is finitary if each \( \mathcal{A}(G, S\cdot): B \rightarrow \mathcal{O} \) is so.

(3.7) In the classical case \( \mathcal{O} = \mathcal{S}_{\text{set}} \), let \( \mathcal{A} \) be an l.f.p. category, and suppose that \( \mathcal{A} \) admits finite (conical) limits - which is in fact automatically true by (7.2) below. Then finite limits commute with filtered colimits in \( \mathcal{A} \).

**PROOF.** We are asserting that the canonical map

\[
p: \text{colim}_p \lim_p F(p, l) \rightarrow \lim_p \text{colim}_p F(p, l)
\]

is an isomorphism, where \( F: \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{A} \) with \( \mathcal{P} \) finite and \( \mathcal{Q} \) filtered. If \( \mathcal{O} \subset \mathcal{A}_f \) is a strong generator for \( \mathcal{A} \), it suffices to show that \( \mathcal{A}(G, \rho) \) is

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an isomorphism for each \( G \in \mathcal{G} \). Since \( \mathcal{G}(G, -) : \mathcal{G} \to \mathcal{S}et \) preserves finite limits and filtered colimits, this follows from (1.2).

(3.8) In the case \( \mathcal{C} = \mathcal{S}et \), let \( \mathcal{A} \) be a category with finite limits and filtered colimits. Then the \( n \)-th power functor \( (\_)^n : \mathcal{A} \to \mathcal{A} \) is finitary for each finite \( n \), if either \( \mathcal{A} \) is l.f.p. or \( \mathcal{A} \) is cartesian closed.

**Proof.** Since the diagonal \( \Delta : \mathcal{A} \to \mathcal{A}^n \) is left adjoint and hence finitary, it suffices to show that the product functor \( \Pi : \mathcal{A}^n \to \mathcal{A} \) is finitary. If \( \mathcal{A} \) is l.f.p., we have this by (3.7). If \( \mathcal{A} \) is cartesian closed, let \( P : \mathcal{L} \to \mathcal{A}^n \) have components \( P_i : \mathcal{L} \to \mathcal{A} \), where \( \mathcal{L} \) is filtered. Since the diagonal \( \Delta : \mathcal{L} \to \mathcal{L}^n \) is final by \([\ast]\) Thm 4.10, we have

\[
\text{colim} \Pi P \cong \text{colim}_{l \in \mathcal{L}^n}(P_1 \times \cdots \times P_n l_n);
\]

and since \(- \times A : \mathcal{A} \to \mathcal{A} \) preserves all colimits (or more trivially when \( n = 1 \) or 0), this is isomorphic to \( \text{colim} P_1 \times \cdots \times \text{colim} P_n = \Pi \text{colim} P \).

(Note that, by the example in Section 3.3 of \([9]\), \((\_)^2 : \mathcal{A} \to \mathcal{A} \) is not finitary when \( \mathcal{A} \) is the category of topological spaces.)

4. Finite Indexed Limits.

(4.1) An indexing-type \( H : K \to \mathcal{C} \) (cf. \([\ast]\) Section 3.1) shall be called finite if

(i) the set of isomorphism classes of \( \text{ob} K \) is finite,

(ii) for each \( k, k' \in K \), we have \( K(k, k') \in \mathcal{C}_f \),

(iii) \( H \) factorizes through the inclusion \( \mathcal{C}_f \subseteq \mathcal{C} \).

A finite limit (or colimit) is one whose indexing-type is finite.

(4.2) It follows from (2.2) that, if \( H : K \to \mathcal{C} \) and \( H' : K' \to \mathcal{C} \) are finite, so is the functor \( H \otimes H' : K \otimes K' \to \mathcal{C} \) sending \((k, k')\) to \( H(k, k') \). It then follows from (3.18) of \([\ast]\) that a repeated finite limit \( \{H?, \{H'?, P(?,-)\}\} \) is a finite limit \( \{H \otimes H', P\} \).

(4.3) **Proposition.** A \( \mathcal{C} \)-category \( \mathcal{A} \) admits all finite (indexed) limits if it admits all conical limits indexed by finite ordinary categories and all cotensor products \( x \otimes A \) with \( x \in \mathcal{C}_f \). The converse is true if \((\_)^n : \mathcal{C}_o \to \mathcal{C}_o \) is finitary for all finite \( n \), and hence by (3.8) if \( \mathcal{C}_o \) is l.f.p. or if \( \mathcal{C} \) is
cartesian closed.

PROOF. Let \( H: K \to \mathcal{V} \) be a finite indexing-type and let \( T: K \to \mathcal{A} \). Replacing \( K \) if necessary by an equivalent category, we may suppose \( \text{ob } K \) to be in fact finite. Since each \( H k \in \mathcal{V}_f \), as does each \( K(k, k') \), we have the cotensor products \( H k \triangleright T k' \) and \( K(k, k') \triangleright (H k \triangleright T k') \). Since \( \mathcal{A} \) admits finite products and equalizers, it admits the equalizer

\[
\prod_{k \in K} H k \triangleright T k \to \prod_{k \in K} H k \triangleright T k \cong \prod_{k, k' \in K} K(k, k') \triangleright (H k \triangleright T k'),
\]

which by \((3.68)\) of [\*] is the indexed limit \( \{H, T\} \).

For the converse, \( x \triangleright A \) is by [\*] Section 3.7 the indexed limit \( \{x, A\} \), where \( x \) and \( A \) are identified with functors \( x: \mathcal{I} \to \mathcal{V} \) and \( A: \mathcal{I} \to \mathcal{A} \). Since \( \text{ob } \mathcal{I} = \{0\} \) and since \( \mathcal{I}(0, 0) = 1 \) is f.p. in \( \mathcal{V} \) by \((2.2)\), \( x: \mathcal{I} \to \mathcal{V} \) is a finite indexing-type if \( x \in \mathcal{V}_f \); thus \( x \triangleright A \) exists. Next, by [\*] Section 3.8, the conical limit in \( \mathcal{A} \) of an ordinary functor \( S: \mathcal{P} \to \mathcal{A} \) is the indexed limit \( \{\Delta l, \vec{S}\} \), where \( \vec{S}: \mathcal{P}_0 \to \mathcal{A} \) is the \( \mathcal{V} \)-functor corresponding to \( S \), its domain \( \mathcal{P}_0 \) being the free \( \mathcal{V} \)-category on the ordinary category \( \mathcal{P} \), and where \( \Delta l: \mathcal{P}_0 \to \mathcal{V} \) corresponds similarly to the ordinary functor \( \Delta l: \mathcal{P} \to \mathcal{V}_0 \) constant at \( l \). When \( \mathcal{P} \) is finite, \( \text{ob } \mathcal{P}_0 = \text{ob } \mathcal{P} \) is finite, and \( \Delta l \) factorizes through \( \mathcal{V}_f \subset \mathcal{V} \) since \( l \in \mathcal{V}_f \) by \((2.2)\). To show that \( \Delta l \) is a finite indexing-type, it remains to show that each \( \mathcal{P}_0(k, k') \) is f.p. in \( \mathcal{V} \). But \( \mathcal{P}_0(k, k') = \mathcal{P}(k, k').1 \), the coproduct of \( \mathcal{P}(k, k') \) copies of \( 1 \); and \( \mathcal{P}(k, k') \) is a finite set \( n \). Since \( \mathcal{V}(n.1, -) \equiv (f^a: \mathcal{V} \to \mathcal{V}) \), to show that \( n.l \in \mathcal{V}_f \) is to show that \( (f^a: \mathcal{V} \to \mathcal{V}) \) is finitary. By \((1.3)\), this is the same as the finitariness of \((f^n: \mathcal{V} \to \mathcal{V}) \).

(4.4) HYPOTHESIS. From now on we suppose at all times that the ordinary category \( \mathcal{V}_0 \) is l.f.p.

(4.5) A \( \mathcal{V} \)-category \( \mathcal{A} \) satisfying the equivalent conditions of \((4.3)\) shall be called finitely complete, or f.c. A \( \mathcal{V} \)-functor \( T: \mathcal{A} \to \mathcal{B} \) shall be called left exact, or lex, if \( \mathcal{A} \) is finitely complete and \( T \) preserves all finite limits. (The duals are finitely cocomplete, and right exact or rex.) It is clear from the proof of \((4.3)\) that:

(4.6) For a finitely complete \( \mathcal{A} \), the functor \( T: \mathcal{A} \to \mathcal{B} \) is left exact if and
only if it preserves finite conical limits and the cotensor products $x \otimes A$ with $x \in \mathcal{O}_f$.

Now (2.6) gives:

\[(4.7)\] In the classical case $\mathcal{V} = \text{Set}$, a category $\mathcal{A}$ is finitely complete precisely when it admits all finite conical limits, and then $T : \mathcal{A} \to \mathcal{B}$ is left exact precisely when it preserves these. For a general $\mathcal{V}$, if the $\mathcal{V}$-category $\mathcal{A}$ is f.c., so is $\mathcal{A}_o$; and if $T : \mathcal{A} \to \mathcal{B}$ is lex, so is $T_o : \mathcal{A}_o \to \mathcal{B}_o$.

\[(4.8)\] Under further hypotheses that we shall later impose on $\mathcal{V}$ we shall show in (7.2) below that every l.f.p. category $\mathcal{A}$ is complete. Hence the hypothesis of finite completeness in such propositions as the next - which generalizes (3.7) - is in fact otiose in practice.

\[(4.9)\] PROPOSITION. Let $\mathcal{A}$ be l.f.p. and f.c., let $H : K \to \mathcal{V}$ be a finite indexing type, and let $P : \mathcal{L} \to [K, \mathcal{A}]_o$ be an ordinary functor with $\mathcal{L}$ small and filtered. Then the evident canonical map

$$\sigma : \text{colim} \{ H, P \} \to \{ H, \text{colim} P \}$$

is an isomorphism. In other words, finite limits commute with filtered colimits in such an l.f.p. $\mathcal{A}$.

PROOF. If $\mathcal{G} \subseteq \mathcal{A}_f$ is a strong generator for $\mathcal{A}$, it suffices to show that $\mathcal{A}(G, \sigma)$ is an isomorphism for all $G \in \mathcal{G}$. Since $\mathcal{A}(G, -) : \mathcal{A} \to \mathcal{V}$ then preserves filtered colimits and all limits, we are reduced to the special case $\mathcal{A} = \mathcal{V}$. Then the functor $x \otimes - : \mathcal{V} \to \mathcal{V}$ is simply $[x, -] = \mathcal{V}(x, -)$, which commutes (by definition) with filtered colimits when $x \in \mathcal{V}_f$. So, by the proof of (4.3), it suffices to show that filtered colimits commute with finite conical limits in $\mathcal{V}$; which is to say that they do so in $\mathcal{V}_o$. But this is so by (3.7), given our hypothesis (4.4) that $\mathcal{V}_o$ is l.f.p.

\[(4.10)\] This result may also be expressed by saying that $\{ H, - \} : [K, \mathcal{A}] \to \mathcal{A}$ is finitary for an l.f.p. and f.c. $\mathcal{A}$ when $H : K \to \mathcal{V}$ is a finite indexing type; or equally by saying that $\text{colim} : [\mathcal{L}_o, \mathcal{A}] \to \mathcal{A}$ is left exact for an l.f.p. and f.c. $\mathcal{A}$ when $\mathcal{L}$ is a small filtered ordinary category. (Recall from [*] Section 2.5 that the $\mathcal{V}$-category $[\mathcal{L}_o, \mathcal{A}]$ has $[\mathcal{L}, \mathcal{A}_o]$ as underlying category.)
(4.11) Taking $\mathcal{A} = \mathcal{V}$ in the first statement of (4.10), we conclude that if $H : K \to \mathcal{V}$ is a finite indexing type, then $H \in [K, \mathcal{V}]_f$.

(4.12) Let $\mathcal{A}$ be l.f.p. and f.c. Then, in the sense made precise in the proof, a filtered colimit of left exact functors into $\mathcal{A}$ is left exact.

Proof. We consider an f.c. category $\mathcal{B}$, not necessarily small, and an ordinary functor $Q : \mathcal{L} \to [\mathcal{B}, \mathcal{A}]_o$ where $\mathcal{L}$ is small and filtered and each $Q_l : \mathcal{B} \to \mathcal{A}$ is left exact; here $[\mathcal{B}, \mathcal{A}]_o$ is the ordinary category of $\mathcal{O}$-functors $\mathcal{B} \to \mathcal{A}$ and $\mathcal{O}$-natural transformations between them, which exists even when $[\mathcal{B}, \mathcal{A}]$ is too big to exist as a $\mathcal{V}$-category; and what we assert is that $\text{colim} Q : \mathcal{B} \to \mathcal{A}$ is left exact. To give $Q$, however, is equivalently to give a left exact $R : \mathcal{B} \to [\mathcal{L}_0, \mathcal{A}]$; and $\text{colim} Q$, being the composite of $R$ with $\text{colim} : [\mathcal{L}_0, \mathcal{A}] \to \mathcal{A}$, is left exact by the second assertion of (4.10).

(4.13) Let $\mathcal{A}$ be l.f.p. and f.c. Then, in the sense made precise in the proof, a finite limit of finitary functors into $\mathcal{A}$ is finitary.

Proof. We consider a $\mathcal{B}$, not necessarily small, which admits filtered colimits, a finite indexing type $H : K \to \mathcal{V}$, and a functor $Q : K \otimes \mathcal{B} \to \mathcal{A}$ such that each $Q(k, \cdot) : \mathcal{B} \to \mathcal{A}$ is finitary; and what we assert is that

$$\{H?, Q(?, \cdot)\} : \mathcal{B} \to \mathcal{A}$$

is finitary. To give $Q$, however, is equivalently to give a finitary $R : \mathcal{B} \to [K, \mathcal{A}]$; and $\{H?, Q(?, \cdot)\}$, being the composite of $R$ with $\{H, \cdot\} : [K, \mathcal{A}] \to \mathcal{A}$, is finitary by the first assertion of (4.10).

(4.14) For any cocomplete $\mathcal{A}$, the full subcategory $\mathcal{A}_f$ is closed under finite colimits.

Proof. Since $\mathcal{A}(\cdot, A)$ sends colimits in $\mathcal{A}$ to limits in $\mathcal{V}$, this follows from the case $\mathcal{A} = \mathcal{V}$ of (4.13).

(4.15) Remark. Note that we used Hypothesis (4.4) in the proof of (4.9).

5. Locally Finitely Presentable Symmetric Monoidal Closed Categories

In accordance with Hypothesis (4.4), we suppose for this section
that $\mathcal{O}_o$ is l.f.p., with strong generator $H \subset \mathcal{O}_o f$.

(5.1) For an object $G$ of the cocomplete $\mathcal{O}$-category $\mathcal{A}$, the following are equivalent:

(i) $G$ is f.p. in $\mathcal{A}$;
(ii) $x \otimes G$ is f.p. in $\mathcal{A}_o$ for all $x \in \mathcal{O}_o f$;
(iii) $z \otimes G$ is f.p. in $\mathcal{A}_o$ for all $z \in H$.

PROOF. To say that $G$ is f.p. in $\mathcal{A}$ is to say that $\mathcal{A}(G, -): \mathcal{A} \to \mathcal{O}$, or equivalently $\mathcal{A}(G, -)_o: \mathcal{A}_o \to \mathcal{O}_o$, is finitary. Since

$$\mathcal{O}_o(x, \mathcal{A}(G, -)_o) \cong \mathcal{O}_o(x \otimes G, -),$$

the result follows from (1.3) and (3.6).

(5.2) The following are equivalent:

(i) $\mathcal{O}_o f \subset \mathcal{O}_o f_0$;
(ii) $x \otimes y \in \mathcal{O}_o f$ whenever $x, y \in \mathcal{O}_o f$;
(iii) $z \otimes y \in \mathcal{O}_o f$ whenever $x, y \in H$.

PROOF. (i) implies (ii) by (5.1) and (ii) implies (iii) trivially. Given (iii), we first use (iii) $\Rightarrow$ (i) of (5.1) to deduce that $H \subset \mathcal{O}_o f_0$; then use (i) $\Rightarrow$ (ii) of (5.1) to deduce that $x \otimes y \in \mathcal{O}_o f$ if $x \in \mathcal{O}_o f$ and $y \in H$; and finally use (iii) $\Rightarrow$ (i) of (5.1), with the symmetry of $\otimes$, to deduce that $\mathcal{O}_o f \subset \mathcal{O}_o f_0$.

(5.3) The following are equivalent:

(i) $\mathcal{O}_o f_0 \subset \mathcal{O}_o f$;
(ii) $l \in \mathcal{O}_o f$;
(iii) $V = \mathcal{O}_o(l, -): \mathcal{O}_o \to \text{Set}$ is finitary;
(iv) $\mathcal{A}_f o \subset \mathcal{A}_o f$ for every cocomplete $\mathcal{O}$-category $\mathcal{A}$.

PROOF. (ii) and (iii) are equivalent by definition; (iv) implies (i) trivially; (i) implies (ii) since $l \in \mathcal{O}_o f_0$ by (2.2); and (ii) implies (iv) by the part (i) $\Rightarrow$ (ii) of (5.1).

(5.4) The example (2.11) shows that neither $\mathcal{O}_o f \subset \mathcal{O}_o f_0$ nor $\mathcal{O}_o f \subset \mathcal{O}_o f$ is automatic when $\mathcal{O}_o$ is l.f.p.; for the $\mathcal{O}_o = [M, \text{Set}]$ of that example is l.f.p. by (3.5). The same example shows that $\mathcal{O}_o f \subset \mathcal{O}_o f$ does not imply $\mathcal{O}_o f \subset \mathcal{O}_o f_0$; for the former holds precisely when $M$ is finitely generated as
a group, and the latter precisely when \( M \) is finite. The author sees no reason to suppose that \( \mathcal{V}_o f \subset \mathcal{V}_o \) implies \( \mathcal{V}_o \subset \mathcal{V}_o f \), but has no counter-example to this. The author is endebted to R. Börger for discussions concerning this example.

(5.5) We shall say that \( \mathcal{V} \) is \textit{locally finitely presentable as a (symmetric monoidal) closed category} if \( \mathcal{V}_o \) is l.f.p. and if \( \mathcal{V}_o f = \mathcal{V}_o f o \). By (5.2) and (5.3), a closed \( \mathcal{V} \) with l.f.p. \( \mathcal{V}_o \) is l.f.p. as a closed category precisely when \( \mathcal{V}_o f \) is \textit{closed under the monoidal structure}, in the sense that \( l \in \mathcal{V}_o f \) and \( x \otimes y \in \mathcal{V}_o f \) when \( x, y \in \mathcal{V}_o f \); while it in fact suffices for the latter that \( x \otimes y \in \mathcal{V}_o f \) whenever \( x, y \in H \), where \( H \subset \mathcal{V}_o f \) is a strong generator for \( \mathcal{V}_o \).

(5.6) EXAMPLES. If \( \mathcal{V}_o \) is the category of algebras for a one-sorted finitary algebraic theory, and if the theory is \textit{commutative} in the sense of Linton [12], then \( \mathcal{V}_o \) admits a symmetric monoidal closed structure \( \mathcal{V} \) in which the tensor product represents the bi-homomorphisms and the unit object \( I \) represents the forgetful functor to \( \text{Set} \). In this case \( F I = I \) is an f.p. strong generator in \( \mathcal{V}_o \) by (3.5), so that \( \mathcal{V} \) is l.f.p. as a closed category by the last remarks of (5.5). The closed categories \( \mathcal{V} = \text{Set}, \text{Set}_*, \text{Ab}, \text{R-Mod} \) of sets, pointed sets, abelian groups, and \( R \)-modules for a commutative ring \( R \), are particular examples.

(5.7) EXAMPLES. The category \( \mathcal{V}_o = [M, \text{Set}] \) is an example of (5.6) when the group \( M \) is abelian. The tensor product here, however, is not the cartesian one; and (5.4) shows that \( [M, \text{Set}] \) with its cartesian closed structure is l.f.p. as a closed category only when \( M \) is finite.

(5.8) EXAMPLES. Consider the closed categories \( \text{C-R-Mod}, \text{DGR-Mod}, \text{Gph} \) of graded \( R \)-modules, differential graded \( R \)-modules, and graphs - the first two with their classical closed structures, and the last with its cartesian one. In each case \( \mathcal{V}_o \) is l.f.p. by (3.5), the f.p. objects \( F_i I \) forming a strong generator. In the first two cases

\[
F_0 I = I \quad \text{and} \quad F_i I \otimes F_j I = F_{i+j} I,
\]

so that \( \mathcal{V} \) is l.f.p. as a closed category by (5.5). In the last case

\[
F_0 I = (0 \rightarrowtail 1) \quad \text{and} \quad F_1 I = (1 \rightarrowtail 2);
\]

\[
\text{where}
\]

\[
\text{in}
\]
the various products $F_i 1 \times F_j 1$ are all finite and hence f.p. by (2.8), as is the unit object $1 = (1 \Rightarrow 1)$; so that again $\mathcal{V}$ is l.f.p. as a closed category.

(5.9) EXAMPLES. Consider the cartesian closed categories $\mathcal{Cat}, \mathcal{Preord}, \mathcal{Ord}, \mathcal{Gpd}, \mathcal{Equiv}$ of (small) categories, preorders, orders, groupoids, and sets-with-an-equivalence-relation. In each case $\mathcal{V}_0$ is l.f.p. by (3.5), an f.p. strong generator being given by 2 in the first three cases and by 1 in the other two. Since $2 \times 2$ and $1 \times 1$ are finite, and hence f.p. by (2.9), as is the unit object 1, it follows by (5.5) that $\mathcal{V}$ is in each case l.f.p. as a closed category.

(5.10) EXAMPLES. A finite complete ordered set, such as 2, with its cartesian closed structure, is clearly l.f.p. as a closed category.

(5.11) For the closed categories $\mathbb{R}^+, \mathcal{CGTop}, \mathcal{Ban}$ of the extended non-negative reals, of compactly-generated topological spaces, and of Banach spaces (see [*] Section 1.1), it is not even true that $\mathcal{V}_0$ is l.f.p.

(5.12) HYPOTHESIS. For the rest of this article we strengthen the hypothesis (4.4) by supposing throughout that $\mathcal{V}$ is l.f.p. as a closed category. Accordingly, both $x \in \mathcal{V}_0$ and $x \in \mathcal{V}_f$ mean the same as $x \in \mathcal{V}_f$.

6. LEFT EXACT AND FLAT FUNCTORS INTO $\mathcal{V}$

(6.1) If $\mathcal{I}$ is a small $\mathcal{V}$-category, we call the functor $F: \mathcal{I} \to \mathcal{V}$ flat if $F^* - : [\mathcal{I}^{op}, \mathcal{V}] \to \mathcal{V}$ is left exact, where * denotes the indexed colimit, as in [*] Section 3.1.

(6.2) A flat functor $F: \mathcal{I} \to \mathcal{V}$ preserves any finite limit that happens to exist in $\mathcal{I}$; in particular, a flat functor is left exact if $\mathcal{I}$ is finitely complete.

PROOF. Let $H : K \to \mathcal{V}$ be a finite indexing type, and let $T : K \to \mathcal{I}$ be such that the limit $\{ H, T \}$ exists in $\mathcal{I}$. Let $Y : \mathcal{I} \to [\mathcal{I}^{op}, \mathcal{V}]$ denote the Yoneda embedding. Since $Y$ preserves limits by [*] Section 3.3, we have a canonical isomorphism $Y\{ H, T \} \cong \{ H, YT \}$; and since $F$ is flat, we have a canonical isomorphism $F\{ H, YT \} \cong \{ H, FYT \}$. Combining these gives...
$F \ast Y \{H, T\} = \{H, F \ast Y T\}$; and since $F \ast Y t = F t$ by (3.9) and (3.10) of [*], we have the desired result $F \{H, T\} = \{H, FT\}$.

(6.3) Any representable $\mathcal{F}(t,-)$ is flat, since by (3.10) of [*] the functor $\mathcal{F}(t,-): [\mathcal{F}^{op}, \mathcal{A}] \to \mathcal{A}$ is isomorphic, for any $\mathcal{A}$, to the evaluation $E_t$, which by [*] Section 3.3 preserves all limits that exist. Since $- \circ T$ is cocontinuous by [*] Section 3.3, it follows from (4.12) that any filtered colimit in $[\mathcal{F}, \mathcal{O}]$ of flat functors is flat, so that in particular every filtered colimit $F$ of representables is flat. In fact (4.12) gives more: for such an $F$ and any l.f.p. and f.c. $\mathcal{A}$, the functor $F^*: [\mathcal{F}^{op}, \mathcal{A}] \to \mathcal{A}$ is left exact.

(6.4) When $\mathcal{O} = \mathcal{S}_{\mathcal{E}}$ it is a classical result (see [*] Thm 5.38, or (6.7) below) that every flat $F: \mathcal{F} \to \mathcal{O}$ is a filtered colimit of representables. The author has no proof of this for a general $\mathcal{O}$ satisfying the Hypothesis (5.12). It is however true when $\mathcal{F}$ is finitely complete ((6.11) below) and for certain special $\mathcal{O}$, if $\mathcal{F}$ at least admits certain cotensor products ((6.10) below).

(6.5) Consider an arbitrary $F: \mathcal{F} \to \mathcal{O}$ with $\mathcal{F}$ small. The composite of $V = \mathcal{O}_0(1, -): \mathcal{O}_0 \to \mathcal{S}_{\mathcal{E}}$ with the underlying functor $F_0: \mathcal{F}_0 \to \mathcal{O}_0$ of $F$ is an ordinary functor $VF_0: \mathcal{F}_0 \to \mathcal{S}_{\mathcal{E}}$. Recall from [*] Section 1.10 that the comma-category $1/VF_0$, whose objects are pairs $(t, a)$ with $t \in \mathcal{F}$ and $a \in VF_0t_1$, is called the category $el(VF_0)$ of elements of $VF_0$. Since a map $\mathcal{F}(t,-) \to F$ from a representable into $F$ corresponds by Yoneda to an element $a \in VF_0t$, we have a canonical inductive cone $\mu_{(t,a)}: \mathcal{F}(t,-) \to F$ in $[\mathcal{F}, \mathcal{O}]_0$, with vertex $F$, whose base is the ordinary functor

$$el(VF_0)^{op} \xrightarrow{d^{op}} \mathcal{F}_0^{op} \xrightarrow{Y_0} [\mathcal{F}, \mathcal{O}]_0,$$

where $d$ is the projection sending $(t,a)$ to $t$; and this cone induces a canonical map $\rho: \text{colim}(Y_0d^{op}) \to F$. When $\mathcal{O} = \mathcal{S}_{\mathcal{E}}$ it is a classical result - see [*] Section 3.3 - that $\mu$ is a colimit cone; so that $\rho$ is an isomorphism. For a general $\mathcal{O}$ - even for one satisfying (5.12) - this is false. In fact, when $\mathcal{F}$ is the unit $\mathcal{O}$-category $\mathcal{I}$ with one object $0$, and with $\mathcal{I}(0,0) = 1$, so that $[\mathcal{I}, \mathcal{O}] \equiv \mathcal{O}$ and $F$ is just an object of $\mathcal{O}$, the cone $\mu$ consists of all maps $1 \to F$ in $\mathcal{O}_0$; and this is a colimit cone for all $F$
precisely when \( \{ I \} \) is dense in \( \mathcal{C}_0 \) - which is false even for \( \mathcal{C} = \mathbb{G} \). In the example \( \mathcal{C} = \text{Cat} \) it is easily seen, when \( \mathcal{T} = \mathcal{J} = I \), that the map \( \rho \) from the colimit into \( F \) is the inclusion into \( F \) of the discrete category with the same objects; and similarly for \( \mathcal{C} = \mathbb{G}_{\text{pd}} \).

(6.6) Proposition. Suppose that \( \mathcal{C} \) is cartesian closed, and that \( V: \mathcal{C}_0 \to \text{Set} \) preserves coproducts and regular epimorphisms; as is true in the examples \( \mathcal{C} = \text{Set}, \mathbb{G}_{\text{ph}}, \text{Cat}, \text{Ord}, \mathbb{G}_{\text{pd}}, \mathbb{G}_{\text{Ord}}, \mathbb{G}_{\text{gpd}}, \) &c. Then, if \( F: \mathcal{T} \to \mathcal{C} \) is flat, the category \( \text{el}(VF_0)^{\text{op}} \) is filtered.

Proof. Let \( T: \mathcal{P} \to \text{el}(VF_0) \) where \( \mathcal{P} \) is a finite ordinary category. To give \( T \) is to give a functor \( S = dT: \mathcal{P} \to \mathcal{T}_0 \) and a cone \((y_P: 1 \to V_{F_0}S_P)\), which is equally an element \( y \in \lim V_{F_0}S \). The flatness of \( F \) gives

\[
F \ast \lim_p T(\cdot, S_p) \cong \lim_p (F \ast T(\cdot, S_p)),
\]
and this latter by Yoneda is \( \lim_p F S_p = \lim F S_0 \). By the formula of [*] Section 3.10 expressing \( F \ast G \) as a coend, we have a regular epimorphism \( f: \Sigma t \in T F_0 t \times \lim_p \mathcal{T}_0(t, S_p) \to \lim F_0 S \). Applying \( V \) to \( f \), and using the hypotheses of the proposition and the preservation of limits by \( V \), we get a surjection \( Vf: \Sigma t \in V F_0 t \times \lim_p \mathcal{T}_0(t, S_p) \to \lim V F_0 S \). Let some inverse image of \( y \) under \( Vf \) be \((\gamma \in VF_0, \beta \in \lim_p \mathcal{T}_0(t, S_p))\). We can regard \( \beta \) as a projective cone in \( \mathcal{T}_0 \) over \( S \) with vertex \( t \). Since \( VF_0 \beta_p: V F_0 t \to V F_0 S_p \) maps \( a \) to \( \gamma_p \) by construction, \( \beta \) is equally a projective cone in \( \text{el}(VF_0) \) over \( T \) with vertex \((t, a)\). Thus \( \text{el}(VF_0)^{\text{op}} \) is filtered.

(6.7) Combining (6.6), (6.5), and (6.3), we regain the classical result in the case \( \mathcal{C} = \text{Set} \) that \( F: \mathcal{T} \to \text{Set} \) is flat if and only if \( (\text{el} F)^{\text{op}} \) is filtered, and if and only if \( F \) is a filtered colimit of representables.

(6.8) We recall also the other classical result that, when \( \mathcal{C} = \text{Set} \) and \( \mathcal{T} \) is finitely complete, \( F: \mathcal{T} \to \text{Set} \) is flat precisely when it is left exact. For flatness implies left exactness by (6.2); while if \( \mathcal{T} \) is finitely complete and \( F \) is left exact, it is immediate that \( \text{el} F \) is finitely complete, so that \( (\text{el} F)^{\text{op}} \) is finitely cocomplete and a fortiori filtered.

(6.9) Proposition. Let \( \mathcal{T} \) admit the cotensor products \( x^y \) - for \( x \in \mathcal{O}_f \), and let \( F: \mathcal{T} \to \mathcal{C} \) preserve them. Then if \( VF_0: \mathcal{T}_0 \to \text{Set} \) is flat, the cam-
nical cone $\mu$ of (6.5) is a colimit cone in $[T, V]$, expressing $F$ as a filtered colimit of representables in $[T, V]$; whence $F$ is flat by (6.3).

**Proof.** Since colimits in $[T, V]$ are formed pointwise by [*] Section 3.3, $V$ being cocomplete by [*] Section 3.10, we have only to prove that $\mu(t, a, s) : T(t, s) \to Fs$ is, for each $s \in T$, a colimit cone in $V$; which by [*] Section 3.8 is to say that $\mu(t, a, s)$ is a colimit cone in $V_0$. Since $V_0$ is l.f.p. by (4.4), it suffices by (3.6) to show that

$$V_0(x, T(t, s)) \to V_0(x, Fs)$$

is a colimit cone in $Set$ for each $x \in V_0$; which by (5.12) means for each $x \in V$. For $x \in V$, however, $V_0(x, T(t, s)) \cong T_0(t, x\otimes s)$ by the definition of the cotensor product $x\otimes s$; while

$$V_0(x, Fs) \cong V[x, Fs] = V(x\otimes Fs),$$

which since $F$ preserves $x\otimes -$ is isomorphic to $VF(x\otimes s)$. Writing $r$ for $x\otimes s$, we are reduced to proving that $T_0(t, r) \to VF_0r$ is a colimit cone in $Set$; which is true by (6.5).

(6.10) **Corollary.** When $V$ satisfies the conditions of (6.6) and $T$ admits the cotensor products $x\otimes t$ for $x \in V$, $F : T \to V$ is flat precisely when $F$ is a filtered colimit of representables.

**Proof.** A filtered colimit of representables is flat by (6.3). If $F$ is flat, it preserves the $x\otimes t$ for $x \in V$ by (6.2), since these are finite limits by the proof of (4.3). Moreover $el(VF_0)^{op}$ is filtered by (6.6), so that $VF_0$ is flat by (6.7). The result now follows by (6.9).

(6.11) **Theorem.** Let $F : T \to V$ where the small $T$ is finitely complete. Then the following are equivalent:

(i) $F$ is left exact;
(ii) the canonical cone $\mu(t, a) : T(t, -) \to F$ of (6.5) expresses $F$ as a filtered colimit in $[T, V]$ of representables;
(iii) $F$ is some filtered colimit in $[T, V]$ of representables;
(iv) $F$ lies in the closure of $T^{op} \subseteq [T, V]$ under filtered colimits (which by [*] Thm 5.35 is the free filtered-colimit completion of $T^{op}$);
(v) $F^{\ast*} : [T^{op}, A] \to A$ is left exact for any l.f.p. and f.c. $A$;
(vi) $F$ is flat.

**Proof.** If $F$ is left exact, so is $F_0$ by (4.7), whence $VF_0$ is left exact since $V = \mathcal{C}_o(I, -)$ preserves all limits; so that $VF_0$ is flat by (4.7) and (6.8). Hence (i) \(\Rightarrow\) (ii) by (6.9). It is trivial that (ii) \(\Rightarrow\) (iii) and (iii) \(\Rightarrow\) (iv); while (iv) implies (v) by the arguments of (6.3) along with (4.12). The implication (v) \(\Rightarrow\) (vi) is trivial on taking $\mathcal{A} = \mathcal{C}$; and we have (vi) \(\Rightarrow\) (i) by (6.2).

(6.12) **Theorem.** Let $M: \mathcal{T} \to \mathcal{S}$ where $\mathcal{T}$ and $\mathcal{S}$ are finitely complete and $\mathcal{T}$ is small. Let $\text{Lan}_M F: \mathcal{S} \to \mathcal{C}$ be the left Kan extension along $M$ of $F: \mathcal{S} \to \mathcal{C}$. Then if $F$ is left exact, so is $\text{Lan}_M F$.

**Proof.** By [*] Section 4.1, $\text{Lan}_M F$ is the composite of $\tilde{M}: \mathcal{S} \to [\mathcal{T}^{op}, \mathcal{C}]$, where $\tilde{M} S = \mathcal{S}(M -, S)$, and $- * F: [\mathcal{T}^{op}, \mathcal{C}] \to \mathcal{C}$. But $\tilde{M}$ is trivially left exact, while $-* F$ is left exact since $F$ is flat by (6.11) and $*$ is symmetric by [*] Section 3.1.

(6.13) **Remark.** Up to this point we have used, besides Hypothesis 4.4, only the part $\mathcal{C}_o f$ of Hypothesis (5.12) - namely, in the proof of (6.9).

7. $\mathcal{A}_f$ as a small dense subcategory of the locally finitely presentable $\mathcal{A}$

(7.1) The set of finite indexing types is small; so that by [*] Section 3.5 the closure under finite colimits of a small full subcategory $\mathcal{G}$ in a finitely-cocomplete $\mathcal{A}$ is again small, and in particular the free finite-colimit completion of a small $\mathcal{G}$ (its finite colimit-closure in $[\mathcal{G}^{op}, \mathcal{C}]$) is small.

**Proof.** By (4.1), it is a matter of observing that $\mathcal{C}_f$ is small. For $\mathcal{C} = \mathcal{S}_{at}$ we have this by (2.6). This justifies the argument in (7.2) below when $\mathcal{C} = \mathcal{S}_{at}$, which then gives for any $\mathcal{C}$ satisfying (5.12) the smallness of $\mathcal{C}_o f$, and hence of $\mathcal{C}_f$. This now gives the general case of (7.1), which is needed for the general case of (7.2).

(7.2) **Theorem.** Let $\mathcal{A}$ be l.f.p., let $\mathcal{G} \subseteq \mathcal{A}_f$ be a strong generator of $\mathcal{A}$, and let $Z: \mathcal{G}_f \to \mathcal{A}$ be the inclusion. Then

(i) $\mathcal{A}_f$ is the closure of $\mathcal{G}$ in $\mathcal{A}$ under finite colimits, and $\mathcal{A}_f$ is small and finitely cocomplete;
(ii) for each $A \in \mathcal{A}$ the totality of maps $g: G \to A$ with $G \in \mathcal{A}_f$ expresses $A$ as the filtered colimit in $\mathcal{A}$ of the functor $Z_\circ d: Z_\circ A \to \mathcal{A}_o$, where $d: Z_\circ A \to \mathcal{A}_f$ is the projection from the comma-category;

(iii) the colimits in (ii) are $\mathcal{Z}$-absolute, and thus present $\mathcal{A}_f$ as a small dense subcategory of $\mathcal{A}$;

(iv) the full embedding $\tilde{Z}: \mathcal{A} \to [\mathcal{A}_f^{op}, \mathcal{V}]$ is finitary, and has the left adjoint $-*Z: [\mathcal{A}_f^{op}, \mathcal{V}] \to \mathcal{A}$;

(v) $\mathcal{A}$ is complete;

(vi) the replete image of $\tilde{Z}$ is the category $\text{Lex}[\mathcal{A}_f^{op}, \mathcal{V}]$ of left-exact functors $\mathcal{A}_f^{op} \to \mathcal{V}$, so that $\tilde{Z}$ induces an equivalence $\mathcal{A} \simeq \text{Lex}[\mathcal{A}_f^{op}, \mathcal{V}]$.

**Proof.** Write $\overline{G}$ for the closure of $G$ in $\mathcal{A}$ under finite colimits; it is small by (7.1), and of course finitely cocomplete; and by (4.14) it is contained in $\mathcal{A}_f$. Write $K: \overline{\mathcal{G}} \to \mathcal{A}$ for the inclusion; since $\overline{G}$ is a strong generator so a fortiori is $\overline{G} \supset \mathcal{G}$, so that by (3.1) the functor $\tilde{K}: \mathcal{A} \to [\overline{\mathcal{G}}^{op}, \mathcal{V}]$ is finitary, conservative, and right-adjoint; its left adjoint is in fact $-*K$ by the definition of the latter. For any $A \in \mathcal{A}$, let $\beta$ denote the canonical cone consisting of all maps $g: G \to A$ with $G \in \overline{G}$, so that $\beta_{G, g} = g$; the indexing category $K_\circ A$ of this cone is filtered, since by (4.1) $K_\circ \mathcal{G}_o$ is closed under finite conical colimits in $\mathcal{A}_o$. The image $\tilde{K}\beta$ of the cone $\beta$ under $\tilde{K}$ has, since $\tilde{K}K \simeq Y$, the components $\tilde{K}g: \overline{\mathcal{G}}(\cdot, G) \to \tilde{K}A$, and is in fact precisely the canonical cone $\mu$ of (6.5) for the functor $\tilde{K}A: \overline{\mathcal{G}}^{op} \to \mathcal{V}$; for the maps $\overline{\mathcal{G}}(\cdot, G) \to \tilde{K}A$ correspond by Yoneda to the elements of

$$V(\tilde{K}A)G = V\mathcal{A}(KG, A) = \mathcal{A}_o(G, A),$$

or the maps $g: G \to A$. But $\tilde{K}A = \mathcal{A}(K\cdot, A)$ is a left exact functor $\overline{\mathcal{G}}^{op} \to \mathcal{V}$, since $K$ preserves finite colimits and $\mathcal{A}(\cdot, A)$ converts them into limits. Hence $\tilde{K}\beta = \mu$ is a colimit cone by (6.11). But the conservative and finitary $\tilde{K}$ reflects filtered colimits; so that $\beta$ is already a colimit cone in $\mathcal{A}$. If $A \in \mathcal{A}_f$, it follows from (2.12) that some $g: G \to A$ with $G \in \overline{G}$ is a retraction, with say $gi = 1$. Then $g: G \to A$ is the coequalizer of $1$, $i g: G \to G$, and hence by (4.3) $A$ belongs to $\overline{G}$, since $\overline{G}$ is closed under finite colimits. As we already have that $\overline{G} \subset \mathcal{A}_f$, we conclude that $\overline{G} = \mathcal{A}_f$, and that $K$ coincides with $Z$. This completes the proof of (i) and of (ii). We
also have (iii), since we have seen that the colimit $B$ is preserved by $\tilde{Z}$; so that $A_f$ is dense in $A$ by [\textsuperscript{5}] Thm 5.19, and $\tilde{Z}$ is fully faithful. We now have (iv), and also (v) because $A$ is reflective in the complete $[A_f^{op}, \mathcal{V}]$. As for (vi), we have already observed that each $\tilde{Z}A$ is left exact. For the converse, let $F: A_f^{op} \to \mathcal{V}$ be left exact. Then, by (6.11), $F$ is a filtered colimit of representables in $[A_f^{op}, \mathcal{V}]$. But the representables lie in the replete image of $\tilde{Z}$, since $\tilde{Z}Z \cong Y$; and this image is closed under filtered colimits in $[A_f^{op}, \mathcal{V}]$, since $\tilde{Z}$ is finitary.

(7.3) COROLLARY. For a cocomplete $\mathcal{A}$, the following are equivalent:

(i) $\mathcal{A}$ is locally finitely presentable;
(ii) $A_f$ is small and strongly generating;
(iii) $A_f$ is small and dense;
(iv) $A$ is a full reflective subcategory of some $[I, \mathcal{V}]$ with $I$ small and with the inclusion $A \to [I, \mathcal{V}]$ finitary. (Here the cocompleteness of $\mathcal{A}$ is automatic.)

(7.4) There are simple generalizations in which «finite» is replaced throughout - except in (5.12) - by «of cardinal $< \alpha$», where $\alpha$ is a small regular cardinal. An ordinary category is an $\alpha$-category if its set of morphisms has cardinal $< \alpha$. A cone or a conical limit is $\alpha$-small, or is an $\alpha$-cone or an $\alpha$-limit, if its indexing category is an $\alpha$-category. An ordinary category $\mathcal{L}$ is $\alpha$-filtered if every functor from an $\alpha$-category into $\mathcal{L}$ is the base of some inductive cone; and a (small) conical limit is $\alpha$-filtered if its indexing category is $\alpha$-filtered. Generalizing (1.2), $\alpha$-filtered colimits commute in $\mathcal{S}et$ with $\alpha$-limits. A $\mathcal{V}$-functor is $\alpha$-ary if it preserves $\alpha$-filtered colimits, and has a rank if it is $\alpha$-ary for some (small) $\alpha$; whereupon its rank is the least such $\alpha$. The object $G$ of $\mathcal{A}$ is $\alpha$-presentable if $\mathcal{A}(G, -)$: $A \to \mathcal{V}$ is $\alpha$-ary, and is presentable if it is $\alpha$-presentable for some $\alpha$. We write $A_\alpha$ for the full subcategory of $\alpha$-presentable objects; and as in (2.6), the $\alpha$-presentable sets are the $\alpha$-small ones. The $\mathcal{V}$-category $\mathcal{A}$ is locally $\alpha$-presentable if it is cocomplete and has a strong generator $G \subset A_\alpha$; and it is locally presentable if it is locally $\alpha$-presentable for some (small) $\alpha$. An indexing type $H: K \to \mathcal{V}$ is $\alpha$-small if $obK$ has fewer than $\alpha$ isomorphism classes,
each $K(k, k')$ lies in $\mathcal{O}_a$, and $H$ factorizes through $\mathcal{O}_a$; a limit indexed by such an $H$ is called an $a$-limit; when $\mathcal{U} = \mathcal{S}_{\text{et}}$ these reduce to the conical $a$-limits above; and for a general $\mathcal{U}$ with $\mathcal{O}_o$ l.f.p. they reduce as in (3.4) to conical $a$-limits and the $xh^*$ with $x \in \mathcal{O}_a$. A $\mathcal{U}$-category is $a$-complete if it admits $a$-limits, and a $\mathcal{U}$-functor is $a$-left-exact if it preserves them; the $a$-analogues of the results of Section 4 are all valid. When $\mathcal{U} = \mathcal{S}_{\text{et}}$ and $\mathcal{A}$ is locally $a$-presentable with strong generator $\mathcal{G} \subset \mathcal{A}_a$, the proof in (7.2) carries over, and in particular exhibits $\mathcal{A}_a$ as the closure of $\mathcal{G}$ in $\mathcal{A}$ under $a$-colimits. Applying this to the l.f.p. and hence locally $a$-presentable $\mathcal{O}_o$, we see that $\mathcal{O}_{o\alpha}$ is the closure of $\mathcal{O}_{o\alpha}$ under $a$-colimits. Always supposing that $\mathcal{U}$ satisfies the hypothesis (5.12), we conclude that $x, y \in \mathcal{O}_a$ if $x, y \in \mathcal{O}_a$; and since $l \in \mathcal{O}_{o\alpha} \subset \mathcal{O}_{o\alpha}$, we get from the $a$-versions of (5.2) and (5.3) the equivalence of $x \in \mathcal{O}_a$ with $x \in \mathcal{O}_a$. Now everything above carries over to the case of general $a$. There is one comment to be made: since every $A$ in the locally $a$-presentable $\mathcal{A}$ is a colimit of objects in $\mathcal{A}_a$, and since this colimit is small and hence $\beta$-small for some $\beta \geq a$, the object $A$ is, in fact, $\beta$-presentable - so that every object in $\mathcal{A}$ is presentable. We continue to write only of the finitary case, leaving the reader to make the easy generalizations - except when the result requires the general case.

(7.5) PROPOSITION. When $\mathcal{A}$ is l.f.p. so is $\mathcal{A}_o$, and $\mathcal{A}_{o\alpha} = \mathcal{A}_{f\alpha}$. Conversely, a cocomplete $\mathcal{A}$ is l.f.p. if $\mathcal{A}_o$ is l.f.p. and if $\mathcal{A}_{o\alpha} \subset \mathcal{A}_{f\alpha}$.

PROOF. When $\mathcal{A}$ is l.f.p., $\mathcal{A}_o$ is cocomplete since $\mathcal{A}$ is, by [*] Section 3.8. Moreover each $A \epsilon \mathcal{A}_o$ is by (7.2) a filtered colimit in $\mathcal{A}_o$ of objects $C \epsilon \mathcal{A}_{f\alpha}$; and these colimits are preserved by the $\mathcal{A}_o(C, -)$ with $C \epsilon \mathcal{A}_{f\alpha}$, since $\mathcal{A}_{f\alpha} \subset \mathcal{A}_{o\alpha}$ by (5.3). By [*] Thm 5.19, therefore, the small subcategory $\mathcal{A}_{f\alpha} \subset \mathcal{A}_{o\alpha}$ is dense, and a fortiori strongly generating, in $\mathcal{A}_o$; so that $\mathcal{A}_o$ is l.f.p. By (7.2) again, $\mathcal{A}_{o\alpha}$ is the closure of $\mathcal{A}_{f\alpha}$ in $\mathcal{A}_o$ under finite colimits; but this is $\mathcal{A}_{f\alpha}$ itself, since $\mathcal{A}_{f\alpha}$ is closed under finite colimits by (7.2) and (4.7).

Conversely, if $\mathcal{A}_o$ is l.f.p., and $\mathcal{A}$ is cocomplete with $\mathcal{A}_{o\alpha} \subset \mathcal{A}_{f\alpha}$, every $A \epsilon \mathcal{A}$ is the filtered colimit in $\mathcal{A}_o$, and hence in $\mathcal{A}$, of objects
$G \in \mathcal{A}_{of}$. Because $\mathcal{A}_{of} \subseteq \mathcal{A}_{of}$, this colimit is preserved by the $\mathcal{A}(G, \cdot)$ with $G \in \mathcal{A}_{of}$. By [*] Thm 5.19, therefore, the full subcategory of $\mathcal{A}$ determined by the objects in $\mathcal{A}_{of}$ is dense in $\mathcal{A}$; and it is contained in $\mathcal{A}_f$, whence $\mathcal{A}$ is l.f.p.

(7.6) **Proposition.** When $\mathcal{A}$ is l.f.p., a functor $T : \mathcal{A} \to \mathcal{B}$ is finitary exactly when $1 : TZ \to TZ$ exhibits $T$ as the left Kan extension of $TZ$ along the inclusion $Z : \mathcal{A}_f \to \mathcal{A}$. When, moreover, $\mathcal{B}$ admits filtered colimits, $\text{Lan}_Z H$ exists for all $H : \mathcal{A}_f \to \mathcal{B}$; and $\text{Lan}_Z$ gives an equivalence between $[\mathcal{A}_f, \mathcal{B}]$ and the $\mathcal{O}$-category $\text{Fin}[[\mathcal{A}, \mathcal{B}]]$ of all finitary functors $\mathcal{A} \to \mathcal{B}$, with the restriction $[Z, 1]$ along $Z$ as its equivalence-inverse.

**Proof.** Given (7.2), this follows from Thm 4.98, Thm 4.99, and Lemma 5.18, of [*].

(7.7) **Remark.** When $\mathcal{A}$ is l.f.p. so is $\mathcal{A}_o$ by (7.5); and then by (1.3), if $\mathcal{B}$ admits filtered colimits, $T : \mathcal{A} \to \mathcal{B}$ is finitary exactly when $T_o : \mathcal{A}_o \to \mathcal{B}_o$ is so. It follows from (7.6) that the identity map expresses $T$ as $\text{Lan}_Z TZ$ if and only if the identity map expresses $T_o$ as $\text{Lan}_{Z_o} T_o Z_o$. Thus for any $H : \mathcal{A}_f \to \mathcal{B}$ where $\mathcal{B}$ admits filtered colimits, we have $(\text{Lan}_Z H)_o = \text{Lan}_{Z_o} H_o$. When $\mathcal{B}$ is cocomplete, it therefore follows from [*] 4.2 an isomorphism

$$\int G \in \mathcal{A}_f \mathcal{A}(ZG, A) \Theta HG \cong \int G \in \mathcal{A}_f \mathcal{A}_o(ZG, A) \cdot HG$$

between functors $\mathcal{A}_o \to \mathcal{B}_o$. This strikes the author as surprising. When $\mathcal{O} = \mathcal{C}_{\text{at}}$ and $\mathcal{A} = \mathcal{O}$, for instance, both $\Theta$ and $\times$ are the cartesian product, if we regard $\mathcal{A}_o(ZG, A)$ as the discrete category formed by the objects of $\mathcal{A}(ZG, A)$; so that here the difference between $\mathcal{A}(ZG, A) \times HG$ and $\mathcal{A}_o(ZG, A) \times HG$ is exactly balanced out by the extra relations (involving the 2-cells) which occur in the passage in the quotient on the left side.

(7.8) **When $\mathcal{A}$ is l.f.p., a functor $S : \mathcal{A} \to \mathcal{B}$ is a left adjoint precisely when it is cocontinuous.**

**Proof.** Since $\mathcal{A}$ has by (7.2) the small dense subcategory $\mathcal{A}_f$, the result follows from [*] Thm 5.33.

(7.9) **When both $\mathcal{A}$ and $\mathcal{B}$ are locally presentable, a functor $T : \mathcal{A} \to \mathcal{B}$ has}
a left adjoint $S$ precisely when it is cocontinuous and has a rank.

**Proof.** For one direction, it suffices to illustrate by the case where $\mathcal{A}$ and $\mathcal{B}$ are l.f.p. and the continuous $T$ is finitary. Then $T$ has a left adjoint by [\*] Thm 5.32, since filtered colimits present $\mathcal{A}_f$ as dense in $\mathcal{A}$, since $\mathcal{B}$ is the closure under small colimits of $\mathcal{B}_f$, since $T$ preserves filtered colimits, and since each $\mathcal{B}(H, -)$ with $H \in \mathcal{B}_f$ also preserves filtered colimits.

For the other direction, it suffices to illustrate by the case where $\mathcal{A}$ and $\mathcal{B}$ are l.f.p. The small subcategory $S(\mathcal{B}_f)$ of $\mathcal{A}$ is then by (7.4) contained in $\mathcal{A}_a$ for some regular cardinal $a$. Then $S(\mathcal{B}_a) \subseteq \mathcal{A}_a$, since by the $a$-analogue of (7.2) the closure of $\mathcal{B}_f$ in $\mathcal{B}$ under $a$-colimits is $\mathcal{B}_a$, and $\mathcal{A}_a$ is closed under $a$-colimits. To show that $T$ preserves $a$-filtered colimits, it suffices by (3.6) to prove that all $\mathcal{B}(H, T-)$ with $H \in \mathcal{B}_a$ does so; but $\mathcal{B}(H, T-)$ $\subseteq \mathcal{A}(SH, -)$ does so because $SH \in \mathcal{A}_a$.

(7.10) Remark. We have now used the full strength of Hypothesis (5.12): as remarked in (4.15), the proof of (4.9) used Hypothesis (4.4), that $\mathcal{C}_o$ is l.f.p.; as remarked in (6.13), the proof of (6.9) used the part of $\mathcal{C}_o_f$ of (5.12); and now the remaining part $\mathcal{C}_o_f \subseteq \mathcal{C}_o_f$ of (5.12) has been used in the proof of (7.1).

**8. SOME PROPERTIES OF LOCALLY FINITELY PRESENTABLE $\mathcal{A}$ WHEN $\mathcal{C} = \mathcal{E}$**

(8.1) Because of (7.5), many of the properties of an l.f.p. $\mathcal{A}$ - including various characterizations of $\mathcal{A}_f$ - can be carried over directly from the classical case $\mathcal{C} = \mathcal{E}$, as studied by Gabriel-Ulmer in [7]. Because the present article is in other respects so near to being self-contained, we take the liberty of recalling some of these here, with sketches of their proofs.

(8.2) Because we are concerned in this section only with $\mathcal{A}$ that are co-complete and finitely complete, a regular epimorphism (in the sense of, say, [8]) is the same thing as a coequalizer of some pair of maps, being the coequalizer of its kernel-pair. The regular factorization $f = nq$ of a map $f: A \to B$ is its factorization through the coequalizer $q$ of the kernel-pair.
of $f$. When $\mathcal{G}$ has a generator $\mathcal{G}_f$, this $q$ is equally the coequalizer of the evident maps $\phi, \psi : \Sigma_{G \in \mathcal{G}} M(G, f) \cdot G \to A$, where $M(G, f)$ is the set of those pairs $u, v : G \to A$ with $fu = fv$. Because $\mathcal{G}$ is finitely complete, the strong epimorphisms of $[8]$ coincide with the extremal epimorphisms - those maps that factorize through no proper subobject of their codomain. The regular epimorphisms are extremal; and the converse is true if and only if the regular epimorphisms are closed under composition, which is further equivalent to the assertion that the $n$ of any regular factorization $f = nq$ is monomorphic (cf. [8]).

(8.3) (Cf. [7] Section 6.6) If $\mathcal{A}$ has a generator $\mathcal{G} \subseteq \mathcal{A}_f$ (not necessarily a strong one - $\mathcal{A}$ could be topological spaces, which is not I.f.p.), write $\mathcal{G}^\sigma$ for the closure of $\mathcal{G}$ in $\mathcal{A}$ under finite coproducts. Let $f : A \to B$ be a regular epimorphism with $A \in \mathcal{A}_f$. Then $B \in \mathcal{A}_f$ precisely when $f$ is the coequalizer of maps $u, v : H \to A$ with $H \in \mathcal{G}^\sigma$.

PROOF. One direction is trivial by (2.5). For the other, $f$ is by (8.2) the coequalizer of $\phi, \psi : K \to A$ where $K$ is a coproduct of objects of $\mathcal{G}$. The coprojections $i_H : H \to K$ of the finite sub-coproducts of its summands express $K$ as a filtered colimit of objects $H \in \mathcal{G}^\sigma$. If $B_H$ is the coequalizer of $\phi i_H$ and $\psi i_H$, we have induced maps $p_H : B_H \to B$; and these express $B$ as a filtered colimit, since colimits commute with colimits, and since the filtered colimit of the functor constant at $A$ is $A$. The connecting maps $B_{H'} \to B_H$, for $H \subseteq H'$ being clearly epimorphisms, and each $B_H$ being f.p. by (2.5), some $p_H$ is an isomorphism by (2.12).

(8.4) (Cf. [7] Section 6.6) For any $f : A \to B$ in $\mathcal{A}$, write $f = n_1 q_1$ for its regular factorization, where $n_1 : A_1 \to B$. Now let $n_1 = n_2 q_2$ be the regular factorization of $n_1$, with $n_2 : A_2 \to B$, and write $p_1$ for $q_1 : A \to A_1$ and $p_2$ for $q_2 q_1 : A \to A_2$. If we continue thus transfinitely, defining $p_\alpha : A \to A_\alpha$ as the colimit of the $p_\beta$ for $\beta < \alpha$ when $\alpha$ is a limit ordinal, we get a sequence of factorizations $n_\alpha p_\alpha$ of $f$, in each of which $p_\alpha$ is an extremal epimorphism. The sequence becomes stationary at some $\alpha$ if and only if $n_\alpha$ is monomorphic, and then $n_\alpha p_\alpha$ is the (unique) factorization of $f$ into an extremal epimorphism and a monomorphism. If $\mathcal{A}$ has a generator
$\mathcal{G} \subset \mathcal{A}_f$, the sequence becomes stationary at the first infinite ordinal $\omega$. For, in proving $n_\omega$ a monomorphism, it suffices to consider pairs $u, v: G \to A_\omega$ with $n_\omega u = n_\omega v$ and $G \in \mathcal{G}$. Because $G$ is f.p. and $A_\omega$ is the filtered colimit of the $A_i$ with $i < \omega$, such $u, v$ factorize through $A_i \to A_\omega$ for some $i < \omega$, say via $x, y: G \to A_i$. Now, since $n_i x = n_i y$, we have $q_{i+1} x = q_{i+1} y$, giving $u = v$ as desired.

It is clear from (8.2) that an object $A$ of such a category $\mathcal{A}$ has but a small set of regular-epimorphic quotients; and now it follows from the above that an object $A$ of such an $\mathcal{A}$ has but a small set of extremal epimorphic quotients. This is true in particular of any l.f.p. $\mathcal{A}$, and of any locally presentable $\mathcal{A}$ by a trivial extension.

(8.5) A generator $\mathcal{G}$ of $\mathcal{A}$ is said to be projective when each $\mathcal{G}(\mathcal{G}, -) : \mathcal{A} \to \mathcal{S}_{\text{et}}$ preserves regular epimorphisms. When this is so it follows easily that the $n$ in any regular factorization $nq$ is a monomorphism, so that regular and extremal epimorphisms coincide by (8.2). When the projective generator $\mathcal{G}$ is a strong generator, say with inclusion $K: \mathcal{G} \to \mathcal{A}$, the existence of the conservative, right-adjoint $\tilde{K}: \mathcal{A} \to [\mathcal{G}^{\text{op}}, \mathcal{S}_{\text{et}}]$ shows that $\mathcal{A}$ has the further property of being a regular category, in the sense that regular epimorphisms are stable under pullback; for this is trivially true in $\mathcal{S}_{\text{et}}$ and hence in $[\mathcal{G}^{\text{op}}, \mathcal{S}_{\text{et}}]$, and a conservative right adjoint clearly reflects extremal epimorphisms. The classical construction of regular quotients, via congruences, in the category of algebras of a (many-sorted) finitary algebraic theory, shows that the $\{F_x l\}$ of (3.5) constitute a projective strong generator, so that such a category is regular. We can infer that $\mathcal{C}_{\text{at}_\omega}$ is not such a category; for there the extremal epimorphism from 2 to the one-object category given by the monoid $\{1, e\}$ with $e^2 = e$ is not regular.

(8.6) We have from [7] Satz 7.14 a result stronger than that at the end of (8.4): if $\mathcal{A}$ is locally presentable, any $A \in \mathcal{A}$ has but a small set of epimorphic quotients - that is, $\mathcal{A}$ is cocomplete; and
the projection $d : A/\mathcal{Q} \to \mathcal{A}$ is conservative, has the left adjoint $B \rightarrow A + B$, and preserves all connected - and hence all filtered - colimits. Accordingly $A/\mathcal{Q}$ is locally $\alpha$-presentable if $\mathcal{A}$ is, by (3.3). Now $f : A \to B$ is epimorphic in $\mathcal{A}$ exactly when $f : I_A \to f$ is epimorphic in $A/\mathcal{Q}$; but $I_A$ is the initial object in $A/\mathcal{Q}$.

It suffices, then, to prove that the initial object $0$ of a locally-presentable $\mathcal{A}$ has but a small set of epimorphic quotients. Call an epimorphic image $K$ of $0$ an atom, and write $\mathcal{K}$ for the full subcategory of $\mathcal{A}$ given by the atoms. Clearly $\mathcal{K}$ is a preordered set, since any two maps $K \to B$ coincide if $K \in \mathcal{K}$. To prove $\mathcal{K}$ small it will more than suffice to prove $\mathcal{K}$ locally presentable; for then it is complete by (7.2) and well-powered by (8.6), and every object is a subobject of the terminal object. In fact, if we choose some regular $\alpha \geq N_1$ such that $\mathcal{A}$ is locally $\alpha$-presentable, then $\mathcal{K}$ too is locally $\alpha$-presentable.

First, $\mathcal{K}$ is closed under colimits in $\mathcal{A}$, since if each $0 \to K_i$ is epimorphic, so is $0 = \text{colim} 0 \to \text{colim} K_i$; hence $\mathcal{K}$ is cocomplete. We show that $\mathcal{K} \cap \mathcal{A}_{\alpha}$, which is clearly contained in $\mathcal{K}_{\alpha}$, is a strong generator for $\mathcal{K}$ - in fact, dense in $\mathcal{K}$. Then, since $\mathcal{K} \cap \mathcal{A}_{\alpha}$ is closed in $\mathcal{K}$ under $\alpha$-colimits, it is in fact by (7.2) (i) the whole of $\mathcal{K}_{\alpha}$. Write $Z : \mathcal{A}_{\alpha} \to \mathcal{A}$ and $Z' : \mathcal{K} \cap \mathcal{A}_{\alpha} \to \mathcal{A}$ for the inclusions; we shall show that for $K \in \mathcal{K}$ the comma-category $Z/K$ has $Z'/K$ as a final (full) subcategory - which will be $\alpha$-filtered since $Z/K$ is. Then, since $\mathcal{K}$ is by (7.2) the canonical $Z/K$-indexed colimit in $\mathcal{A}$ of the objects in $\mathcal{A}_{\alpha}$, it is also the canonical $Z'/K$-indexed colimit, in $\mathcal{A}$ and hence in $\mathcal{K}$, of the objects in $\mathcal{K} \cap \mathcal{A}_{\alpha}$. This latter $\alpha$-filtered colimit is preserved by $\mathcal{K}(H, -)$ for all $H \in \mathcal{K} \cap \mathcal{A}_{\alpha}$; so that by [\*] Thm 5.19 these colimits present $\mathcal{K} \cap \mathcal{A}_{\alpha}$ as dense in $\mathcal{K}$.

Since $Z/K$ is filtered, to prove the finality of $Z'/K$ it suffices by [\*] Prop. 4.71 to show that any $f : G \to K$ with $G \in \mathcal{A}_{\alpha}$ factorizes through some $H \in \mathcal{K} \cap \mathcal{A}_{\alpha}$. To say that $K$ is an atom is equally to say that the two coprojections $i_K, j_K : K \to K + K$ coincide. It follows that the composites of $i_G, j_G : G \to G + G$ with $f + f : G + G \to K + K$ coincide. Since $K$ is the $\alpha$-filtered colimit of all the $h : G' \to K$ with $G' \in \mathcal{A}_{\alpha}$, so $K + K$ is the $\alpha$-
filtered colimit of the $h + h: G' + G' \to K + K$. Since $\mathfrak{A}(G, -)$ preserves this colimit, $f$ is the composite of some $g_1: G \to G_1$ and some $f_1: G_1 \to K$, with $G_1 \in \mathfrak{A}_\alpha$, such that $(g_1 + g_1) i_G = (g_1 + g_1) i_{G_1}$. Now repeat this process, replacing $f: G \to K$ by $f_1: G_1 \to K$, to get $g_2: G_1 \to G_2$ with

$$(g_2 + g_2) i_{G_1} = (g_2 + g_2) i_{G_1}$$

and $f_2: G_2 \to K$ with $f_1 = f_2 g_2$; and so on. Finally let the colimit of

$$G \xrightarrow{g_1} G_1 \xrightarrow{g_2} G_2 \xrightarrow{g_3} G_3 \to \ldots$$

be $q_n: G_n \to H$, inducing $\tilde{f}: H \to K$. Then $H \in \mathfrak{A}_\alpha$, since $\mathfrak{A}_\alpha$ is closed under countable colimits. Moreover, since

$$i_H q_n = (q_n + q_n) i_{G_n} = (q_{n+1} + q_{n+1}) i_{G_n},$$

and since $(g_{n+1} + g_{n+1}) i_{G_n} = (g_{n+1} + g_{n+1}) i_{G_n}$, we have $i_H q_n = i_H q_n$ for all $n$; so that $i_H = i_H$, and $H \in \mathfrak{K}$. Thus $H \in \mathfrak{K} \cap \mathfrak{A}_\alpha$, as required.

(8.7) When we regard a small set $\mathfrak{G}$ of objects of $\mathfrak{A}$ as a discrete category $|\mathfrak{G}|$, with inclusion $Z: |\mathfrak{G}| \to \mathfrak{A}$, the counit $\epsilon$ of the adjunction

$$\ast \Rightarrow Z: \mathfrak{A} \to [|\mathfrak{G}|^{op}, \mathfrak{S}_{df}]$$

has components $\epsilon_A: \Sigma_{G \in \mathfrak{G}} \mathfrak{A}(G, A). G \to A$, whose components in turn are given by $\epsilon_A, G, f = f$ for $f: G \to A$. By the definitions of [\*] Section 3.6, $\mathfrak{G}$ is a generator [resp. a strong generator] when $\tilde{Z}$ is faithful [resp. conservative], which by [\*] Section 3.4 is the case precisely when each $\epsilon_A$ is an epimorphism [resp. an extremal epimorphism]. We call $\mathfrak{G}$ a regular generator when each $\epsilon_A$ is a regular epimorphism. Then, in the notation of (8.2), we have for each $A \in \mathfrak{A}$ a coequalizer diagram

$$\Sigma_{G \in \mathfrak{G}} M(G, \epsilon_A). G \xrightarrow{\delta} \Sigma_{G \in \mathfrak{G}} \mathfrak{A}(G, A). G \xrightarrow{\epsilon_A} A,$$

so that our definition of regular generator agrees with that in [\*] Section 4.10. It was observed there that each $\mathfrak{A}(G, -)$ with $G \in \mathfrak{G}$ preserves the colimit (8.8), sending it in fact to a split coequalizer diagram in $\mathfrak{S}_{df}$. Finally, a small dense full subcategory $\mathfrak{G}$ is a fortiori a regular generator (cf. [\*] Section 5.3), since we have the regular epimorphism
\( \sum_{G \in \mathcal{G}} \mathfrak{A}(G, A). G \to f^{G \in \mathcal{G}} \mathfrak{A}(G, A). G = A. \)

(8.9) We have as in [*] Prop. 5.24 that, if \( \mathcal{G} \) is a regular generator in \( \mathfrak{A} \), the closure \( \mathcal{G}^{\Sigma} \) of \( \mathcal{G} \) under small coproducts is dense in \( \mathfrak{A} \); this follows from [*] Thm 5.19, for the coequalizer diagrams (8.8) are preserved by the \( \mathfrak{A}(H, -) \) for \( H \in \mathcal{G}^{\Sigma} \), a product of split coequalizer diagrams being a split coequalizer diagram. More importantly, we have as in [*] Prop. 5.24 that, if \( \mathcal{G} \) is a regular generator contained in \( \mathcal{A}_f \) - so that \( \mathfrak{A} \) is l.f.p. - the closure \( \mathcal{G}^{\sigma} \) of \( \mathcal{G} \) under finite coproducts is already dense in \( \mathfrak{A} \); this follows from [*] Thm 5.19 because the infinite coproducts in (8.8) are the filtered colimits of their finite sub-coproducts, and these filtered colimits are preserved by the \( \mathfrak{A}(H, -) \) with \( H \in \mathcal{G}^{\sigma} \), since \( \mathcal{G}^{\sigma} \subset \mathcal{A}_f \) by (2.5).

\( \) In fact, \( \mathcal{G} \) itself is dense in \( \mathfrak{A} \) if coproducts in \( \mathfrak{A} \) are universal; for this, see [7] Satz 3.7.

(8.10) It is immediate that, if \( \mathcal{G} \) is a generator or a strong generator, so is any small \( \mathcal{G}' \supset \mathcal{G} \). For regular generators this is still true but no longer immediate. It suffices to show that, when the \( \epsilon_A \) of (8.7) are regular epimorphisms, so is \( (\epsilon_A, f): (\sum \mathfrak{A}(G, A). G) + C \to A \) for any \( f: C \to A \). However the composite of \( (\epsilon_A, f) \) with

\[
1 + \epsilon_C: (\sum \mathfrak{A}(G, A). G) + (\sum \mathfrak{A}(G, C). G) \to (\sum \mathfrak{A}(G, A). G) + C
\]

is, by the naturality of \( \epsilon \), the composite of \( \epsilon_A \) with

\[
(1, \sum \mathfrak{A}(G, f). G): (\sum \mathfrak{A}(G, A). G) + (\sum \mathfrak{A}(G, C). G) \to \sum \mathfrak{A}(G, A). G.
\]

Since the latter map is a retraction, its composite with \( \epsilon_A \) is a regular epimorphism by [8] Prop. 2.1; thus \( (\epsilon_A, f)(1 + \epsilon_C) \) is a regular epimorphism; and, since \( 1 + \epsilon_C \) is an epimorphism, \( (\epsilon_A, f) \) is a regular epimorphism by [8] Prop. 2.2.

(8.11) (Cf. [7] Satz 7.6) Let \( \mathcal{G} \subset \mathcal{A}_f \) be a regular generator of the l.f.p. \( \mathfrak{A} \), and write \( \mathcal{G}^e \) for the set of objects which are coequalizers of pairs \( u, v: H \to K \) with \( H, K \in \mathcal{G}^{\sigma} \). Then \( \mathcal{A}_f \) consists of the retracts of the objects of \( \mathcal{G}^e \); and \( \mathcal{A}_f \) coincides with \( \mathcal{G}^e \) if all extremal epimorphisms in \( \mathfrak{A} \) are regular.

\textbf{P R O O F.} By (2.5), \( \mathcal{A}_f \) certainly contains all the retracts of the objects of
For the converse, let $A \in \mathcal{G}_f$, and express $A$ as in (8.8) as the coequalizer of maps $\phi, \psi : P \to Q$ with $P, Q \in \mathcal{G}$. In the functor category $[\mathcal{G}, \mathcal{G}]$ consider the canonical cone of vertex $(\phi, \psi)$ given by all maps $(i, j) : (u, v) \to (\phi, \psi)$ of the form

$$
\begin{array}{ccc}
H & & K \\
\downarrow i & & \downarrow j \\
P & & Q \\
\phi & & \psi \\
\end{array}
$$

where $H, K \in \mathcal{G}$. It is easy to see that the comma category indexing this cone is filtered; for, since the coproduct $P$ is the filtered colimit of its finite sub-coproducts, every $f : B \to P$ with $B \in \mathcal{G}_f$ factorizes through one of these finite sub-coproducts, which lies in $\mathcal{G}$; and similarly for maps $g : B \to Q$. The same reasoning shows that the $i : H \to P$ which occur in this cone include all the coprojections from finite sub-coproducts of $P$; while the possibility of taking $H = 0$ shows that the $j : K \to Q$ which occur in this cone include all the coprojections from finite sub-coproducts of $Q$; and from these observations it follows that this filtered cone is a colimit-cone. If $A_{u,v} \in \mathcal{G}$ denotes the coequalizer of $u, v$ in the diagram above, we conclude that the coequalizer $A$ of $\phi, \psi$ is the filtered colimit of the $A_{u,v}$; so that, by (2.12), the f.p. $A$ is a retract of some $A_{u,v}$. If the composite extremal epimorphism $K \to A_{u,v} \to A$ is in fact a regular epimorphism, we have by (8.3) that $K \in \mathcal{G}$. 

(8.12) When $\mathcal{G}$ is the category of algebras for an $X$-sorted finitary algebraic theory, as in (2.8), it follows from (8.5) that extremal epimorphisms are regular and that the $\{ F_x 1 \}_{x \in X}$ constitute a regular generator. By (8.11), therefore, $\mathcal{G}_f$ consists of the coequalizers of the finite coproducts of the $F_x 1$, as stated in (2.7) and (2.8). When $\mathcal{G} = \mathcal{C}_{a_o}$, the ordinary category of small categories, it is easy to verify (cf. [*] Section 5.3) that the ordered set $3 = \{0, 1, 2\}$ is dense in $\mathcal{G}$. A fortiori the set $\mathcal{G}$ of all free categories $F_g$ on some finite graph $g$ is dense, since $3$ is such a category. Hence $\mathcal{G}$ is certainly a regular generator, and $\mathcal{G} \subset \mathcal{G}_f$ by (2.9). Thus, by
(8.11), any $A \in \mathcal{G}$ is a retract of an object of $\mathcal{G}_e$; so that there is some extremal epimorphism $p : F g \to A$. If $h$ is the image in $A$ of the finite graph $g$ under $p$, then $h$ too is a finite graph, with the same objects as $A$, which generates $A$, the induced $q : F h \to A$ being an extremal epimorphism. But such a $q$, which is the identity on objects, is easily seen to be a regular epimorphism. Hence by (8.3) we conclude that $A \in \mathcal{G}_e$ as asserted in (2.9).

Similar arguments apply to the category $\mathcal{G}_{pd_o}$ of small groupoids, and to the subcategories of $\mathcal{C}_o$ and $\mathcal{G}_{pd_o}$ mentioned in (3.5).

9. FINITARY ESSENTIALLY-ALGEBRAIC $\mathcal{C}$-THEORIES

(9.1) Whenever $\mathcal{C}_o$ is l.f.p., the closed category $\mathcal{C}$ is locally bounded in the sense of [*] Section 6.1, as is pointed out there; so that all the results of [*] Ch. 6 apply. Our aim now is to give more delicate results under our present stronger hypothesis of (5.12), that $\mathcal{C}$ is l.f.p. as a closed category. In addition, some of the results of [*] admit simpler proofs under this hypothesis.

(9.2) By a finitary essentially-algebraic $\mathcal{C}$-theory we mean a small finitely-complete $\mathcal{C}$-category $\mathcal{F}$. If $\mathcal{T}$ denotes the set of finite indexing types, this is the same thing as an $\mathcal{T}$-theory in the sense of [*] Section 6.3. We shall henceforth omit, as understood, the adjective "essentially-algebraic", and call $\mathcal{F}$ a finitary $\mathcal{C}$-theory, or just a finitary theory. A model of $\mathcal{F}$ in $\mathcal{B}$ is a left-exact functor $\mathcal{F} \to \mathcal{B}$; and these form a $\mathcal{C}$-category

$$\text{Mod}(\mathcal{F}, \mathcal{B}) = \text{Lex}(\mathcal{F}, \mathcal{B}) \subset [\mathcal{F}, \mathcal{B}].$$

A model of $\mathcal{F}$ in $\mathcal{C}$ is called a $\mathcal{F}$-algebra; and these form a $\mathcal{C}$-category

$$\mathcal{T} \text{-Alg} = \text{Mod}(\mathcal{T}, \mathcal{C}) = \text{Lex}(\mathcal{T}, \mathcal{C}) \subset [\mathcal{T}, \mathcal{C}].$$

Similarly, for a regular cardinal $\alpha$, we define an $\alpha$-ary $\mathcal{C}$-theory to be a small $\alpha$-complete $\mathcal{C}$-category $\mathcal{F}$; and now a model or an algebra is to be $\alpha$-left-exact. The results below for finitary theories extend at once to $\alpha$-ary ones.

(9.3) For any small $\mathcal{C}$-category $\mathcal{K}$, let $K : \mathcal{K} \to \overline{\mathcal{K}}$ be the full embedding into its free completion under finite limits; see [*] Section 5.7. Then, by
(7.1) above, $\overline{K}$ is a finitary theory. By [*] Thm 5.35, restriction along $K$ gives for every finitely-complete $B$ an equivalence $\text{Mod}[\overline{K}, B] \cong [K, B]$, whose equivalence-inverse is $\text{Ran}_K$ (right Kan extension along $K$). When $K$ is the unit $\mathcal{O}$-category $\mathbb{1}$, it follows from [*] Thm 5.35 together with (3.4) and (7.2) above that $\overline{K} = \mathcal{O}_{f}^{op}$; and now, since $\text{Mod}[\mathcal{O}_{f}^{op}, B] \cong B$ for any finitely-complete $B$, we may call $\mathcal{O}_{f}^{op}$ the finitary theory of an object.

(9.4) For any $\mathcal{O}$-theory $\mathcal{T}$, the representable $\mathcal{T}(t, -) : \mathcal{T} \to \mathcal{O}$ preserves all limits and is therefore a $\mathcal{T}$-algebra. We conclude from [*] Prop. 5.16 that the Yoneda embedding $\text{Y} : \mathcal{T}_{f}^{op} \to [\mathcal{T}, \mathcal{O}]$ factorizes through $\mathcal{T}_{-}\text{-Alg}$, say via the fully-faithful $Z : \mathcal{T}_{f}^{op} \to \mathcal{T}_{-}\text{-Alg}$; the full inclusion $\mathcal{T}_{-}\text{-Alg} \to [\mathcal{T}, \mathcal{O}]$ is isomorphic to $\tilde{Z}$ where $\tilde{Z} F = \mathcal{T}_{-}\text{-Alg}(Z -, F)$; and $Z$ is a dense embedding of $\mathcal{T}_{f}^{op}$ in $\mathcal{T}_{-}\text{-Alg}$. It also follows from (4.12) that $\mathcal{T}_{-}\text{-Alg}$ is closed in $[\mathcal{T}, \mathcal{O}]$ under filtered colimits, so that $\mathcal{T}_{-}\text{-Alg}$ admits filtered colimits and $\tilde{Z}$ is finitary. (We know from [*] Thm 6.11 that $\mathcal{T}_{-}\text{-Alg}$ is in fact reflective in $[\mathcal{T}, \mathcal{O}]$, and hence complete and cocomplete; but in the present hypotheses we get a simpler proof of this in (9.7) below.)

(9.5) By a map $M : \mathcal{T} \to \mathcal{T}'$ of finitary $\mathcal{O}$-theories we mean a left-exact functor. Since $FM : \mathcal{T} \to \mathcal{O}$ is left exact when $F : \mathcal{T}' \to \mathcal{O}$ is, the functor $[M, 1] : [\mathcal{T}', \mathcal{O}] \to [\mathcal{T}, \mathcal{O}]$ restricts to a functor $M^* : \mathcal{T}'_{-}\text{-Alg} \to \mathcal{T}_{-}\text{-Alg}$; such a functor between categories of algebras induced by a map of theories is called an algebraic functor. Note that $M^*$ is finitary; for $[M, 1]$ preserves all colimits, while $\mathcal{T}'_{-}\text{-Alg}$ and $\mathcal{T}_{-}\text{-Alg}$ are closed under filtered colimits. Now by (6.12), the left adjoint $\text{Lan}_M : [\mathcal{T}, \mathcal{O}] \to [\mathcal{T}', \mathcal{O}]$ of $[M, 1]$ restricts to a functor $M_* : \mathcal{T}_{-}\text{-Alg} \to \mathcal{T}'_{-}\text{-Alg}$ left adjoint to $M^*$. Since

$$\text{Lan}_M \mathcal{T}'(s, -) = (\hat{M} - \circ \mathcal{T}'(s, -)) \cong (\hat{M} -) s = \mathcal{T}(M s, -),$$

we have commutativity to within isomorphism in each square of the diagram

(9.6)

\[
\begin{array}{ccc}
\mathcal{T}^{op} & \xrightarrow{Z} & \mathcal{T}_{-}\text{-Alg} & \xrightarrow{\tilde{Z}} & [\mathcal{T}, \mathcal{O}] \\
M^{op} \downarrow & & M_* \downarrow & & \downarrow \text{Lan}_M \\
\mathcal{T}'^{op} & \xrightarrow{Z'} & \mathcal{T}'_{-}\text{-Alg} & \xrightarrow{\tilde{Z}'} & [\mathcal{T}', \mathcal{O}].
\end{array}
\]

The commutativity of the left square here is, by [*] Prop. 6.16, a general
fact about maps of f-theories for any \( F \) and any \( \mathcal{V} \); that of the right square, however, depending on (6.12), is special to l.f.p. \( \mathcal{V} \) and finitary (or more generally \( a \)-ary) theories.

(9.7) For the finitary theory \( F \), let \( K: F \to F \) be, as in (9.3), the embedding into the free finite-limit completion of \( F \). Since \( F \) is finitely complete, there is a left-exact \( L: F \to F \), unique to within isomorphism, for which \( LK \cong 1: F \to F \). The composite of \( L*: F-\text{Alg} \to F-\text{Alg} \) (which is just composition with \( L \)) and the equivalence \( F-\text{Alg} \cong [F, \mathcal{V}] \) of (9.3) (which is given by composition with \( K \)) is just the inclusion \( \tilde{Z}: F-\text{Alg} \to [F, \mathcal{V}] \). Since \( L* \) has a left adjoint by (9.5), so has \( \tilde{Z} \); so that \( F-\text{Alg} \) is a reflective full subcategory of \([F, \mathcal{V}]\).

(9.8) **Theorem.** For a finitary theory \( F \), the category \( F-\text{Alg} \) is l.f.p.; and \((F-\text{Alg})_f \) is the replete image of the embedding \( Z: F^{op} \to F-\text{Alg} \), and is hence equivalent to \( F^{op} \). Moreover a category \( \mathcal{A} \) is equivalent to \( F-\text{Alg} \) for some finitary theory \( F \) precisely when \( \mathcal{A} \) is l.f.p., whereupon \( F \cong \mathcal{A}^{op} \).

**Proof.** \( F-\text{Alg} \) is l.f.p. by (3.1) and (9.7), since \( \tilde{Z} \) is finitary by (9.4). The representables \( F(t, -) \) being f.p. in \([F, \mathcal{V}]\) by (2.2), their images under the left adjoint \( *Z \) of \( \tilde{Z} \) are f.p. in \( F-\text{Alg} \) by (2.4). But they are their own images, since the representables already lie in \( F-\text{Alg} \); whence \( F^{op} \subset (F-\text{Alg})_f \). Since \( F^{op} \) is dense in \( F-\text{Alg} \) by (9.4), it follows from (7.2) that \((F-\text{Alg})_f \) is the closure of \( F^{op} \) in \( F-\text{Alg} \) under finite colimits. This, however, is just the repletion of \( F^{op} \); for \( Z: F^{op} \to F-\text{Alg} \) preserves finite colimits. To see this, it suffices to observe that \( F-\text{Alg}(Z-, F): F \to \mathcal{V} \) preserves finite limits for each \( F \in F-\text{Alg} \); but \( F-\text{Alg}(Z-, F) \) is just \( \tilde{Z} F \), or \( F \) seen as an object of \([F, \mathcal{V}]\) - and the algebra \( F \) is by definition left exact. The final assertion now follows from (7.2).

(9.9) **Theorem.** Let \( F \) be a finitary theory and let \( \mathcal{B} \) be cocomplete. A functor \( S: F-\text{Alg} \to \mathcal{B} \) has a left adjoint \( T \) precisely when it is cocontinuous; and then its restriction \( G = SZ: F^{op} \to \mathcal{B} \) is a comodel for \( F \), in the sense that \( G^{op}: F \to \mathcal{B}^{op} \) is a model. Moreover \( 1: \mathcal{V} \to SZ \) then expresses \( S \) as \( \text{Lan}_Z G \). On the other hand, a given functor \( G: F^{op} \to \mathcal{B} \) is a comodel precisely when \( G: \mathcal{B} \to [F, \mathcal{V}] \) factorizes through the full em-
bedding $\tilde{Z}: \mathcal{I}\text{-Alg} \to [\mathcal{I}, \mathcal{V}]$, say as $G \cong \tilde{Z} T$; then $T$ has the left adjoint $S = \text{Lan}_Z G$, and $SZ \cong G$. In this way we obtain an equivalence of $\mathcal{V}$-categories

$$\text{Cocts}[\mathcal{I}\text{-Alg}, \mathcal{B}] = \text{Ladj}[\mathcal{I}\text{-Alg}, \mathcal{B}] \cong \text{Com}[\mathcal{I}^{\text{op}}, \mathcal{B}] = \text{Mod}[\mathcal{I}, \mathcal{B}^{\text{op}}]^{\text{op}},$$

the equivalence from left to right being composition with $Z$ and that from right to left being $\text{Lan}_Z$.

**Proof.** When account is taken of $[*]$ Thm 6.11, this is a special case of $[*]$ Thm 5.56. The latter result, however, is very general; accordingly, we give a simple direct proof in the present case. $Z$ being fully faithful, so is $\text{Lan}_Z$ by $[*]$ Thm 4.99, and $(\text{Lan}_Z G)Z \cong G$ for any $G$ by $[*]$ Prop. 4.23. We have therefore only to identify the image under $\text{Lan}_Z$ of the comodels $G$. Since $SZ$ is trivially a comodel if $S$ is cocontinuous, this image contains the cocontinuous $S$ by $[*]$ Thm 5.29; and a cocontinuous $S$ is left adjoint by (7.8) and (9.8). For the other direction, we observe that $G$ is a comodel when $G$ preserves finite colimits, which is to say that $\mathcal{V}(G \cdot, B): \mathcal{I} \to \mathcal{V}$ preserves finite limits for each $B \in \mathcal{B}$, or that $\tilde{G}: \mathcal{B} \to [\mathcal{I}, \mathcal{V}]$ factorizes through $\mathcal{I}\text{-Alg}$ as $\tilde{G} \cong \tilde{Z} T$ for some $T$. But then $T$ has as left adjoint $S$ the restriction $(- \cdot G)\tilde{Z} = \tilde{Z} - \cdot G$ of the left adjoint $- \cdot G$ of $\tilde{G}$; so that $S = \text{Lan}_Z G$ by $[*]$ Section 4.1. Thus for each model $G$, $\text{Lan}_Z G$ is cocontinuous.

(9.10) **Theorem.** Let $\mathcal{I}$ be a finitary theory, let $\mathcal{B}$ be cocomplete, and let the adjunction $S \dashv T: \mathcal{B} \to \mathcal{I}\text{-Alg}$ be related to the comodel $G: \mathcal{I}^{\text{op}} \to \mathcal{B}$ as in (9.9), by $S \cong \text{Lan}_Z G$ and $G \cong SZ$. Then $T$ is finitary if and only if $G(\mathcal{I}^{\text{op}}) \subset \mathcal{B}_f$.

**Proof.** If $T$ is finitary, (2.4) gives $S(\mathcal{I}\text{-Alg})_f \subset \mathcal{B}_f$, and hence $G(\mathcal{I}^{\text{op}}) \subset \mathcal{B}_f$, by (9.9) and (9.8). For the converse, since $Z$ is finitary and conservative, $T$ is finitary by (1.3) if $\tilde{G}$ is so. Since colimits in $[\mathcal{I}, \mathcal{V}]$ are formed pointwise, $\tilde{G}$ is finitary when each $E_t \tilde{G}: \mathcal{B} \to \mathcal{V}$ is finitary. However $E_t \tilde{G} = \mathcal{B}(Gt, -)$, which is finitary since $Gt \in \mathcal{B}_f$.

(9.11) **Proposition** (cf. [7] Satz 7.8). Suppose that the $G: \mathcal{I}^{\text{op}} \to \mathcal{B}$ of (9.10) not only takes its values in $\mathcal{B}_f$, but is fully faithful. Then $S$ is...
fully faithful, and its image is the closure of $G(\mathcal{T}^{op})$ in $\mathcal{B}$ under small colimits.

**Proof.** $S$ is fully faithful precisely when the unit $\eta: 1 \to TS$ is an isomorphism. Since both $T$ and $S$ are finitary, and since $\mathcal{T}^{op}$-$\text{Alg}$ is by (6.11) the closure of $\mathcal{T}^{op}$ under filtered colimits, it suffices to prove that $\eta Z: Z \to TSZ$ is an isomorphism; or equivalently that $\tilde{Z}\eta Z: \tilde{Z}Z \to \tilde{Z}TSZ$ is an isomorphism in $[\mathcal{J}, \mathcal{C}]$. But $\tilde{Z}Z \cong Y$, and $\tilde{Z}TSZ \cong \tilde{G}G$ by (9.9); and $\tilde{Z}\eta Z$ is the canonical map $Y \to \tilde{G}G$, which is an isomorphism (see [*] Section 4.2) if $G$ is fully faithful. Since $\mathcal{T}^{op}$-$\text{Alg}$ is the closure of $\mathcal{T}^{op}$ under filtered colimits, the image of $S$ is contained in the closure in $\mathcal{B}$ of $G(\mathcal{T}^{op})$ under small colimits, and therefore coincides with this closure.

(9.12) **Example.** (cf. [7], loc. cit.). With $\mathcal{C} = \text{Set}$, let $\mathcal{B}$ be the dual of the category of compact Hausdorff spaces. By [7] Section 6.5, $\mathcal{B}_f$ consists of the finite spaces. Taking $\mathcal{T} = \text{Set}_f = (\mathcal{B}_f)^{op}$ and $G$ to be the inclusion, we get by (9.11) an identification of $(\text{Set}_f$-$\text{Alg})^{op}$ with the small-limit closure in $\mathcal{B}^{op}$ of the finite spaces: that is, the category of totally-disconnected compact spaces. On the other hand, $\text{Set}_f$-$\text{Alg}$ is easily seen to be the category of boolean algebras.

(9.13) **Proposition.** If $\mathcal{T}$ and $\mathcal{T}'$ are finitary theories, a functor $T: \mathcal{T}'$-$\text{Alg} \to \mathcal{T}$-$\text{Alg}$ is algebraic - that is, of the form $M^*$ for some map $M: \mathcal{T}' \to \mathcal{T}$ of theories - precisely when it is finitary and has a left adjoint.

**Proof.** One direction comes from (9.5), and the other from (9.9) and (9.10) on taking $\mathcal{B} = \mathcal{T}'$-$\text{Alg}$.

(9.14) We can improve (9.13) by observing that every $\mathcal{V}$-natural transformation $M_1^* \to M_2^*$ between algebraic functors is induced by a unique $\mathcal{V}$-natural transformation $M_1 \to M_2$. In fact the situation is still richer than this. Denote by $\mathcal{V}$-$\text{Th}_k$ the ($\mathcal{V}$-$\text{Cat}$)-category whose objects are the finitary $\mathcal{V}$-theories and whose hom-object $\mathcal{V}$-$\text{Th}_k(\mathcal{J}, \mathcal{J}')$ is the $\mathcal{V}$-category $\text{Lex}[\mathcal{J}, \mathcal{J}]$, a full subcategory of $[\mathcal{J}, \mathcal{J}]$. Denote by $\mathcal{V}$-$\text{Lfp}_k$ the ($\mathcal{V}$-$\text{Cat}$)-category whose objects are the l.f.p. $\mathcal{V}$-categories and whose hom-object $\mathcal{V}$-$\text{Lfp}_k(\mathcal{A}, \mathcal{A}')$ is the full sub-$\mathcal{V}$-category of $[\mathcal{A}, \mathcal{A}']$ given by the finitary $\mathcal{V}$-functors with
left adjoints. Then (9.9) and (9.10) give an equivalence

\[(9.15) \quad \mathcal{V}\mathcal{T}\mathcal{r}(\mathcal{I}, \mathcal{I}') \cong \mathcal{V}\mathcal{L}_{\text{fp}}(\mathcal{I}'\text{-Alg}, \mathcal{I}\text{-Alg}) ; \]

so that, by (9.8), we have in fact a biequivalence

\[(9.16) \quad (\mathcal{V}\mathcal{T}\mathcal{r})^{op} = \mathcal{V}\mathcal{L}_{\text{fp}} . \]

The functor sending the \(\mathcal{V}\)-theory \(\mathcal{I}\) to \(\mathcal{I}\text{-Alg}\) may be called the semantics functor \(\text{Sem}\), and that sending the l.f.p. \(\mathcal{A}\) to \(\mathcal{A}_{f}^{op}\) may be called the structure functor \(\text{Str}\). We can in fact define \(\text{Str}\) on a category \(\mathcal{V}\text{-Fin}\) bigger than \(\mathcal{V}\mathcal{L}_{\text{fp}}\); the objects are now all cocomplete \(\mathcal{A}\) with \(\mathcal{A}_{f}\) small, and the hom-object is still the full subcategory of \([\mathcal{A}, \mathcal{A}']\) given by the finitary functors with left adjoints; this hom-object is in general not a \(\mathcal{V}\)-category, but a \(\mathcal{V}'\)-category for some larger \(\mathcal{V}'\) in a higher universe. If \(\text{Str}\mathcal{A}\) is still defined as \(\mathcal{A}_{f}^{op}\), we have a biadjunction

\[(9.17) \quad \text{Str} \dashv \text{Sem} : (\mathcal{V}\mathcal{T}\mathcal{r})^{op} \to \mathcal{V}\text{-Fin} , \]

since (9.9) and (9.10) give \(\mathcal{V}\mathcal{T}\mathcal{r}(\mathcal{I}, \mathcal{A}_{f}^{op}) \cong \mathcal{V}\text{-Fin}(\mathcal{A}, \mathcal{I}\text{-Alg}) .\]

(9.18) It follows from (7.5) that if \(\mathcal{I}\) is a finitary \(\mathcal{V}\)-theory, then \(\mathcal{I}_{0}\) is a finitary \(\text{Set}\)-theory and \((\mathcal{I}\text{-Alg})_{0} = \mathcal{I}_{0}\text{-Alg} .\) Using (6.11), (7.2), (7.5) and (9.8), we can describe the equivalence in detail: the \(\mathcal{I}\)-algebra \(F : \mathcal{I} \to \mathcal{V}\) is sent to the \(\mathcal{I}_{0}\)-algebra \(V F_{0} : \mathcal{I}_{0} \to \text{Set}\); while the equivalence-inverse sends the \(\mathcal{I}_{0}\)-algebra \(H : \mathcal{I}_{0} \to \text{Set}\) to the colimit \(F\) in \([\mathcal{I}, \mathcal{V}]\) of the canonical functor \(Y_{0} d^{op} : (e l H)^{op} \to [\mathcal{I}, \mathcal{V}]_{0}\) of (6.5). Not every finitary \(\text{Set}\)-theory need admit an enrichment to a finitary \(\mathcal{V}\)-theory, since an l.f.p. ordinary category need admit no enrichment to a \(\mathcal{V}\)-category. Thus, when \(\mathcal{V} = \text{Grpd}\), the l.f.p. category of all groups admits no additive structure. When \(\mathcal{V} = \text{Cat}\) or \(\text{Gpd}\), however, every finitary \(\text{Set}\)-theory \(\mathcal{S}\) admits at least one enrichment to a finitary \(\mathcal{V}\)-theory \(\mathcal{I}\); it suffices to take \(\mathcal{I}\) to be \(\mathcal{S}\), made into a 2-category with only identity 2-cells; then \(\mathcal{I}\) trivially admits the cotensor product \(2_Ht\), and is hence finitely complete as a 2-category and \textit{a fortiori} as a \(\text{Gpd}\)-category. Such an \(\mathcal{S}\) may admit two different enrichments to a finitary \(\mathcal{V}\)-theory: taking \(\mathcal{V} = \text{Cat}\) again, we have the \(\mathcal{V}\)-theory \(\text{Cat}_{f}^{op}\) of an object; and if \(\mathcal{S}\) is the underlying category \(\text{Cat}_{f}^{op}\) of this, the \(\mathcal{I}\) constructed above is not \(\text{Cat}_{f}^{op}\). In fact,
(\text{Cat}^{oP})\text{-Alg} = \text{Cat}, \text{while } \mathcal{T}\text{-Alg} \text{ is the 2-category of all small categories, all functors, but only identity 2-cells.}

10. FINITARY \mathcal{V}\text{-SKETCHES}

(10.1) By a finitary \mathcal{V}\text{-sketch } (\mathcal{S}, \Phi), \text{ or just } \Phi \text{ for short, we mean a small } \mathcal{V}\text{-category } \mathcal{S} \text{ together with a small set } \Phi = \{ \Phi_\gamma : \mathcal{H}_\gamma \to \mathcal{S}(A_\gamma, P_\gamma \cdot) \}_{\gamma \in \Gamma}, \text{ where the } \mathcal{H}_\gamma : \mathcal{K}_\gamma \to \mathcal{V} \text{ are finite indexing types, where } P_\gamma : \mathcal{K}_\gamma \to \mathcal{S}, \text{ and where } A_\gamma \in \mathcal{S}. \text{ A model of } \Phi \text{ in } \mathcal{B} \text{ is a functor } F : \mathcal{S} \to \mathcal{B} \text{ such that each cylinder}

$$H_\gamma \xrightarrow{\Phi_\gamma} \mathcal{S}(A_\gamma, P_\gamma \cdot) \xrightarrow{F} \mathcal{B}(FA_\gamma, FP_\gamma \cdot)$$

is a limit-cylinder in \mathcal{B}; \text{ and a model of } \Phi \text{ in } \mathcal{V} \text{ is called a } \Phi\text{-algebra; see } [*] \text{ Section 6.3. The results of } [*] \text{ Ch. 6 all apply; and in fact the proof of } [*] \text{ Thm 6.5 can be simplified in our present case, replacing the appeal to Theorem 10.2 of [9] by one to the simpler Theorem 6.2 of [9]. In particular, from } [*] \text{ Section 6.4, we get}

(10.2) \text{ THEOREM. Consider the composite of the Yoneda embedding } \mathcal{S}^{op} \to [\mathcal{S}, \mathcal{V}] \text{ and the reflexion of } [\mathcal{S}, \mathcal{V}] \text{ onto } \Phi\text{-Alg, and let } \mathcal{I}^{op} \text{ be the closure of its image in } \Phi\text{-Alg under finite colimits, so that we have a functor } K : \mathcal{S} \to \mathcal{I}. \text{ Then } K \text{ is a model of } \Phi, \text{ and is fully faithful precisely when the cylinders } \Phi_\gamma \text{ are already limi-cylinders in } \mathcal{S}. \text{ The sketch } \Phi \text{ and the finitary theory } \mathcal{I} \text{ have the same models in any finitely-complete } \mathcal{B}, \text{ restriction along } K \text{ giving an equivalence } \text{Mod}[\mathcal{I}, \mathcal{B}] \cong \Phi\text{-Mod}[\mathcal{S}, \mathcal{B}], \text{ whose equivalence-inverse is given by right Kan extension along } K.

(10.3) \text{ If } \mathcal{I} \text{ and } \mathcal{I}' \text{ are finitary } \mathcal{V}\text{-theories, we have as in } [*] \text{ Section 6.5 the sketch } (\mathcal{I} \Theta \mathcal{I}', \Phi), \text{ where } \Phi \text{ is so chosen that a } \Phi\text{-model in } \mathcal{B} \text{ is a functor } F : \mathcal{I} \Theta \mathcal{I}' \to \mathcal{B} \text{ with each } F(t, \cdot) \text{ and each } F(\cdot, s) \text{ left exact. The corresponding finitary theory in the sense of (10.2), which was denoted in } [*] \text{ by } \mathcal{I} \Theta_{\text{lex}} \mathcal{I}' \text{ when the appropriate set of indexing types was } \mathcal{I}, \text{ may be denoted here by } \mathcal{I} \Theta_{\text{lex}} \mathcal{I}'.

(10.4) \text{ PROPOSITION. If } \mathcal{A} \text{ is l.f.p. and } \mathcal{I} \text{ is a finitary theory, then }

\text{Mod}[\mathcal{I}, \mathcal{A}] \text{ is l.f.p.}
PROOF. $\mathcal{A}$ is equivalent by (9.8) to $\mathcal{T}'-\text{Alg}$ for some finitary theory $\mathcal{T}'$. Now $\text{Mod}[^J, \mathcal{A}] \cong \text{Mod}[^J, \text{Mod}[\mathcal{T}', \mathcal{O}]]$ is equivalent by [*] Section 6.5 to the l.f.p. category $(\mathcal{T} \otimes_{\text{lex}} \mathcal{T}')-\text{Alg}$.

(10.5) Let $\mathcal{F}$ and $\mathcal{F}'$ be finitary $\mathcal{O}$-theories as in (10.3), and let $K: \mathcal{F} \otimes \mathcal{F}' \rightarrow \mathcal{F} \otimes_{\text{lex}} \mathcal{F}'$ be the «generic model» of $\Phi$, as in (10.2). At the level of ordinary categories, let $(\mathcal{J}_o \times \mathcal{J}'_o, \Psi)$ be the sketch whose algebras in $\mathcal{C}$ are the functors $\mathcal{J}_o \times \mathcal{J}'_o \rightarrow \mathcal{C}$ left exact in each variable separately, and let $L: \mathcal{J}_o \times \mathcal{J}'_o \rightarrow \mathcal{J}_o \otimes_{\text{lex}} \mathcal{J}'_o$ be the generic model. Let $N: \mathcal{J}_o \times \mathcal{J}'_o \rightarrow (\mathcal{J} \otimes \mathcal{J}')_o$ be the canonical functor of [*] Section 1.4, and recall that, if $F: \mathcal{J} \otimes \mathcal{J}' \rightarrow \mathcal{A}$ is any $\mathcal{O}$-functor, the partial functors of $F_0 N$ are $(F(t, -))_o$ and $(F(-, s))_o$. We see at once that the composite $K_0 N: \mathcal{J}_o \times \mathcal{J}'_o \rightarrow (\mathcal{J} \otimes_{\text{lex}} \mathcal{J}')_o$ is left exact in each variable separately, and therefore factorizes (to within isomorphism) through $L$ to give a map $M: \mathcal{J}_o \otimes_{\text{lex}} \mathcal{J}'_o \rightarrow (\mathcal{J} \otimes_{\text{lex}} \mathcal{J}')_o$ of theories. This map is not an equivalence of theories: the theory $\mathcal{U}_f^\circ \circ p$ of an object is the identity for the $\otimes_{\text{lex}}$ of $\mathcal{O}$-theories, so that

$$(\mathcal{U}_f^\circ \circ p \otimes_{\text{lex}} \mathcal{U}_f^\circ \circ p)_o \cong \mathcal{U}_f^\circ \circ p = \mathcal{U}_f^\circ \circ p,$$

which is different from $\mathcal{U}_f^\circ \circ p \otimes_{\text{lex}} \mathcal{U}_f^\circ \circ p$; when $\mathcal{O} = \text{Cat}$, the first is the theory of a category, while the second is the theory of a double category.

More generally, we have for any finitary $\mathcal{O}$-theory $\mathcal{F}$ a map of theories

$$M: \mathcal{U}_f^\circ \circ p \otimes_{\text{lex}} \mathcal{J}_o \rightarrow (\mathcal{U}_f^\circ \circ p \otimes \mathcal{J})_o \cong \mathcal{J}_o$$

the corresponding functor $\mathcal{U}_f^\circ \circ p \times \mathcal{J}_o \rightarrow \mathcal{J}_o$ left exact in each variable is that sending $(x, t)$ to $x \circ t$. The relation between theory-maps

$$\mathcal{U}_f^\circ \circ p \otimes_{\text{lex}} S \rightarrow S$$

and $\mathcal{O}$-enrichments of the finitary $\mathcal{Sel}$-theory $\mathcal{S}$ will be discussed in the second part of this article.
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