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Structures defined by finite limits in the enriched context, I

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INTRODUCTION

Ehresmann's notion of the category of algebras defined by a (projective) sketch, along with the elementary properties of such categories and the basic relations between such categories, are generalized to the case of enriched categories in Chapter 6 of the author's forthcoming book [*]. That generalization, more precisely, is to the case of $\mathcal{V}$-categories where $\mathcal{V}$ is a symmetric monoidal closed category whose underlying ordinary category $\mathcal{V}_0$ is locally small, complete, and cocomplete, and where, moreover, $\mathcal{V}$ is locally bounded in the sense of [*] - as most base-categories of interest seem to be. In the present article we set about giving more precise results when the sketch, or the corresponding theory, is finitary: generalizing the results of Gabriel-Ulmer [7] on locally finitely presentable (ordinary) categories.

It turns out that finitariness has a good definition in the enriched case, leading to results analogous to the classical ones, only when the base category $\mathcal{V}$ itself is suitably special. The analogy works most perfectly when the ordinary category $\mathcal{V}_0$ is locally finitely presentable in the classical sense, and when the finitely-presentable objects of $\mathcal{V}_0$ are closed under the monoidal structure of $\mathcal{V}$; that is, when the unit-object $I$ of $\mathcal{V}$ is finitely presentable in $\mathcal{V}_0$, and when the object $x \otimes y$ is so whenever $x$ and $y$ are; in which case we say that $\mathcal{V}$ is locally finitely presentable as a closed category.

For such a $\mathcal{V}$, a $\mathcal{V}$-category $\mathcal{A}$ is the category of algebras for a
finitary $\mathcal{V}$-sketch precisely when $\mathcal{A}$ is \emph{locally finitely presentable as a $\mathcal{V}$-category}; and the corresponding finitary $\mathcal{V}$-theory $\mathcal{I}$ is $(\mathcal{A})^{\mathcal{V}}$, where $\mathcal{A}_{\mathcal{V}}$ is the full subcategory of $\mathcal{A}$ determined by the finitely-presentable objects - whereupon $\mathcal{A} = \mathcal{I}$. A functor $T: \mathcal{I} \to \mathcal{I}'$ is induced by a morphism $M: \mathcal{I} \to \mathcal{I}'$ of theories precisely when $T$ is finitary and admits a left adjoint $S$; whereupon $M^{\mathcal{V}}$ is the restriction of $S$ to the finitely-presentable objects. Here a \emph{finitary $\mathcal{V}$-theory} is a small $\mathcal{V}$-category $\mathcal{I}$ that is finitely-complete in the appropriate $\mathcal{V}$-enriched sense; a morphism of theories $\mathcal{I} \to \mathcal{I}'$ and a $\mathcal{I}$-algebra $\mathcal{I} \to \mathcal{V}$ are $\mathcal{V}$-functors which are \emph{left exact}, in that they preserve the appropriate finite $\mathcal{V}$-limits; and a $\mathcal{V}$-functor is \emph{finitary} if it preserves (classical, conical) filtered colimits.

The underlying ordinary category $\mathcal{I}_{\mathcal{V}}$ of a finitary $\mathcal{V}$-theory $\mathcal{I}$ is itself finitely complete in the classical sense; and $\mathcal{I}_{\mathcal{V}}$-Alg is precisely the underlying category $(\mathcal{I}_{\mathcal{V}})$ of $\mathcal{I}_{\mathcal{V}}$. A given classical finitary theory may or may not be of the form $\mathcal{I}_{\mathcal{V}}$ for some finitary $\mathcal{V}$-theory $\mathcal{I}$; and when it is so, $\mathcal{I}$ need not be unique.

Examples of such base-categories $\mathcal{V}$, other than $\text{Set}$, include $R$-modules, graded $R$-modules, and differential graded $R$-modules, for any commutative ring $R$; and the categories $\text{Cat}$, $\text{Spd}$, $\text{Ord}$ of (small) categories, groupoids, and ordered sets.

Since many of the arguments below are direct generalizations of what is true when $\mathcal{V} = \text{Set}$, it was very little extra trouble to make the article self-contained, assuming no prior knowledge of that classical case. We have therefore done this, but kept these references to the classical case brief, since so much detail is available in [7] and [4].

A second part of this article, to appear later, will study the special case of those finitary $\mathcal{V}$-theories which arise from finitary $\mathcal{V}$-monads; and will use this study to give a syntactic description of finitary $\mathcal{V}$-theories, as is done for the classical case in the thesis [3] of M. Coste or the book [13] of Makkai-Reyes.
Categories with essentially-algebraic extra structure are algebras for a finitary \( \mathcal{V} \)-theory \( \mathcal{F} \), where \( \mathcal{V} \) is usually \( \mathcal{C}at \) but sometimes \( \mathcal{C}at \). The morphisms in \( \mathcal{F}-\text{Alg} \) are those that preserve the structure strictly; and it is more natural to study a bigger category \( \mathcal{F}-\text{Alg}^* \), with the same objects, but with morphisms that preserve the structure only to within isomorphism. In a forthcoming paper [10], the author will examine the relation between \( \mathcal{F}-\text{Alg} \) and \( \mathcal{F}-\text{Alg}^* \), and combine this with the results of the present article, on the left adjoint of the \( \mathcal{V} \)-functor \( \mathcal{F}'-\text{Alg} \rightarrow \mathcal{F}-\text{Alg} \) induced by a map \( \mathcal{F} \rightarrow \mathcal{F}' \) of theories, to describe the left bi-adjoint of the corresponding functor \( \mathcal{F}'-\text{Alg}^* \rightarrow \mathcal{F}-\text{Alg}^* \).

0. REVISION OF NOMENCLATURE

Our general reference for enriched category theory is [\*]. We suppose that our chosen base-category \( \mathcal{V} \) is a (symmetric monoidal) closed category whose underlying category \( \mathcal{V}_0 \) is locally small, complete, and cocomplete. An important special case is that where \( \mathcal{V} \) is the cartesian-closed category \( \mathcal{S}et \) of small sets; note that a \( \mathcal{S}et \)-category is a locally-small ordinary category. The tensor product, unit object, and internal hom of \( \mathcal{V} \) are \( X \otimes Y \), \( I \), and \( [X,Y] \); and we use \( V: \mathcal{V}_0 \rightarrow \mathcal{S}et \) for the canonical representable functor \( \mathcal{V}_0 (I,-) \).

Recall from Section 1.3 of [\*] the careful distinction we make between a \( \mathcal{V} \)-category \( \mathcal{A} \) and the underlying ordinary category \( \mathcal{A}_0 \), which has the same objects but has \( \mathcal{A}_0 (A,B) = V (\mathcal{A} (A,B)) \); and between a \( \mathcal{V} \)-functor \( T: \mathcal{A} \rightarrow \mathcal{B} \) and its underlying functor \( T_0: \mathcal{A}_0 \rightarrow \mathcal{B}_0 \). It is precisely this notational distinction that allows us the simplification of writing «category» for «\( \mathcal{V} \)-category» and «functor» for «\( \mathcal{V} \)-functor» when the context makes clear which base-category is meant.

Recall too that a \( \mathcal{V} \)-category is small if the set of isomorphism classes of its objects is small. Then an ordinary category is small if it is a small \( \mathcal{S}et \)-category. We often use lower-case letters for the objects of small categories.

Finally, as is appropriate for enriched categories, we use the un-
qualified word limit to mean indexed limit in the sense of [*] Ch. 3, and similarly for colimit - except where $\mathcal{O} = \text{Set}$ and the context makes clear that we are referring to classical conical limits.

1. FILTERED COLIMITS AND FINITARY FUNCTORS

(1.1) By a filtered colimit in the $\mathcal{O}$-category $\mathcal{A}$ we mean the conical colimit in $\mathcal{A}$ (see [*] Section 3.8) of an ordinary functor $P : \mathcal{L} \to \mathcal{A}_0$, where the ordinary category $\mathcal{L}$ is small and filtered. Since $\text{colim} P$, if it exists in $\mathcal{A}$, is a fortiori the colimit of $P$ in $\mathcal{A}_0$, it follows that if $\mathcal{A}$ admits all filtered colimits, so does $\mathcal{A}_0$.

(1.2) The notion of a filtered category can be found in [*] Section 4.6, where it is shown that filtered colimits in $\text{Set}$ commute with finite (conical) limits.

(1.3) We call a $\mathcal{O}$-functor $T : \mathcal{A} \to \mathcal{B}$ finitary if $\mathcal{A}$ admits filtered colimits and $T$ preserves them. Clearly $T$ is finitary if and only if $T_0$ is so, provided that $\mathcal{A}$ and $\mathcal{B}$ admit filtered colimits. Of course a composite $TS$ is finitary if $T$ and $S$ are ; while, since a conservative (= isomorphism-reflecting) functor reflects such colimits as it preserves ([*] Section 3.6), $S$ is finitary if $TS$ is finitary and $T$ is finitary and conservative.

(1.4) Let the fully-faithful $T : \mathcal{A} \to \mathcal{B}$ have the left adjoint $S$, where $\mathcal{B}$ (and hence $\mathcal{A}$) admits filtered colimits. Then $T$ is finitary if and only if $TS$ is so.

PROOF. For the non-trivial part let $\text{colim} P$ be a filtered colimit in $\mathcal{A}$. We have isomorphisms

$$\text{colim} T P \cong \text{colim} T S T P \cong T S \text{colim} T P \cong T \text{colim} S T P \cong T \text{colim} P,$$

the first because $ST \cong 1$, the second because $TS$ is finitary, the third because $S$ is left adjoint, and the fourth because $ST \cong 1$; and the composite isomorphism is easily verified to be the canonical map $\text{colim} T P \Rightarrow T \text{colim} P$.

2. FINITELY-PRESENTABLE OBJECTS

(2.1) We shall call the object $G$ of the $\mathcal{O}$-category $\mathcal{A}$ finitely-presentable (or f.p.) if the representable $\mathcal{O}$-functor $\mathcal{A}(G, -) : \mathcal{A} \to \mathcal{O}$ is finitary. We
write $\mathbb{A}_f$ for the full subcategory of $\mathbb{A}$ given by the f.p. objects.

(2.2) EXAMPLES. The unit object $I$ is f.p. in the $\mathcal{C}$-category $\mathbb{C}$, since

$$\mathbb{C}(I, -) = [I, -] \cong I : \mathcal{C} \to \mathbb{C}. $$

If $x$ and $y$ are f.p. objects in $\mathbb{C}$, so is $x \otimes y$, since

$$[x \otimes y, -] \cong [x, [y, -]].$$

More generally, if $x \in \mathbb{C}_f$ and $G \in \mathbb{A}_f$, the tensor product $x \otimes G$ is f.p. if it exists; for $\mathbb{A}(x \otimes G, -) \cong [x, \mathbb{A}(G, -)]$. For any small $\mathcal{C}$-category $\mathcal{F}$, the representable $Y_t = \mathcal{F}(t, -)$ is f.p. in $[\mathcal{F}, \mathcal{C}]$, since $[\mathcal{F}, \mathcal{C}](Y_t, -)$ is isomorphic by Yoneda to the evaluation $E_t : [\mathcal{F}, \mathcal{C}] \to \mathcal{C}$, which preserves all small colimits by [§] Section 3.6. More generally, if $G \in \mathbb{A}_f$ and if the tensor product $\mathcal{F}(t, s) \otimes G$ exists in $\mathbb{A}$ for all $t, s \in \mathcal{F}$, the object $Y_t \otimes G = \mathcal{F}(t, -) \otimes G$ is f.p. in $[\mathcal{F}, \mathbb{A}]$; for now Yoneda gives

$$[\mathcal{F}, \mathbb{A}](Y_t \otimes G, -) \cong \mathbb{A}(G, E_t \cdot -).$$

In fact the finite presentability of $Y_t \otimes G$ could also be deduced from that of $Y_t$ and

(2.4) PROPOSITION. If $S \to T : \mathbb{B} \to \mathbb{C}$ where $T$ is finitary, we have $S(\mathbb{B}_f) \subset \mathbb{A}_f$.

PROOF. For $G \in \mathbb{B}_f$ we have $\mathbb{A}(SG, -) \cong \mathbb{B}(G, T \cdot -)$, the composite of the finitary functors $T$ and $\mathbb{B}(G, -)$.

(2.5) It follows from (1.2) that, when $\mathcal{C} = \mathbb{Set}$ and $\mathbb{A}$ admits filtered colimits, the category $\mathbb{A}_f$ is closed in $\mathbb{A}$ under finite (conical) limits.

Before we can exhibit other «suitable» $\mathcal{C}$ for which an analogue of this is true, we need more examples of finite presentability in the case $\mathcal{C} = \mathbb{Set}$: to which we now restrict ourselves up to the end of (2.9) below.

(2.6) $\mathbb{Set}$ is the category of finite sets.

PROOF. $1 \in \mathbb{Set}$ being f.p. by (2.2), so is every finite set $n$ by (2.5). For the converse, since every set $X$ is the filtered colimit of the diagram $n \to X$ of its finite subsets, we have $\mathbb{Set}(X, X) \cong \operatorname{colim} \mathbb{Set}(X, n)$, if $X$ is f.p.; but then $1 : X \to X$ factorizes as $X \to n \to X$ for some finite $n$, whence $X$ is finite.
(2.7) Let $\mathcal{A}$ be the category of algebras for a one-sorted finitary algebraic theory in the sense of Lawvere [11], and let $U: \mathcal{A} \to \mathbf{Set}$ be the underlying-set functor. It was observed in [*] Section 4.6 that a filtered colimit in $\mathcal{A}$ is formed by taking the colimit of the underlying sets and giving this the algebra structure which it inherits by (1.2). Thus $U$ is finitary: so that, if $F \to U$, the free algebra $F_n$ on a finite set $n$ is f.p. by (2.6) and (2.4). By (2.5), an algebra $A$ is f.p. if it is the coequalizer of two maps $F_m \xrightarrow{\sim} F_n$ with $m$ and $n$ finite. [It will follow from (8.12) below that every f.p. algebra has this form: so that our definition of finite presentability agrees here with the classical one of universal algebra.]

(2.8) The results are similar when $\mathcal{A}$ is the category of algebras for a many-sorted finitary algebraic theory in the sense of Bénabou [2]. If $X$ is the set of sorts, the forgetful functor $U: \mathcal{A} \to \mathbf{Set}^X$ is finitary, since its component $U_x: \mathcal{A} \to \mathbf{Set}$ is so for each $x \in X$. Hence $F_x n$ is f.p. for finite $n$, where $F_x \to U_x$. A two-sorted example is the category of (small) graphs, where by a graph $A$ we mean a diagram $A_1 \to A_0$. The graph $A$ is called finite if both $A_0$ and $A_1$ are finite; and it follows much as in (2.6) - essentially because the theory of graphs has no axioms - that the f.p. graphs are exactly the finite ones. Two $\aleph_0$-sorted examples are the categories of graded and of differential graded $R$-modules; it follows from the above that $A = (A_i)_{i \in \mathbb{Z}}$ is f.p. if each $A_i$ is f.p. and if $A_i = 0$ for all but a finite number of $i$. [It will follow from (8.12) below that these are the only f.p. objects.]

(2.9) The structure of a category or of a groupoid, not being given by operations defined on a finite product, is not algebraic in the sense of (2.8); it is however what Freyd [6] called essentially algebraic, in that it can be described in terms of finite limits; more precisely, as was first pointed out by Ehresmann ([5]; see also [1]), both categories and groupoids are algebras for a finitary essentially-algebraic theory in the sense of [*] Section 6.3 - which theories are the case $\mathcal{O} = \mathbf{Set}$ of our present object of study in this article. It follows easily using (1.2) that the forgetful functor
from the category of (small) categories [resp. groupoids] to that of graphs is finitary; and hence that a category [resp. a groupoid] is f.p. if it is the coequalizer of a diagram \( Fg \xrightarrow{g} Fh \), where \( Fg \) and \( Fh \) are the free categories [resp. groupoids] on the finite graphs \( g \) and \( h \). {Once again it will follow from (8.12) below that these are the only f.p. objects.} In particular the category \( 2 = (0 \to 1) \) and the groupoid \( I = (0 \xrightarrow{>} 1) \) are f.p.; the set-valued functors they represent send a category or a groupoid to its set of morphisms, and are not only finitary but also conservative.

(2.10) We now return to the case of a general \( \mathcal{C} \), and consider a \( \mathcal{C} \)-category \( \mathcal{A} \) admitting filtered colimits. For an object \( G \) of \( \mathcal{A} \), we must carefully distinguish between its finite presentability in \( \mathcal{A} \) - the finitariness of \( \mathcal{A}(G, \cdot) : \mathcal{A} \to \mathcal{C} \), or equally of \( \mathcal{A}_o(G, \cdot) : \mathcal{A}_o \to \mathcal{C}_o \) - and its finite presentability in \( \mathcal{A}_o \) - the finitariness of \( \mathcal{A}_o(G, \cdot)_o : \mathcal{A}_o \to \mathcal{S}_\text{et} \), which is the composite of \( \mathcal{A}(G, \cdot)_o \) with \( V : \mathcal{C}_o \to \mathcal{S}_\text{et} \). Neither implies the other in general, so that neither of \( \mathcal{A}_f_0 \) and \( \mathcal{A}_o_f \) need contain the other; and this even for \( \mathcal{A} = \mathcal{C} \).

(2.11) As an example of this, let \( M \) be a group, and let \( \mathcal{O} \) be the category \( \mathcal{O}_o = [M, \text{Set}] \) of \( M \)-sets with its cartesian closed structure. It follows easily from (2.7) that \( G \in \mathcal{O} \) is f.p. in \( \mathcal{O}_o \) if and only if (i) the set of orbits of \( G \) is finite, and (ii) the stabilizer of each \( g \in G \) is a finitely-generated group. On the other hand, since the finitary and conservative forgetful functor \( U : \mathcal{O}_o \to \mathcal{S}_\text{et} \) is represented by the \( M \)-set \( M \), the object \( G \) is f.p. in \( \mathcal{O} \) precisely when \( \mathcal{O}_o(M, [G, \cdot]) \) is finitary; that is, when \( \mathcal{O}_o(M \times G, \cdot) \) is finitary, or when \( M \times G \) is f.p. in \( \mathcal{O}_o \); which by (i) and (ii) above is the case exactly when \( G \) is finite. Thus the unit object \( 1 = 1 \) is f.p. in \( \mathcal{O} \), as it must be by (2.2); but it is f.p. in \( \mathcal{O}_o \) only when \( M \) is finitely generated as a group. On the other hand the \( M \)-set \( M \), since it represents \( U \), is always f.p. in \( \mathcal{O}_o \); but it is f.p. in \( \mathcal{O} \) only when \( M \) is finite.

(2.12) In the case \( \mathcal{O} = \mathcal{S}_\text{et} \), let \( q_l : P l \to C \) be the colimit of \( P : \mathcal{L} \to \mathcal{A} \), where \( \mathcal{L} \) is small and filtered. Then if \( C \) is f.p., some \( q_l \) is a retraction. If, moreover, every \( P l \) is f.p., and if \( P \phi \) is epimorphic for every \( \phi : l \to m \)
then some $q_l$ is an isomorphism.

**Proof.** Since $C$ is f.p. the $\mathcal{A}(C, q_l): \mathcal{A}(C, Pl) \to \mathcal{A}(C, C)$ constitute a colimit cone in $\mathcal{S}et$, so that $l_C = q_l i$ for some $l$ and for some $i: C \to Pl$. For such an $i$ we have

$$q_l i q_l = q_l 1: Pl \to C;$$

and since the $\mathcal{A}(Pl, q_m): \mathcal{A}(Pl, Pm) \to \mathcal{A}(Pl, C)$ also constitute a colimit in $\mathcal{S}et$ if $Pl$ is f.p., it follows (see [*] Thm. 4.72) that there is some $\phi: l \to m$ in $\mathcal{L}$ with $P\phi . i q_l = P\phi$. Since $q_l = q_m . P\phi$, this gives

$$P\phi . i q_m . P\phi = P\phi,$$

whence $P\phi . i q_m = 1$

since $P\phi$ is epimorphic. But $q_m . P\phi . i = q_l . i = 1$; so that $q_m$ is an isomorphism.

3. **Locally Finitely Presentable Categories**

(3.1) **Proposition.** For a cocomplete $\mathcal{V}$-category $\mathcal{A}$, the following are equivalent:

(i) $\mathcal{A}$ has a (small) strong generator $\mathcal{G} \subseteq \mathcal{A}$;

(ii) there is a small $\mathcal{G}$ and a strongly-generating $K: \mathcal{G} \to \mathcal{A}$ with $K(\mathcal{G}) \subseteq \mathcal{A}$;

(iii) there is a small $\mathcal{G}$ and a right-adjoint, finitary, conservative functor $T: \mathcal{A} \to [\mathcal{G}^{op}, \mathcal{V}]$.

**Proof.** (i) and (ii) are equivalent since, by [*] Section 3.6, the full image of a strongly generating functor is a strong generator. By [*] Thm 4.51, there is an equivalence between functors $K: \mathcal{G} \to \mathcal{A}$ and right adjoints $T: \mathcal{A} \to [\mathcal{G}^{op}, \mathcal{V}]$, given by $T = \tilde{K}$ where $\tilde{K} A = \mathcal{A}(K-, A)$. By definition, $K$ is strongly generating exactly when $\tilde{K}$ is conservative; moreover, since small colimits in $[\mathcal{G}^{op}, \mathcal{V}]$ are formed pointwise, $\tilde{K}$ is finitary exactly when $K(\mathcal{G}) \subseteq \mathcal{A}$.

(3.2) We shall call a $\mathcal{V}$-category $\mathcal{A}$ locally finitely presentable (or l.f.p.) when it is cocomplete and satisfies the equivalent conditions of (3.1).

(3.3) It follows from (3.1) (iii) that, if $I: \mathcal{B} \to \mathcal{A}$ is right-adjoint, finitary and conservative, and if $\mathcal{A}$ is l.f.p., then $\mathcal{B}$ is l.f.p., provided that $\mathcal{B}$ is
cocomplete. Since the cocompleteness of \( B \) follows from that of \( A \) if the right-adjoint \( J \) is fully faithful, we conclude that a reflective full subcategory \( B \) of an l.f.p. \( A \) is l.f.p. if the inclusion \( f: B \to A \) is finitary.

(3.4) EXAMPLES. The \( \mathcal{O} \)-category \( \mathcal{O} \) itself is always l.f.p., since the object \( I \), which is f.p. by (2.2), is (not merely a strong generator but) dense, by (5.17) of [*]. More generally the functor category \( [\mathcal{T}, \mathcal{O}] \) is l.f.p. for a small \( \mathcal{T} \), since the representables \( \mathcal{T}(t, -) \) are f.p. by (2.2) and dense by Proposition 5.16 of [*]. Still more generally, if \( A \) is l.f.p. with strong generator \( G \subset A_f \), then \( [\mathcal{T}, A] \) is l.f.p.; for the \( \{ \mathcal{T}(t, -) \otimes G \}_{t \in \mathcal{T}, G \in G} \), which are f.p. by (2.2), form a strong generator by (2.3), the \( A(G, E_t, -) \) jointly reflecting isomorphisms.

(3.5) EXAMPLES OF L.F.P. CATEGORIES WHEN \( \mathcal{O} = \text{Set} \). The category of algebras for a finitary algebraic theory, one-sorted or many-sorted, is l.f.p.; the objects \( \{ F_{xI} \}_{x \in X} \) of (2.8) are f.p., and constitute a strong generator - indeed, a regular one. The category of small categories is l.f.p. by (2.9), the f.p. object \( 2 \) being a strong generator; and the same argument applies to the full subcategories of preordered sets and of ordered sets. The category of small groupoids is again l.f.p. by (2.9), the f.p. object \( I \) being a strong generator; and the same argument applies to the full subcategory of sets-with-an-equivalence-relation.

(3.6) By applying the last remark of (1.3) to the finitary and conservative \( A \to [\mathcal{G}^\text{op}, \mathcal{O}] \) of (3.1), we deduce that if \( A \) is l.f.p. with the strong generator \( G \subset A_f \), the functors \( A(G, -) : A \to \mathcal{O} \) for \( G \in G \) jointly reflect filtered colimits; so that \( S: B \to A \) is finitary if each \( A(G, S-): B \to \mathcal{O} \) is so.

(3.7) In the classical case \( \mathcal{O} = \text{Set} \), let \( A \) be an l.f.p. category, and suppose that \( A \) admits finite (conical) limits - which is in fact automatically true by (7.2) below. Then finite limits commute with filtered colimits in \( A \).

PROOF. We are asserting that the canonical map

\[
\rho: \text{colim}_l \lim_p F(p, l) \to \lim_p \text{colim}_l F(p, l)
\]

is an isomorphism, where \( F: P \times \mathcal{L} \to A \) with \( P \) finite and \( \mathcal{L} \) filtered. If \( G \subset A_f \) is a strong generator for \( A \), it suffices to show that \( A(G, \rho) \) is
an isomorphism for each $G \in \mathfrak{G}$. Since $\mathfrak{A}(G, -) : \mathfrak{A} \to \mathcal{S}_{\text{set}}$ preserves finite limits and filtered colimits, this follows from (1.2).

(3.8) In the case $\mathcal{C} = \mathcal{S}_{\text{set}}$, let $\mathfrak{A}$ be a category with finite limits and filtered colimits. Then the $n$-th power functor $(^n) : \mathfrak{A} \to \mathfrak{A}$ is finitary for each finite $n$, if either $\mathfrak{A}$ is l.f.p. or $\mathfrak{A}$ is cartesian closed.

**Proof.** Since the diagonal $\Delta : \mathfrak{A} \to \mathfrak{A}^n$ is left adjoint and hence finitary, it suffices to show that the product functor $\Pi : \mathfrak{A}^n \to \mathfrak{A}$ is finitary. If $\mathfrak{A}$ is l.f.p., we have this by (3.7). If $\mathfrak{A}$ is cartesian closed, let $P : \mathfrak{L} \to \mathfrak{A}^n$ have components $P_i : \mathfrak{L} \to \mathfrak{A}$, where $\mathfrak{L}$ is filtered. Since the diagonal $\Delta : \mathfrak{L} \to \mathfrak{L}^n$ is final by [**] Thm 4.10, we have

$$\text{colim} \Pi P \cong \text{colim}_{l \in \mathcal{L}^n} (P_1 \times \cdots \times P_n \times l_n);$$

and since $\times A : \mathfrak{A} \to \mathfrak{A}$ preserves all colimits (or more trivially when $n = 1$ or 0), this is isomorphic to $\text{colim} P_1 \times \cdots \times \text{colim} P_n = \prod \text{colim} P$.

(Note that, by the example in Section 3.3 of [9], $(^2) : \mathfrak{A} \to \mathfrak{A}$ is not finitary when $\mathfrak{A}$ is the category of topological spaces.)

4. **FINITE INDEXED LIMITS.**

(4.1) An indexing-type $H : K \to \mathcal{O}$ (cf. [*] Section 3.1) shall be called finite if

(i) the set of isomorphism classes of $\text{ob} K$ is finite,

(ii) for each $k, k' \in K$, we have $K(k, k') \in \mathcal{O}_f$,

(iii) $H$ factorizes through the inclusion $\mathcal{O}_f \subset \mathcal{O}$.

A finite limit (or colimit) is one whose indexing-type is finite.

(4.2) It follows from (2.2) that, if $H : K \to \mathcal{O}$ and $H' : K' \to \mathcal{O}$ are finite, so is the functor $H \circ H' : K \circ K' \to \mathcal{O}$ sending $(k, k')$ to $H k \circ H' k'$. It then follows from (3.18) of [*] that a repeated finite limit $\{H ?, \{H' ? , P (?, ?)\}$ is a finite limit $\{H \circ H', P\}$.

(4.3) **Proposition.** A $\mathcal{O}$-category $\mathfrak{A}$ admits all finite (indexed) limits if it admits all conical limits indexed by finite ordinary categories and all cotensor products $x H A$ with $x \in \mathcal{O}_f$. The converse is true if $(^n) : \mathcal{O}_o \to \mathcal{O}_o$ is finitary for all finite $n$, and hence by (3.8) if $\mathcal{O}_o$ is l.f.p. or if $\mathcal{O}$ is
cartesian closed.

PROOF. Let \( H : K \to \mathcal{V} \) be a finite indexing-type and let \( T : K \to \mathcal{A} \). Replacing \( K \) if necessary by an equivalent category, we may suppose \( \text{ob } K \) to be in fact finite. Since each \( H k \in \mathcal{V}_f \), as does each \( K(k, k') \), we have the cotensor products \( H k \otimes T k' \) and \( K(k, k') \otimes (H k \otimes T k') \). Since \( \mathcal{A} \) admits finite products and equalizers, it admits the equalizer

\[
\bigcap_{k \in K} H k \otimes T k \
\to \prod_{k \in K} H k \otimes T k \quad \cap \quad \prod_{k, k' \in K} K(k, k') \otimes (H k \otimes T k'),
\]

which by (3.68) of [*] is the indexed limit \( \{H, T\} \).

For the converse, \( x \otimes A \) is by [*] Section 3.7 the indexed limit \( \{x, A\} \), where \( x \) and \( A \) are identified with functors \( x : A \to \mathcal{V} \) and \( A : A \to \mathcal{A} \).

Since \( \text{ob } A = \{0\} \) and since \( \mathcal{V}(0, 0) = I \) is f.p. in \( \mathcal{V} \) by (2.2), \( x : A \to \mathcal{V} \) is a finite indexing-type if \( x \in \mathcal{V}_f \); thus \( x \otimes A \) exists. Next, by [*] Section 3.8, the conical limit in \( \mathcal{A} \) of an ordinary functor \( S : P \to \mathcal{A} \) is the indexed limit \( \{\Delta I, S\} \), where \( \Delta : \mathcal{V}_f \to \mathcal{V} \) is the \( \mathcal{V} \)-functor corresponding to \( S \), its domain \( \mathcal{V}_f \) being the free \( \mathcal{V} \)-category on the ordinary category \( P \), and where \( \Delta I : \mathcal{V}_f \to \mathcal{V} \) corresponds similarly to the ordinary functor \( \Delta I : P \to \mathcal{V}_f \) constant at \( I \). When \( P \) is finite, \( \text{ob } \mathcal{V}_f = \text{ob } P \) is finite, and \( \Delta I \) factorizes through \( \mathcal{V}_f \subset \mathcal{V} \) since \( l \in \mathcal{V}_f \) by (2.2). To show that \( \Delta I \) is a finite indexing-type, it remains to show that each \( \mathcal{V}_f (k, k') \) is f.p. in \( \mathcal{V} \). But \( \mathcal{V}_f (k, k') = \mathcal{V}(k, k') \cdot I \), the coproduct of \( \mathcal{V}(k, k') \) copies of \( I \); and \( \mathcal{V}(k, k') \) is a finite set \( n \). Since \( \mathcal{V}(n, l, -) \simeq (\mathcal{V}, \mathcal{V}) : \mathcal{V} \to \mathcal{V} \), to show that \( n \cdot l \in \mathcal{V}_f \) is to show that \( (\mathcal{V}, \mathcal{V}) : \mathcal{V} \to \mathcal{V} \) is finitary. By (1.3), this is the same as the finitariness of \( (\mathcal{V}, \mathcal{V}) \).

(4.4) HYPOTHESIS. From now on we suppose at all times that the ordinary category \( \mathcal{V}_f \) is l.f.p.

(4.5) A \( \mathcal{V} \)-category \( \mathcal{A} \) satisfying the equivalent conditions of (4.3) shall be called finitely complete, or f.c.. A \( \mathcal{V} \)-functor \( T : \mathcal{A} \to \mathcal{B} \) shall be called left exact, or lex, if \( \mathcal{A} \) is finitely complete and \( T \) preserves all finite limits. (The duals are finitely cocomplete, and right exact or rex.) It is clear from the proof of (4.3) that:

(4.6) For a finitely complete \( \mathcal{A} \), the functor \( T : \mathcal{A} \to \mathcal{B} \) is left exact if and
only if it preserves finite conical limits and the cotensor products $x^A$ with $x \in \mathcal{V}_f$.

Now (2.6) gives:

(4.7) In the classical case $\mathcal{V} = \mathbf{Set}$, a category $\mathcal{A}$ is finitely complete precisely when it admits all finite conical limits, and then $T : \mathcal{A} \to \mathcal{B}$ is left exact precisely when it preserves these. For a general $\mathcal{V}$, if the $\mathcal{V}$-category $\mathcal{A}$ is f.c., so is $\mathcal{A}_o$; and if $T : \mathcal{A} \to \mathcal{B}$ is lex, so is $T_o : \mathcal{A}_o \to \mathcal{B}_o$.

(4.8) Under further hypotheses that we shall later impose on $\mathcal{V}$ we shall show in (7.2) below that every l.f.p. category $\mathcal{A}$ is complete. Hence the hypothesis of finite completeness in such propositions as the next - which generalizes (3.7) - is in fact otiose in practice.

(4.9) PROPOSITION. Let $\mathcal{A}$ be l.f.p. and f.c., let $H : \mathcal{K} \to \mathcal{V}$ be a finite indexing type, and let $P : \mathcal{L} \to [\mathcal{K}, \mathcal{A}]_o$ be an ordinary functor with $\mathcal{L}$ small and filtered. Then the evident canonical map

$$\sigma : \text{colim} \{ H, P \} \to \{ H, \text{colim} P \}$$

is an isomorphism. In other words, finite limits commute with filtered colimits in such an l.f.p. $\mathcal{A}$.

PROOF. If $\mathcal{G} \subset \mathcal{A}_f$ is a strong generator for $\mathcal{A}$, it suffices to show that $\mathcal{A}(G, \sigma)$ is an isomorphism for all $G \in \mathcal{G}$. Since $\mathcal{A}(G, \cdot) : \mathcal{A} \to \mathcal{V}$ then preserves filtered colimits and all limits, we are reduced to the special case $\mathcal{A} = \mathcal{V}$. Then the functor $x^\cdot : \mathcal{V} \to \mathcal{V}$ is simply $[x, \cdot] = \mathcal{V}(x, \cdot)$, which commutes (by definition) with filtered colimits when $x \in \mathcal{V}_f$. So, by the proof of (4.3), it suffices to show that filtered colimits commute with finite conical limits in $\mathcal{V}$; which is to say that they do so in $\mathcal{V}_o$. But this is so by (3.7), given our hypothesis (4.4) that $\mathcal{V}_o$ is l.f.p.

(4.10) This result may also be expressed by saying that $\{ H, \cdot \} : [\mathcal{K}, \mathcal{A}] \to \mathcal{A}$ is finitary for an l.f.p. and f.c. $\mathcal{A}$ when $H : \mathcal{K} \to \mathcal{V}$ is a finite indexing type; or equally by saying that $\text{colim} : [\mathcal{L}], \mathcal{A} \to \mathcal{A}$ is left exact for an l.f.p. and f.c. $\mathcal{A}$ when $\mathcal{L}$ is a small filtered ordinary category. (Recall from [*] Section 2.5 that the $\mathcal{V}$-category $[\mathcal{L}], \mathcal{A}$ has $[\mathcal{L}, \mathcal{A}_o]$ as underlying category.)
(4.11) Taking $\mathcal{A} = \mathcal{C}$ in the first statement of (4.10), we conclude that if $H : K \to \mathcal{C}$ is a finite indexing type, then $H \in [K, \mathcal{C}]_f$.

(4.12) Let $\mathcal{A}$ be l.f.p. and f.c. Then, in the sense made precise in the proof, a filtered colimit of left exact functors into $\mathcal{A}$ is left exact.

**PROOF.** We consider an f.c. category $\mathcal{B}$, not necessarily small, and an ordinary functor $Q : \mathcal{L} \to [\mathcal{B}, \mathcal{A}]_o$ where $\mathcal{L}$ is small and filtered and each $Q_l : \mathcal{B} \to \mathcal{A}$ is left exact; here $[\mathcal{B}, \mathcal{A}]_o$ is the ordinary category of $\mathcal{C}$-functors $\mathcal{B} \to \mathcal{A}$ and $\mathcal{C}$-natural transformations between them, which exists even when $[\mathcal{B}, \mathcal{A}]$ is too big to exist as a $\mathcal{C}$-category; and what we assert is that $\text{colim} Q : \mathcal{B} \to \mathcal{A}$ is left exact. To give $Q$, however, is equivalently to give a left exact $R : \mathcal{B} \to [\mathcal{L}_0, \mathcal{A}]$; and $\text{colim} Q$, being the composite of $R$ with $\text{colim} : [\mathcal{L}_0, \mathcal{A}] \to \mathcal{A}$, is left exact by the second assertion of (4.10).

(4.13) Let $\mathcal{A}$ be l.f.p. and f.c. Then, in the sense made precise in the proof, a finite limit of finitary functors into $\mathcal{A}$ is finitary.

**PROOF.** We consider a $\mathcal{B}$, not necessarily small, which admits filtered colimits, a finite indexing type $H : K \to \mathcal{C}$, and a functor $Q : K \otimes \mathcal{B} \to \mathcal{A}$ such that each $Q(k, -) : \mathcal{B} \to \mathcal{A}$ is finitary; and what we assert is that $\{ H?, Q(?,-) \} : \mathcal{B} \to \mathcal{A}$ is finitary. To give $Q$, however, is equivalently to give a finitary $R : \mathcal{B} \to [K, \mathcal{A}]$; and $\{ H?, Q(?,-) \}$, being the composite of $R$ with $\{ H,- \} : [K, \mathcal{A}] \to \mathcal{A}$, is finitary by the first assertion of (4.10).

(4.14) For any cocomplete $\mathcal{A}$, the full subcategory $\mathcal{A}_f$ is closed under finite colimits.

**PROOF.** Since $\mathcal{A}(\cdot, A)$ sends colimits in $\mathcal{A}$ to limits in $\mathcal{C}$, this follows from the case $\mathcal{A} = \mathcal{C}$ of (4.13).

(4.15) **REMARK.** Note that we used Hypothesis (4.4) in the proof of (4.9).

5. **LOCALLY FINITELY PRESENTABLE SYMMETRIC MONOIDAL CLOSED CATEGORIES**

In accordance with Hypothesis (4.4), we suppose for this section...
that $\mathcal{O}_0$ is l.f.p., with strong generator $\mathcal{H} \subset \mathcal{O}_0$.

(5.1) For an object $G$ of the cocomplete $\mathcal{V}$-category $\mathcal{A}$, the following are equivalent:

(i) $G$ is f.p. in $\mathcal{A}$;
(ii) $x \otimes G$ is f.p. in $\mathcal{A}_0$ for all $x \in \mathcal{O}_0$;
(iii) $x \otimes G$ is f.p. in $\mathcal{A}_0$ for all $x \in \mathcal{H}$.

**Proof.** To say that $G$ is f.p. in $\mathcal{A}$ is to say that $\mathcal{A}(G, -): \mathcal{A} \to \mathcal{V}$, or equivalently $\mathcal{A}(G, -)_0: \mathcal{A}_0 \to \mathcal{V}_0$, is finitary. Since

$$\mathcal{V}_0(x, \mathcal{A}(G, -)_0) \cong \mathcal{V}_0(x \otimes G, -),$$

the result follows from (1.3) and (3.6).

(5.2) The following are equivalent:

(i) $\mathcal{V}_0 \subset \mathcal{V}_0$;
(ii) $x \otimes y \in \mathcal{V}_0$ whenever $x, y \in \mathcal{V}_0$;
(iii) $x \otimes y \in \mathcal{V}_0$ whenever $x, y \in \mathcal{H}$.

**Proof.** (i) implies (ii) by (5.1) and (ii) implies (iii) trivially. Given (iii), we first use (iii) $\Rightarrow$ (i) of (5.1) to deduce that $\mathcal{H} \subset \mathcal{V}_0$; then use (i) $\Rightarrow$ (ii) of (5.1) to deduce that $x \otimes y \in \mathcal{V}_0$ if $x \in \mathcal{V}_0$ and $y \in \mathcal{H}$; and finally use (iii) $\Rightarrow$ (i) of (5.1), with the symmetry of $\otimes$, to deduce that $\mathcal{V}_0 \subset \mathcal{V}_0$.

(5.3) The following are equivalent:

(i) $\mathcal{V}_0 \subset \mathcal{V}_0$;
(ii) $I \in \mathcal{V}_0$;
(iii) $\mathcal{V}_0(1, -): \mathcal{V}_0 \to \text{Set}$ is finitary;
(iv) $\mathcal{A}_0 \subset \mathcal{A}_0$ for every cocomplete $\mathcal{V}$-category $\mathcal{A}$.

**Proof.** (ii) and (iii) are equivalent by definition; (iv) implies (i) trivially; (i) implies (ii) since $I \in \mathcal{V}_0$ by (2.2); and (ii) implies (iv) by the part (i) $\Rightarrow$ (ii) of (5.1).

(5.4) The example (2.11) shows that neither $\mathcal{V}_0 \subset \mathcal{V}_0$ nor $\mathcal{V}_0 \subset \mathcal{V}_0$ is automatic when $\mathcal{V}_0$ is l.f.p.; for the $\mathcal{V}_0 = \text{[M, Set]}$ of that example is l.f.p. by (3.5). The same example shows that $\mathcal{V}_0 \subset \mathcal{V}_0$ does not imply $\mathcal{V}_0 \subset \mathcal{V}_0$; for the former holds precisely when $M$ is finitely generated as
a group, and the latter precisely when $M$ is finite. The author sees no reason to suppose that $\mathcal{O}_{of} \subseteq \mathcal{O}_{fo}$ implies $\mathcal{O}_{fo} \subseteq \mathcal{O}_{of}$, but has no counter-example to this. The author is indebted to R. Börger for discussions concerning this example.

(5.5) We shall say that $\mathcal{O}$ is locally finitely presentable as a (symmetric monoidal) closed category if $\mathcal{O}_{o}$ is l.f.p. and if $\mathcal{O}_{of} = \mathcal{O}_{f o}$. By (5.2) and (5.3), a closed $\mathcal{O}$ with l.f.p. $\mathcal{O}_{o}$ is l.f.p. as a closed category precisely when $\mathcal{O}_{of}$ is closed under the monoidal structure, in the sense that $I \in \mathcal{O}_{of}$ and $x \otimes y \in \mathcal{O}_{of}$ when $x, y \in \mathcal{O}_{of}$; while it in fact suffices for the latter that $x \otimes y \in \mathcal{O}_{of}$ whenever $x, y \in \mathcal{H}$, where $\mathcal{H} \subseteq \mathcal{O}_{of}$ is a strong generator for $\mathcal{O}_{o}$.

(5.6) EXAMPLES. If $\mathcal{O}_{o}$ is the category of algebras for a one-sorted finitary algebraic theory, and if the theory is commutative in the sense of Linton [12], then $\mathcal{O}_{o}$ admits a symmetric monoidal closed structure $\mathcal{O}$ in which the tensor product represents the bi-homomorphisms and the unit object $I$ represents the forgetful functor to $\mathcal{S}_{et}$. In this case $F I = I$ is an f.p. strong generator in $\mathcal{O}_{o}$ by (3.5), so that $\mathcal{O}$ is l.f.p. as a closed category by the last remarks of (5.5). The closed categories $\mathcal{O} = \mathcal{S}_{et}, \mathcal{S}_{et}, \mathcal{A}_{b}, R-\mathcal{M}_{od}$ of sets, pointed sets, abelian groups, and $R$-modules for a commutative ring $R$, are particular examples.

(5.7) EXAMPLES. The category $\mathcal{O}_{o} = [M, \mathcal{S}_{et}]$ is an example of (5.6) when the group $M$ is abelian. The tensor product here, however, is not the cartesian one; and (5.4) shows that $[M, \mathcal{S}_{et}]$ with its cartesian closed structure is l.f.p. as a closed category only when $M$ is finite.

(5.8) EXAMPLES. Consider the closed categories $\mathcal{G}_{-R-M}_{od}, \mathcal{D}_{R-\mathcal{M}_{od}}, \mathcal{G}_{R\mathcal{d}}$ of graded $R$-modules, differential graded $R$-modules, and graphs - the first two with their classical closed structures, and the last with its cartesian one. In each case $\mathcal{O}_{o}$ is l.f.p. by (3.5), the f.p. objects $F_{i} I$ forming a strong generator. In the first two cases

$$F_{0} I = I \quad \text{and} \quad F_{i} I \otimes F_{j} I = F_{i+j} I,$$

so that $\mathcal{O}$ is l.f.p. as a closed category by (5.5). In the last case

$$F_{0} I = (0 \xrightarrow{1}) \quad \text{and} \quad F_{L} I = (1 \xrightarrow{2});$$
the various products $F_i 1 \times F_j 1$ are all finite and hence f.p. by (2.8), as is the unit object $1 = (1 \Rightarrow 1)$; so that again $\mathcal{V}$ is l.f.p. as a closed category.

(5.9) EXAMPLES. Consider the cartesian closed categories $\text{Cat}$, $\text{Preord}$, $\text{Ord}$, $\text{Gpd}$, $\text{Equiv}$ of (small) categories, preorders, orders, groupoids, and sets-with-an-equivalence-relation. In each case $\mathcal{V}_0$ is l.f.p. by (3.5), an f.p. strong generator being given by 2 in the first three cases and by 1 in the other two. Since $2 \times 2$ and $1 \times 1$ are finite, and hence f.p. by (2.9), as is the unit object 1, it follows by (5.5) that $\mathcal{V}$ is in each case l.f.p. as a closed category.

(5.10) EXAMPLES. A finite complete ordered set, such as 2, with its cartesian closed structure, is clearly l.f.p. as a closed category.

(5.11) For the closed categories $\mathbb{R}_+^+$, $\mathbb{C}G\text{Top}$, $\mathbb{B}an$ of the extended non-negative reals, of compactly-generated topological spaces, and of Banach spaces (see [*] Section 1.1), it is not even true that $\mathcal{V}_0$ is l.f.p.

(5.12) HYPOTHESIS. For the rest of this article we strengthen the hypothesis (4.4) by supposing throughout that $\mathcal{V}$ is l.f.p. as a closed category. Accordingly, both $x \in \mathcal{V}_0 f$ and $x \in \mathcal{V}_f$ mean the same as $x \in \mathcal{V}_f$.

6. LEFT EXACT AND FLAT FUNCTORS INTO $\mathcal{V}$

(6.1) If $\mathcal{T}$ is a small $\mathcal{V}$-category, we call the functor $F: \mathcal{T} \to \mathcal{V}$ flat if $F^\ast: [\mathcal{T}^{\text{op}}, \mathcal{V}] \to \mathcal{V}$ is left exact, where $\ast$ denotes the indexed colimit, as in [*] Section 3.1.

(6.2) A flat functor $F: \mathcal{T} \to \mathcal{V}$ preserves any finite limit that happens to exist in $\mathcal{T}$; in particular, a flat functor is left exact if $\mathcal{T}$ is finitely complete.

PROOF. Let $H: \mathcal{K} \to \mathcal{V}$ be a finite indexing type, and let $T: \mathcal{K} \to \mathcal{T}$ be such that the limit $\{H, T\}$ exists in $\mathcal{T}$. Let $Y: \mathcal{T} \to [\mathcal{T}^{\text{op}}, \mathcal{V}]$ denote the Yoneda embedding. Since $Y$ preserves limits by [*] Section 3.3, we have a canonical isomorphism $Y \{H, T\} \cong \{H, YT\}$; and since $F$ is flat, we have a canonical isomorphism $F^\ast \{H, YT\} \cong \{H, FYT\}$. Combining these gives
\[ F \ast Y\{H, T\} \cong \{H, F \ast Y T\}; \text{ and since } F \ast Y t \cong F t \text{ by (3.9) and (3.10) of } [*], \text{ we have the desired result } F\{H, T\} \cong \{H, FT\}. \]

6.3 Any representable \( \mathcal{F}(t, -) \) is flat, since by (3.10) of [*] the functor \( \mathcal{F}(t, -) \ast : [\mathcal{F}^{op}, \mathcal{A}] \to \mathcal{A} \) is isomorphic, for any \( \mathcal{A} \), to the evaluation \( E_t \), which by [*] Section 3.3 preserves all limits that exist. Since \( - \ast T \) is cocontinuous by [*] Section 3.3, it follows from (4.12) that any filtered colimit in \([\mathcal{F}, \mathcal{C}]\) of flat functors is flat, so that in particular every filtered colimit \( F \) of representables is flat. In fact (4.12) gives more: for such an \( F \) and any l.f.p. and f.c. \( \mathcal{A} \), the functor \( F \ast : [\mathcal{F}^{op}, \mathcal{A}] \to \mathcal{A} \) is left exact.

6.4 When \( \mathcal{C} = \mathcal{S}_{et} \) it is a classical result (see [*] Thm 5.38, or (6.7) below) that every flat \( F : \mathcal{I} \to \mathcal{C} \) is a filtered colimit of representables. The author has no proof of this for a general \( \mathcal{C} \) satisfying the Hypothesis (5.12). It is however true when \( \mathcal{I} \) is finitely complete ((6.11) below) and for certain special \( \mathcal{C} \), if \( \mathcal{I} \) at least admits certain cotensor products ((6.10) below).

6.5 Consider an arbitrary \( F : \mathcal{I} \to \mathcal{C} \) with \( \mathcal{I} \) small. The composite of \( V = \mathcal{C}o(1, -): \mathcal{C}o \to \mathcal{S}_{et} \) with the underlying functor \( F_0 : \mathcal{I}o \to \mathcal{C}o \) of \( F \) is an ordinary functor \( VF_0 : \mathcal{I}o \to \mathcal{S}_{et} \). Recall from [*] Section 1.10 that the comma-category \[ V = \mathcal{I}o(1, -): \mathcal{C}o \to \mathcal{S}_{et} \] with the underlying functor \( F_0 : \mathcal{I}o \to \mathcal{C}o \) of \( F \) is an ordinary functor \( VF_0 : \mathcal{I}o \to \mathcal{S}_{et} \). Recall from [*] Section 1.10 that the comma-category \( I/VF_0 \), whose objects are pairs \((t, a)\) with \( t \in \mathcal{I} \) and \( a \in VF_0 t \), is called the category \( el(VF_0) \) of elements of \( VF_0 \). Since a map \( \mathcal{I}(t, -) \to F \) from a representable into \( F \) corresponds by Yoneda to an element \( a \in VF_0 t \), we have a canonical inductive cone \( \mu(t, a) : \mathcal{I}(t, -) \to F \) in \([\mathcal{I}, \mathcal{C}]_o\), with vertex \( F \), whose base is the ordinary functor

\[
\begin{align*}
\text{el}(VF_0)^{op} & \xrightarrow{d^{op}} \mathcal{I}o^{op} \xrightarrow{Y_o} [\mathcal{I}, \mathcal{C}]_o,
\end{align*}
\]

where \( d \) is the projection sending \((t, a)\) to \( t \); and this cone induces a canonical map \( \rho : \text{colim}(Y_o d^{op}) \to F \). When \( \mathcal{C} = \mathcal{S}_{et} \) it is a classical result - see [*] Section 3.3 - that \( \mu \) is a colimit cone; so that \( \rho \) is an isomorphism. For a general \( \mathcal{C} \) - even for one satisfying (5.12) - this is false. In fact, when \( \mathcal{I} \) is the unit \( \mathcal{C} \)-category \( 1 \) with one object \( 0 \), and with \( 1(0, 0) = 1 \), so that \([1, \mathcal{C}] \cong \mathcal{C} \) and \( F \) is just an object of \( \mathcal{C} \), the cone \( \mu \) consists of all maps \( l \to F \) in \( \mathcal{C}_o \); and this is a colimit cone for all \( F \).
precisely when \( \{ I \} \) is dense in \( \mathcal{O}_o \) which is false even for \( \mathcal{O} = \mathcal{G}_k \). In the example \( \mathcal{O} = \mathcal{C}at \) it is easily seen, when \( \mathcal{T} = \mathcal{J} = I \), that the map \( \rho \) from the colimit into \( F \) is the inclusion into \( F \) of the discrete category with the same objects; and similarly for \( \mathcal{O} = \mathcal{G}_p d \).

(6.6) **Proposition.** Suppose that \( \mathcal{O} \) is cartesian closed, and that \( V: \mathcal{O}_o \to \mathcal{Set} \) preserves coproducts and regular epimorphisms; as is true in the examples \( \mathcal{O} = \mathcal{Set}, \mathcal{G}_p h, \mathcal{C}at, \mathcal{P}_{ad}, \mathcal{O}_d, \mathcal{G}_p d, \mathcal{G}_q i. \) Then, if \( F: \mathcal{T} \to \mathcal{O} \) is flat, the category \( el(VF_0)^{op} \) is filtered.

**Proof.** Let \( T: \mathcal{P} \to el(VF_0) \) where \( \mathcal{P} \) is a finite ordinary category. To give \( T \) is to give a functor \( S = dT: \mathcal{P} \to \mathcal{T} \) and a cone \( (\gamma_p: I \to VF_0Sp) \), which is equally an element \( \gamma \in \lim VF_0S \). The flatness of \( F \) gives

\[
F \ast \lim_p \mathcal{T}(\ast, Sp) \cong \lim_p (F \ast \mathcal{T}(\ast, Sp)),
\]

and this latter by Yoneda is \( \lim_p FSp = \lim_FoS \). By the formula of \([\ast]\) Section 3.10 expressing \( F \ast G \) as a coend, we have a regular epimorphism \( f: \Sigma_{t \in \mathcal{T}} (VF_0t \times \lim_p \mathcal{T}(t, Sp)) \to \lim_VFoS \). Applying \( V \) to \( f \), and using the hypotheses of the proposition and the preservation of limits by \( V \), we get a surjection \( Vf: \Sigma_{t \in \mathcal{T}} VF_0t \times \lim_p \mathcal{T}(t, Sp) \to \lim_VFoS \). Let some inverse image of \( \gamma \) under \( Vf \) be \( (\gamma \in VF_0t, \beta \in \lim_p \mathcal{T}(t, Sp)) \). We can regard \( \beta \) as a projective cone in \( \mathcal{T}_o \) over \( S \) with vertex \( t \). Since \( VF_0\beta_p: VF_0t \to VF_0Sp \) maps \( a \) to \( \gamma_p \) by construction, \( \beta \) is equally a projective cone in \( el(VF_0) \) over \( T \) with vertex \( (t, a) \). Thus \( el(VF_0)^{op} \) is filtered.

(6.7) Combining (6.6), (6.5), and (6.3), we regain the classical result in the case \( \mathcal{O} = \mathcal{Set} \) that \( F: \mathcal{T} \to \mathcal{Set} \) is flat if and only if \( (elF)^{op} \) is filtered, and if and only if \( F \) is a filtered colimit of representables.

(6.8) We recall also the other classical result that, when \( \mathcal{O} = \mathcal{Set} \) and \( \mathcal{T} \) is finitely complete, \( F: \mathcal{T} \to \mathcal{Set} \) is flat precisely when it is left exact. For flatness implies left exactness by (6.2); while if \( \mathcal{T} \) is finitely complete and \( F \) is left exact, it is immediate that \( elF \) is finitely complete, so that \( (elF)^{op} \) is finitely cocomplete and a fortiori filtered.

(6.9) **Proposition.** Let \( \mathcal{T} \) admit the cotensor products \( x \lhd - \) for \( x \in \mathcal{O}_f \), and let \( F: \mathcal{T} \to \mathcal{O} \) preserve them. Then if \( VF_0: \mathcal{T}_o \to \mathcal{Set} \) is flat, the cam-
Onical cone $\mu$ of (6.5) is a colimit cone in $[T, \mathcal{C}]$, expressing $F$ as a filtered colimit of representables in $[T, \mathcal{C}]$; whence $F$ is flat by (6.3).

**Proof.** Since colimits in $[T, \mathcal{C}]$ are formed pointwise by [*] Section 3.3, $\mathcal{C}$ being cocomplete by [*] Section 3.10, we have only to prove that $\mu(t, a, s) : T(t, s) \rightarrow F_s$ is, for each $s \in T$, a colimit cone in $\mathcal{C}$; which by [*] Section 3.8 is to say that $\mu(t, a, s)$ is a colimit cone in $\mathcal{C}_o$. Since $\mathcal{C}_o$ is l.f.p. by (4.4), it suffices by (3.6) to show that

$$\mathcal{C}_o(x, T(t, s)) \rightarrow \mathcal{C}_o(x, F_s)$$

is a colimit cone in $\mathcal{S}_{\text{et}}$ for each $x \in \mathcal{C}_{o_f}$; which by (5.12) means for each $x \in \mathcal{C}_f$. For $x \in \mathcal{C}_f$, however, $\mathcal{C}_o(x, T(t, s)) \simeq T_o(t, x^*s)$ by the definition of the cotensor product $x^*s$; while

$$\mathcal{C}_o(x, F_s) \simeq V[x, F_s] = V(x^*F_s),$$

which since $F$ preserves $x^*F_s$ is isomorphic to $VF(x^*s)$. Writing $r$ for $x^*s$, we are reduced to proving that $T_o(t, r) \rightarrow VF_o r$ is a colimit cone in $\mathcal{S}_{\text{et}}$; which is true by (6.5).

(6.10) **Corollary.** When $\mathcal{C}$ satisfies the conditions of (6.6) and $T$ admits the cotensor products $x^*t$ for $x \in \mathcal{C}_f$, $F : T \rightarrow \mathcal{C}$ is flat precisely when $F$ is a filtered colimit of representables.

**Proof.** A filtered colimit of representables is flat by (6.3). If $F$ is flat, it preserves the $x^*t$ for $x \in \mathcal{C}_f$ by (6.2), since these are finite limits by the proof of (4.3). Moreover $el(VF_o)^{op}$ is filtered by (6.6), so that $VF_o$ is flat by (6.7). The result now follows by (6.9).

(6.11) **Theorem.** Let $F : T \rightarrow \mathcal{C}$ where the small $T$ is finitely complete. Then the following are equivalent:

(i) $F$ is left exact;

(ii) the canonical cone $\mu(t, a) : T(t, \cdot) \rightarrow F$ of (6.5) expresses $F$ as a filtered colimit in $[T, \mathcal{C}]$ of representables;

(iii) $F$ is some filtered colimit in $[T, \mathcal{C}]$ of representables;

(iv) $F$ lies in the closure of $T^{op} \subset [T, \mathcal{C}]$ under filtered colimits (which by [*] Thm 5.35 is the free filtered-colimit completion of $T^{op}$);

(v) $F^{*} : [T^{op}, \mathcal{A}] \rightarrow \mathcal{A}$ is left exact for any l.f.p. and f.c. $\mathcal{A}$;
(vi) F is flat.

PROOF. If F is left exact, so is \( F_0 \) by (4.7), whence \( VF_0 \) is left exact since \( V = \mathcal{C}_0(I, -) \) preserves all limits; so that \( VF_0 \) is flat by (4.7) and (6.8). Hence (i) \( \Rightarrow \) (ii) by (6.9). It is trivial that (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv); while (iv) implies (v) by the arguments of (6.3) along with (4.12). The implication (v) \( \Rightarrow \) (vi) is trivial on taking \( \mathcal{A} = \mathcal{C} \); and we have (vi) \( \Rightarrow \) (i) by (6.2).

(6.12) THEOREM. Let \( M : \mathcal{I} \to \mathcal{S} \) where \( \mathcal{I} \) and \( \mathcal{S} \) are finitely complete and \( \mathcal{S} \) is small. Let \( \text{Lan}_M F : \mathcal{S} \to \mathcal{C} \) be the left Kan extension along \( M \) of \( F : \mathcal{I} \to \mathcal{S} \). Then if \( F \) is left exact, so is \( \text{Lan}_M F \).

PROOF. By [*] Section 4.1, \( \text{Lan}_M F \) is the composite of \( \tilde{M} : \mathcal{S} \to [\mathcal{S}^{\text{op}}, \mathcal{C}] \), where \( \tilde{M} S = \mathcal{S}(M -, S) \), and \( - * F : [\mathcal{S}^{\text{op}}, \mathcal{C}] \to \mathcal{C} \). But \( \tilde{M} \) is trivially left exact, while \( - * F \) is left exact since \( F \) is flat by (6.11) and \( * \) is symmetric by [*] Section 3.1.

(6.13) REMARK. Up to this point we have used, besides Hypothesis 4.4, only the part \( \mathcal{C}_f \) of Hypothesis (5.12) - namely, in the proof of (6.9).

7. \( \mathcal{G}_f \) AS A SMALL DENSE SUBCATEGORY OF THE LOCALLY FINITELY PRESENTABLE \( \mathcal{A} \)

(7.1) The set of finite indexing types is small; so that by [*] Section 3.5 the closure under finite colimits of a small full subcategory \( \mathcal{G} \) in a finitely-cocomplete \( \mathcal{A} \) is again small, and in particular the free finite-colimit completion of a small \( \mathcal{G} \) (its finite colimit-closure in \( [\mathcal{G}^{\text{op}}, \mathcal{C}] \) is small.

PROOF. By (4.1), it is a matter of observing that \( \mathcal{G}_f \) is small. For \( \mathcal{G} = \mathcal{S}_{\text{et}} \) we have this by (2.6). This justifies the argument in (7.2) below when \( \mathcal{G} = \mathcal{S}_{\text{et}} \), which then gives for any \( \mathcal{G} \) satisfying (5.12) the smallness of \( \mathcal{G}_f \), and hence of \( \mathcal{G}_f \). This now gives the general case of (7.1), which is needed for the general case of (7.2).

(7.2) THEOREM. Let \( \mathcal{A} \) be l.f.p., let \( \mathcal{G} \subset \mathcal{A}_f \) be a strong generator of \( \mathcal{A} \), and let \( Z : \mathcal{G}_f \to \mathcal{A} \) be the inclusion. Then

(i) \( \mathcal{A}_f \) is the closure of \( \mathcal{G} \) in \( \mathcal{A} \) under finite colimits, and \( \mathcal{A}_f \) is small and finitely cocomplete;
(ii) for each $A \in \mathcal{A}$ the totality of maps $g : G \to A$ with $G \in \mathcal{A}_f$ expresses $A$ as the filtered colimit in $\mathcal{A}$ of the functor $Z_0 d : Z_0/A \to \mathcal{A}_0$, where $d : Z_0/A \to \mathcal{A}_f$ is the projection from the comma-category;

(iii) the colimits in (ii) are $\mathcal{Z}$-absolute, and thus present $\mathcal{A}_f$ as a small dense subcategory of $\mathcal{A}$;

(iv) the full embedding $\tilde{Z} : \mathcal{A} \to [\mathcal{A}_f^{op}, \mathcal{V}]$ is finitary, and has the left adjoint $\ast_Z : [\mathcal{A}_f^{op}, \mathcal{V}] \to \mathcal{A}$;

(v) $\mathcal{A}$ is complete;

(vi) the replete image of $\tilde{Z}$ is the category $\text{Lex}[\mathcal{A}_f^{op}, \mathcal{V}]$ of left-exact functors $\mathcal{A}_f^{op} \to \mathcal{V}$, so that $\tilde{Z}$ induces an equivalence $\mathcal{A} \sim \text{Lex}[\mathcal{A}_f^{op}, \mathcal{V}]$.

**Proof.** Write $\bar{G}$ for the closure of $G$ in $\mathcal{A}$ under finite colimits; it is small by (7.1), and of course finitely cocomplete; and by (4.14) it is contained in $\mathcal{A}_f$. Write $K : \bar{G} \to \mathcal{A}$ for the inclusion; since $\bar{G}$ is a strong generator so a fortiori is $\bar{G} \supset G$, so that by (3.1) the functor $K : \mathcal{A} \to [\mathcal{A}_f^{op}, \mathcal{V}]$ is finitary, conservative, and right-adjoint; its left adjoint is in fact $\ast K$ by the definition of the latter. For any $A \in \mathcal{A}$, let $\beta$ denote the canonical cone consisting of all maps $g : G \to A$ with $G \in \bar{G}$, so that $\beta G, g = g$; the indexing category $K_0/A$ of this cone is filtered, since by (4.7) $\bar{G}$ is closed under finite conical colimits in $\mathcal{A}_0$. The image $\tilde{K}\beta$ of the cone $\beta$ under $\tilde{K}$ has, since $\tilde{K}K \cong Y$, the components $\tilde{K}g : \bar{G}(\cdot, G) \to \tilde{K}A$, and is in fact precisely the canonical cone $\mu$ of (6.5) for the functor $\tilde{K}A : \bar{G}^{op} \to \mathcal{V}$; for the maps $\bar{G}(\cdot, G) \to \tilde{K}A$ correspond by Yoneda to the elements of $V(\tilde{K}A)G = V\mathcal{A}(KG, A) = \mathcal{A}_0(G, A)$, or the maps $g : G \to A$. But $\tilde{K}A = \mathcal{A}(K\ast, A)$ is a left exact functor $\bar{G}^{op} \to \mathcal{V}$, since $K$ preserves finite colimits and $\mathcal{A}(\cdot, A)$ converts them into limits. Hence $\tilde{K}\beta = \mu$ is a colimit cone by (6.11). But the conservative and finitary $\tilde{K}$ reflects filtered colimits; so that $\beta$ is already a colimit cone in $\mathcal{A}$. If $A \in \mathcal{A}_f$, it follows from (2.12) that some $g : G \to A$ with $G \in \bar{G}$ is a retraction, with say $g i = 1$. Then $g : G \to A$ is the coequalizer of $1, ig : G \to G$, and hence by (4.3) $A$ belongs to $\bar{G}$, since $\bar{G}$ is closed under finite colimits. As we already have that $\bar{G} \subseteq \mathcal{A}_f$, we conclude that $\bar{G} = \mathcal{A}_f$, and that $K$ coincides with $Z$. This completes the proof of (i) and of (ii). We
also have (iii), since we have seen that the colimit $\beta$ is preserved by $\tilde{Z}$; so that $\tilde{A}_f$ is dense in $\tilde{A}$ by [*] Thm 5.19, and $\tilde{Z}$ is fully faithful. We now have (iv), and also (v) because $\tilde{A}$ is reflective in the complete $[\tilde{A}_f^{op}, \mathcal{V}]$. As for (vi), we have already observed that each $\tilde{Z}A$ is left exact. For the converse, let $F: \tilde{A}_f^{op} \to \mathcal{V}$ be left exact. Then, by (6.11), $F$ is a filtered colimit of representables in $[\tilde{A}_f^{op}, \mathcal{V}]$. But the representables lie in the replete image of $\tilde{Z}$, since $\tilde{Z}Z \cong Y$; and this image is closed under filtered colimits in $[\tilde{A}_f^{op}, \mathcal{V}]$, since $\tilde{Z}$ is finitary.

(7.3) COROLLARY. For a cocomplete $\tilde{A}$, the following are equivalent:

(i) $\tilde{A}$ is locally finitely presentable;
(ii) $\tilde{A}_f$ is small and strongly generating;
(iii) $\tilde{A}_f$ is small and dense;
(iv) $\tilde{A}$ is a full reflective subcategory of some $[\tilde{T}, \mathcal{V}]$ with $\tilde{T}$ small and with the inclusion $\tilde{A} \to [\tilde{T}, \mathcal{V}]$ finitary. (Here the cocompleteness of $\tilde{A}$ is automatic.)

(7.4) There are simple generalizations in which «finite» is replaced throughout - except in (5.12) - by «of cardinal $< \alpha$», where $\alpha$ is a small regular cardinal. An ordinary category is an $\alpha$-category if its set of morphisms has cardinal $< \alpha$. A cone or a conical limit is $\alpha$-small, or is an $\alpha$-cone or an $\alpha$-limit, if its indexing category is an $\alpha$-category. An ordinary category $\mathcal{L}$ is $\alpha$-filtered if every functor from an $\alpha$-category into $\mathcal{L}$ is the base of some inductive cone; and a (small) conical limit is $\alpha$-filtered if its indexing category is $\alpha$-filtered. Generalizing (1.2), $\alpha$-filtered colimits commute in $\mathbf{Set}$ with $\alpha$-limits. A $\mathcal{V}$-functor is $\alpha$-ary if it preserves $\alpha$-filtered colimits, and has a rank if it is $\alpha$-ary for some (small) $\alpha$; whereupon its rank is the least such $\alpha$. The object $G$ of $\tilde{A}$ is $\alpha$-presentable if $\tilde{A}/(G, -)$; $\tilde{A} \to \mathcal{V}$ is $\alpha$-ary, and is presentable if it is $\alpha$-presentable for some $\alpha$. We write $\tilde{A}_\alpha$ for the full subcategory of $\alpha$-presentable objects; and as in (2.6), the $\alpha$-presentable sets are the $\alpha$-small ones. The $\mathcal{V}$-category $\tilde{A}$ is locally $\alpha$-presentable if it is cocomplete and has a strong generator $\mathcal{G} \subset \tilde{A}_\alpha$; and it is locally presentable if it is locally $\alpha$-presentable for some (small) $\alpha$. An indexing type $H: \mathcal{K} \to \mathcal{V}$ is $\alpha$-small if $\text{ob}\mathcal{K}$ has fewer than $\alpha$ isomorphism classes,
each $K(k, k')$ lies in $\mathcal{V}_a$, and $H$ factorizes through $\mathcal{V}_a$; a limit indexed
by such an $H$ is called an $\alpha$-limit; when $\mathcal{V} = \mathcal{S}_{\text{set}}$ these reduce to the conical $\alpha$-limits above; and for a general $\mathcal{V}$ with $\mathcal{V}_0$ l.f.p. they reduce as in (3.4) to conical $\alpha$-limits and the $x^\perp \cdot \cdot \cdot$ with $x \in \mathcal{V}_a$. A $\mathcal{V}$-category is $\alpha$-complete if it admits $\alpha$-limits, and a $\mathcal{V}$-functor is $\alpha$-left-exact if it preserves them; the $\alpha$-analogues of the results of Section 4 are all valid. When $\mathcal{V} = \mathcal{S}_{\text{set}}$ and $\mathcal{A}$ is locally $\alpha$-presentable with strong generator $\mathcal{G} \subset \mathcal{A}_a$, the proof in (7.2) carries over, and in particular exhibits $\mathcal{A}_a$ as the closure of $\mathcal{G}$ in $\mathcal{A}$ under $\alpha$-colimits. Applying this to the l.f.p. and hence locally $\alpha$-presentable $\mathcal{V}_0$, we see that $\mathcal{V}_{\alpha 0}$ is the closure of $\mathcal{V}_{\alpha f}$ under $\alpha$-colimits. Always supposing that $\mathcal{V}$ satisfies the hypothesis (5.12), we conclude that $x^\perp y \in \mathcal{V}_{\alpha 0}$ if $x, y \in \mathcal{V}_{\alpha 0}$; and since $\mathcal{L} \in \mathcal{V}_{\alpha f} \subset \mathcal{V}_{\alpha 0}$, we get from the $\alpha$-versions of (5.2) and (5.3) the equivalence of $x \in \mathcal{V}_{\alpha 0}$ with $x \in \mathcal{V}_a$. Now everything above carries over to the case of general $\alpha$. There is one comment to be made: since every $A$ in the locally $\alpha$-presentable $\mathcal{A}$ is a colimit of objects in $\mathcal{A}_a$, and since this colimit is small and hence $\beta$-small for some $\beta \geq \alpha$, the object $A$ is, in fact, $\beta$-presentable - so that every object in $\mathcal{A}$ is presentable. We continue to write only of the finitary case, leaving the reader to make the easy generalizations - except when the result requires the general case.

(7.5) PROPOSITION. When $\mathcal{G}$ is l.f.p. so is $\mathcal{A}_0$, and $\mathcal{A}_{\alpha f} = \mathcal{A}_{f 0}$. Conversely, a cocomplete $\mathcal{A}$ is l.f.p. if $\mathcal{A}_0$ is l.f.p. and if $\mathcal{A}_{\alpha f} \subset \mathcal{A}_{f 0}$.

PROOF. When $\mathcal{A}$ is l.f.p., $\mathcal{A}_0$ is cocomplete since $\mathcal{A}$ is, by [*] Section 3.8. Moreover each $A \in \mathcal{A}_0$ is by (7.2) a filtered colimit in $\mathcal{A}_0$ of objects $G \in \mathcal{A}_{f 0}$; and these colimits are preserved by the $\mathcal{A}_0(G, \cdot)$ with $G \in \mathcal{A}_{f 0}$, since $\mathcal{A}_{f 0} \subset \mathcal{A}_{\alpha f}$ by (5.3). By [*] Thm 5.19, therefore, the small subcategory $\mathcal{A}_{f 0} \subset \mathcal{A}_{\alpha f}$ is dense, and a fortiori strongly generating, in $\mathcal{A}_0$; so that $\mathcal{A}_0$ is l.f.p. By (7.2) again, $\mathcal{A}_{\alpha f}$ is the closure of $\mathcal{A}_{f 0}$ in $\mathcal{A}_0$ under finite colimits; but this is $\mathcal{A}_{f 0}$ itself, since $\mathcal{A}_{f 0}$ is closed under finite colimits by (7.2) and (4.7).

Conversely, if $\mathcal{A}_0$ is l.f.p., and $\mathcal{A}$ is cocomplete with $\mathcal{A}_{\alpha f} \subset \mathcal{A}_{f 0}$, every $A \in \mathcal{A}$ is the filtered colimit in $\mathcal{A}_0$, and hence in $\mathcal{A}$, of objects
Because $\mathcal{A}_o \subseteq \mathcal{A}_f$, this colimit is preserved by the $\mathcal{A}(G, -)$ with $G \in \mathcal{A}_o$. By [•] Thm 5.19, therefore, the full subcategory of $\mathcal{A}$ determined by the objects in $\mathcal{A}_o$ is dense in $\mathcal{A}$; and it is contained in $\mathcal{A}_f$, whence $\mathcal{A}$ is l.f.p.

(7.6) PROPOSITION. When $\mathcal{A}$ is l.f.p., a functor $T: \mathcal{A} \to \mathcal{B}$ is finitary exactly when $1: TZ \to TZ$ exhibits $T$ as the left Kan extension of $TZ$ along the inclusion $Z: \mathcal{A}_f \to \mathcal{A}$. When, moreover, $\mathcal{B}$ admits filtered colimits, $\text{Lan}_Z H$ exists for all $H: \mathcal{A}_f \to \mathcal{B}$; and $\text{Lan}_Z$ gives an equivalence between $[\mathcal{A}_f, \mathcal{B}]$ and the $\mathcal{O}$-category $\text{Fin}([\mathcal{A}, \mathcal{B}]$ of all finitary functors $\mathcal{A} \to \mathcal{B}$, with the restriction $[Z, 1]$ along $Z$ as its equivalence-inverse.

PROOF. Given (7.2), this follows from Thm 4.98, Thm 4.99, and Lemma 5.18, of [•].

(7.7) REMARK. When $\mathcal{A}$ is l.f.p. so is $\mathcal{A}_o$ by (7.5); and then by (1.3), if $\mathcal{B}$ admits filtered colimits, $T: \mathcal{A} \to \mathcal{B}$ is finitary exactly when $T_o: \mathcal{A}_o \to \mathcal{B}_o$ is so. It follows from (7.6) that the identity map expresses $T$ as $\text{Lan}_Z TZ$ if and only if the identity map expresses $T_o$ as $\text{Lan}_{Z_o} T_o Z_o$. Thus for any $H: \mathcal{A}_f \to \mathcal{B}$ where $\mathcal{B}$ admits filtered colimits, we have $(\text{Lan}_Z H)_o = \text{Lan}_{Z_o} H_o$. When $\mathcal{B}$ is cocomplete, it therefore follows from [•] 4.2 an isomorphism

$$\int G \in \mathcal{A}_f \mathcal{A}(Z G, A) \otimes H G \cong \int G \in \mathcal{A}_o(Z G, A) H G$$

between functors $\mathcal{A}_o \to \mathcal{B}_o$. This strikes the author as surprising. When $\mathcal{V} = \mathcal{C}_{\text{at}}$ and $\mathcal{A} = \mathcal{V}$, for instance, both $\otimes$ and $\cdot$ are the cartesian product, if we regard $\mathcal{A}_o(Z G, A)$ as the discrete category formed by the objects of $\mathcal{A}(Z G, A)$; so that here the difference between $\mathcal{A}(Z G, A) \times H G$ and $\mathcal{A}_o(Z G, A) \times H G$ is exactly balanced out by the extra relations (involving the 2-cells) which occur in the passage in the quotient on the left side.

(7.8) When $\mathcal{A}$ is l.f.p., a functor $S: \mathcal{A} \to \mathcal{B}$ is a left adjoint precisely when it is cocontinuous.

PROOF. Since $\mathcal{A}$ has by (7.2) the small dense subcategory $\mathcal{A}_f$, the result follows from [•] Thm 5.33.

(7.9) When both $\mathcal{A}$ and $\mathcal{B}$ are locally presentable, a functor $T: \mathcal{A} \to \mathcal{B}$ has
a left adjoint \( S \) precisely when it is cocontinuous and has a rank.

**Proof.** For one direction, it suffices to illustrate by the case where \( \mathcal{A} \) and \( \mathcal{B} \) are I.f.p. and the continuous \( T \) is finitary. Then \( T \) has a left adjoint by \([*]\) Thm 5.32, since filtered colimits present \( \mathcal{A}_f \) as dense in \( \mathcal{A} \), since \( \mathcal{B} \) is the closure under small colimits of \( \mathcal{B}_f \), since \( T \) preserves filtered colimits, and since each \( \mathcal{B}(H, -) \) with \( H \in \mathcal{B}_f \) also preserves filtered colimits.

For the other direction, it suffices to illustrate by the case where \( \mathcal{A} \) and \( \mathcal{B} \) are I.f.p. The small subcategory \( S(\mathcal{B}_f) \) of \( \mathcal{A} \) is then by (7.4) contained in \( \mathcal{A}_a \) for some regular cardinal \( a \). Then \( S(\mathcal{B}_a) \subset \mathcal{A}_a \), since by the \( a \)-analogue of (7.2) the closure of \( \mathcal{B}_f \) in \( \mathcal{B} \) under \( a \)-colimits is \( \mathcal{B}_a \), and \( \mathcal{A}_a \) is closed under \( a \)-colimits. To show that \( T \) preserves \( a \)-filtered colimits, it suffices by (3.6) to prove that all \( \mathcal{B}(H, T-) \) with \( H \in \mathcal{B}_a \) does so; but \( \mathcal{B}(H, T-) \cong \mathcal{A}(SH, -) \) does so because \( SH \in \mathcal{A}_a \).

(7.10) Remark. We have now used the full strength of Hypothesis (5.12): as remarked in (4.15), the proof of (4.9) used Hypothesis (4.4), that \( \mathcal{C}_o \) is I.f.p.; as remarked in (6.13), the proof of (6.9) used the part of \( \mathcal{C}_o(f \circ -) \) of (5.12); and now the remaining part \( \mathcal{C}_o(f \circ -) \) of (5.12) has been used in the proof of (7.1).

8. SOME PROPERTIES OF LOCALLY FINITELY PRESENTABLE \( \mathcal{A} \)

**When \( \mathcal{C} = \mathcal{S}_{\text{et}} \)**

(8.1) Because of (7.5), many of the properties of an I.f.p. \( \mathcal{A} \) - including various characterizations of \( \mathcal{A}_f \) - can be carried over directly from the classical case \( \mathcal{C} = \mathcal{S}_{\text{et}} \), as studied by Gabriel-Ulmer in [7]. Because the present article is in other respects so near to being self-contained, we take the liberty of recalling some of these here, with sketches of their proofs.

(8.2) Because we are concerned in this section only with \( \mathcal{A} \) that are co-complete and finitely complete, a regular epimorphism (in the sense of, say, [8]) is the same thing as a coequalizer of some pair of maps, being the coequalizer of its kernel-pair. The regular factorization \( f = nq \) of a map \( f \colon A \to B \) is its factorization through the coequalizer \( q \) of the kernel-pair.
of $f$. When $\mathcal{A}$ has a generator $\mathcal{G}$, this $q$ is equally the coequalizer of the evident maps $\phi, \psi: \Sigma_{G \in \mathcal{G}} M(G, f). G \to A$, where $M(G, f)$ is the set of those pairs $u, v: G \to A$ with $fu = fv$. Because $\mathcal{A}$ is finitely complete, the strong epimorphisms of $[\mathcal{A}]$ coincide with the extremal epimorphisms - those maps that factorize through no proper subobject of their codomain. The regular epimorphisms are extremal; and the converse is true if and only if the regular epimorphisms are closed under composition, which is further equivalent to the assertion that the $n$ of any regular factorization $f = nq$ is monomorphic (cf. [8]).

(8.3) (Cf. [7] Section 6.6) If $\mathcal{A}$ has a generator $\mathcal{G} \subseteq \mathcal{A}$ (not necessarily a strong one - $\mathcal{A}$ could be topological spaces, which is not l.f.p.), write $\mathcal{G}^\sigma$ for the closure of $\mathcal{G}$ in $\mathcal{A}$ under finite coproducts. Let $f: A \to B$ be a regular epimorphism with $A \in \mathcal{A}$. Then $B \in \mathcal{A}$ precisely when $f$ is the coequalizer of maps $u, v: H \to A$ with $H \in \mathcal{G}$. The proof. One direction is trivial by (2.5). For the other, $f$ is by (8.2) the coequalizer of $\phi, \psi: K \to A$ where $K$ is a coproduct of objects of $\mathcal{G}$. The coprojections $i_H: H \to K$ of the finite sub-coproducts of its summands express $K$ as a filtered colimit of objects $H \in \mathcal{G}^\sigma$. If $B_H$ is the coequalizer of $\phi i_H$ and $\psi i_H$, we have induced maps $p_H: B_H \to B$; and these express $B$ as a filtered colimit, since colimits commute with colimits, and since the filtered colimit of the functor constant at $A$ is $A$. The connecting maps $B_H \to B_{H'}$, for $H \subseteq H'$ being clearly epimorphisms, and each $B_H$ being f.p. by (2.5), some $p_H$ is an isomorphism by (2.12).

(8.4) (Cf. [7] Section 6.6) For any $f: A \to B$ in $\mathcal{A}$, write $f = n_1 q_1$ for its regular factorization, where $n_1: A \to B$. Now let $n_1 = n_2 q_2$ be the regular factorization of $n_1$, with $n_2: A_2 \to B$, and write $p_1$ for $q_1: A \to A_1$ and $p_2$ for $q_2 q_1: A \to A_2$. If we continue thus transfinitely, defining $p_\alpha: A \to A_\alpha$ as the colimit of the $p_\beta$ for $\beta < \alpha$ when $\alpha$ is a limit ordinal, we get a sequence of factorizations $n_\alpha p_\alpha$ of $f$, in each of which $p_\alpha$ is an extremal epimorphism. The sequence becomes stationary at some $\alpha$ if and only if $n_\alpha$ is monomorphic, and then $n_\alpha p_\alpha$ is the (unique) factorization of $f$ into an extremal epimorphism and a monomorphism. If $\mathcal{A}$ has a generator
the sequence becomes stationary at the first infinite ordinal $\omega$.

For, in proving $n_\omega$ a monomorphism, it suffices to consider pairs $u, v : G \to A_\omega$ with $n_\omega u = n_\omega v$ and $G \in \mathcal{G}$. Because $G$ is f.p. and $A_\omega$ is the filtered colimit of the $A_i$ with $i < \omega$, such $u, v$ factorize through $A_i \to A_\omega$ for some $i < \omega$, say via $x, y : G \to A_i$. Now, since $n_i x = n_i y$, we have $q_{i+1} x = q_{i+1} y$, giving $u = v$ as desired.

It is clear from (8.2) that an object $A$ of such a category $\mathcal{A}$ has but a small set of regular-epimorphic quotients; and now it follows from the above that an object $A$ of such an $\mathcal{A}$ has but a small set of extremal epimorphic quotients. This is true in particular of any l.f.p. $\mathcal{A}$, and of any locally presentable $\mathcal{A}$ by a trivial extension.

(8.5) A generator $\mathcal{G}$ of $\mathcal{A}$ is said to be projective when each $\mathcal{G}(G, -) : \mathcal{A} \to \mathbb{Set}$ preserves regular epimorphisms. When this is so it follows easily that the $n$ in any regular factorization $n q$ is a monomorphism, so that regular and extremal epimorphisms coincide by (8.2). When the projective generator $\mathcal{G}$ is a strong generator, say with inclusion $K : \mathcal{G} \to \mathcal{A}$, the existence of the conservative, right-adjoint $\bar{K} : \mathcal{A} \to [\mathcal{G}^{\text{op}}, \mathbb{Set}]$ shows that $\mathcal{A}$ has the further property of being a regular category, in the sense that regular epimorphisms are stable under pullback; for this is trivially true in $\mathbb{Set}$ and hence in $[\mathcal{G}^{\text{op}}, \mathbb{Set}]$, and a conservative right adjoint clearly reflects extremal epimorphisms. The classical construction of regular quotients, via congruences, in the category of algebras of a (many-sorted) finitary algebraic theory, shows that the $\{ F_x l \}$ of (3.5) constitute a projective strong generator, so that such a category is regular. We can infer that $\mathcal{C}_{\text{at}}$ is not such a category; for there the extremal epimorphism from 2 to the one-object category given by the monoid $\{ 1, e \}$ with $e^2 = e$ is not regular.

(8.6) We have from [7] Satz 7.14 a result stronger than that at the end of (8.4): if $\mathcal{A}$ is locally presentable, any $A \in \mathcal{A}$ has but a small set of epimorphic quotients - that is, $\mathcal{A}$ is cowellpowered. ($\mathcal{A}$ is trivially wellpowered, since it has a strong generator.) This may be seen by first passing to the comma-category $A/\mathcal{A}$ of objects under $A$. This is cocomplete; and
the projection \( d: A/\mathfrak{A} \to \mathfrak{A} \) is conservative, has the left adjoint \( B \to A + B \), and preserves all connected - and hence all filtered - colimits. Accordingly \( A/\mathfrak{A} \) is locally \( \alpha \)-presentable if \( \mathfrak{A} \) is, by (3.3). Now \( f: A \to B \) is epimorphic in \( \mathfrak{A} \) exactly when \( f: I_A \to f \) is epimorphic in \( A/\mathfrak{A} \); but \( I_A \) is the initial object in \( A/\mathfrak{A} \).

It suffices, then, to prove that the initial object \( 0 \) of a locally-presentable \( \mathfrak{A} \) has but a small set of epimorphic quotients. Call an epimorphic image \( K \) of \( 0 \) an atom, and write \( K \) for the full subcategory of \( \mathfrak{A} \) given by the atoms. Clearly \( K \) is a preordered set, since any two maps \( K \to B \) coincide if \( K \in K \). To prove \( K \) small it will more than suffice to prove \( K \) locally presentable; for then it is complete by (7.2) and well-powered by (8.6), and every object is a subobject of the terminal object. In fact, if we choose some regular \( \alpha \geq \aleph_1 \) such that \( A \) is locally \( \alpha \)-presentable, then \( K \) too is locally \( \alpha \)-presentable.

First, \( K \) is closed under colimits in \( \mathfrak{A} \), since if each \( 0 \to K_i \) is epimorphic, so is \( 0 = \text{colim} 0 \to \text{colim} K_i \); hence \( K \) is cocomplete. We show that \( K \cap \mathfrak{A}_\alpha \), which is clearly contained in \( K_\alpha \), is a strong generator for \( K \) - in fact, dense in \( K \). Then, since \( K \cap \mathfrak{A}_\alpha \) is closed in \( K \) under \( \alpha \)-colimits, it is in fact by (7.2) (i) the whole of \( K_\alpha \). Write \( Z: \mathfrak{A}_\alpha \to \mathfrak{A} \) and \( Z': K \cap \mathfrak{A}_\alpha \to \mathfrak{A} \) for the inclusions; we shall show that for \( K \in K \) the comma-category \( Z/K \) has \( Z'/K \) as a final (full) subcategory - which will be \( \alpha \)-filtered since \( Z/K \) is. Then, since \( K \) is by (7.2) the canonical \( Z/K \)-indexed colimit in \( \mathfrak{A} \) of the objects in \( \mathfrak{A}_\alpha \), it is also the canonical \( Z'/K \)-indexed colimit, in \( \mathfrak{A} \) and hence in \( K \), of the objects in \( K \cap \mathfrak{A}_\alpha \). This latter \( \alpha \)-filtered colimit is preserved by \( K(H, -) \) for all \( H \in K \cap \mathfrak{A}_\alpha \); so that by \([\ast]\) Thm 5.19 these colimits present \( K \cap \mathfrak{A}_\alpha \) as dense in \( K \).

Since \( Z/K \) is filtered, to prove the finality of \( Z'/K \) it suffices by \([\ast]\) Prop. 4.71 to show that any \( f: G \to K \) with \( G \in \mathfrak{A}_\alpha \) factorizes through some \( H \in K \cap \mathfrak{A}_\alpha \). To say that \( K \) is an atom is equally to say that the two coprojections \( i_K, j_K: K \to K + K \) coincide. It follows that the composites of \( i_G, j_G: G \to G + G \) with \( f + f: G + G \to K + K \) coincide. Since \( K \) is the \( \alpha \)-filtered colimit of all the \( h: G' \to K \) with \( G' \in \mathfrak{A}_\alpha \), so \( K + K \) is the \( \alpha \)-
filtered colimit of the $h + h: G' + G' \to K + K$. Since $\bar{\alpha}(G, -)$ preserves this colimit, $f$ is the composite of some $g_1: G \to G_1$ and some $f_1: G_1 \to K$, with $G_1 \in \bar{\alpha}_\alpha$, such that $(g_1 + g_1)i_{G_1} = (g_1 + g_1)i_G$. Now repeat this process, replacing $f: G \to K$ by $f_1: G_1 \to K$, to get $g_2: G_1 \to G_2$ with

$$(g_2 + g_1)i_{G_1} = (g_2 + g_1)i_{G_1}$$

and $f_2: G_2 \to K$ with $f_1 = f_2 g_2$; and so on. Finally let the colimit of

$$G \xrightarrow{g_1} G_1 \xrightarrow{g_2} G_2 \xrightarrow{g_3} G_3 \to \ldots$$

be $q_n: G_n \to H$, inducing $\bar{f}: H \to K$. Then $H \in \bar{\alpha}_\alpha$, since $\bar{\alpha}_\alpha$ is closed under countable colimits. Moreover, since

$$i_H q_n = (q_n + q_n)i_{G_n} = (q_{n+1} + q_{n+1})(g_{n+1} + g_{n+1})i_{G_n},$$

and since $(g_{n+1} + g_{n+1})i_{G_n} = (g_{n+1} + g_{n+1})i_{G_n}$, we have $i_H q_n = i_H q_n$ for all $n$; so that $i_H = i_H$, and $H \in \bar{K}$. Thus $H \in \bar{K} \cap \bar{\alpha}_\alpha$, as required.

(8.7) When we regard a small set $\mathcal{G}$ of objects of $\bar{\alpha}$ as a discrete category $|\mathcal{G}|$, with inclusion $Z: |\mathcal{G}| \to \bar{\alpha}$, the counit $\epsilon$ of the adjunction

$$\epsilon: \bar{\alpha} \to [|\mathcal{G}|^{op}, \mathcal{S}_{\mathcal{E}}]$$

has components $\epsilon_A: \Sigma_G \mathcal{G} \bar{\alpha}(G, A). G \to A$, whose components in turn are given by $\epsilon_{A,G,f} = \bar{f}$ for $f: G \to A$. By the definitions of $[\ast]$ Section 3.6, $\mathcal{G}$ is a generator [resp. a strong generator] when $\bar{Z}$ is faithful [resp. conservative], which by $[\ast]$ Section 3.4 is the case precisely when each $\epsilon_A$ is an epimorphism [resp. an extremal epimorphism]. We call $\mathcal{G}$ a regular generator when each $\epsilon_A$ is a regular epimorphism. Then, in the notation of (8.2), we have for each $A \in \bar{\alpha}$ a coequalizer diagram

$$\Sigma_G \mathcal{G} \mathcal{M}(G, \epsilon_A). G \xrightarrow{\phi} \Sigma_G \mathcal{G} \bar{\alpha}(G, A). G \xrightarrow{\epsilon_A} A,$$

so that our definition of regular generator agrees with that in $[\ast]$ Section 4.10. It was observed there that each $\bar{\alpha}(G, -)$ with $G \in \mathcal{G}$ preserves the colimit (8.8), sending it in fact to a split coequalizer diagram in $\mathcal{S}_{\mathcal{E}}$. Finally, a small dense full subcategory $\mathcal{G}$ is a fortiori a regular generator (cf. $[\ast]$ Section 5.3), since we have the regular epimorphism
\[ \sum_{G \in \mathcal{G}} \mathcal{G}(G, A). G \rightarrow f^{G \in \mathcal{G}} \mathcal{G}(G, A). G = A. \]

(8.9) We have as in [\*] Prop. 5.24 that, if \( \mathcal{G} \) is a regular generator in \( \mathcal{A} \), the closure \( \mathcal{G}^{\Sigma} \) of \( \mathcal{G} \) under small coproducts is dense in \( \mathcal{A} \); this follows from [\*] Thm 5.19, for the coequalizer diagrams (8.8) are preserved by the \( \mathcal{A}(H, \cdot) \) for \( H \in \mathcal{G}^{\Sigma} \), a product of split coequalizer diagrams being a split coequalizer diagram. More importantly, we have as in [\*] Prop. 5.24 that, if \( \mathcal{G} \) is a regular generator contained in \( \mathcal{A}_f \) so that \( \mathcal{A} \) is l.f.p. - the closure \( \mathcal{G}^{\sigma} \) of \( \mathcal{G} \) under finite coproducts is already dense in \( \mathcal{A} \); this follows from [\*] Thm 5.19 because the infinite coproducts in (8.8) are the filtered colimits of their finite sub-coproducts, and these filtered colimits are preserved by the \( \mathcal{A}(H, \cdot) \) with \( H \in \mathcal{G}^{\sigma} \), since \( \mathcal{G}^{\sigma} \subset \mathcal{A}_f \) by (2.5). In fact, \( \mathcal{G} \) itself is dense in \( \mathcal{A} \) if coproducts in \( \mathcal{A} \) are universal; for this, see [7] Satz 3.7.

(8.10) It is immediate that, if \( \mathcal{G} \) is a generator or a strong generator, so is any small \( \mathcal{G}' \supset \mathcal{G} \). For regular generators this is still true but no longer immediate. It suffices to show that, when the \( \epsilon_A \) of (8.7) are regular epimorphisms, so is \( (\epsilon_A, f): (\Sigma \mathcal{A}(G, A). G) + C \rightarrow A \) for any \( f: C \rightarrow A \). However the composite of \( (\epsilon_A, f) \) with

\[ 1 + \epsilon_C : (\Sigma \mathcal{A}(G, A). G) + (\Sigma \mathcal{A}(G, C). G) \rightarrow (\Sigma \mathcal{A}(G, A). G) + C \]

is, by the naturality of \( \epsilon \), the composite of \( \epsilon_A \) with

\[ (1, \Sigma \mathcal{A}(G, f). G): (\Sigma \mathcal{A}(G, A). G) + (\Sigma \mathcal{A}(G, C). G) \rightarrow \Sigma \mathcal{A}(G, A). G. \]

Since the latter map is a retraction, its composite with \( \epsilon_A \) is a regular epimorphism by [8] Prop. 2.1; thus \( (\epsilon_A, f)(1 + \epsilon_C) \) is a regular epimorphism; and, since \( 1 + \epsilon_C \) is an epimorphism, \( (\epsilon_A, f) \) is a regular epimorphism by [8] Prop. 2.2.

(8.11) (Cf. [7] Satz 7.6) Let \( \mathcal{G} \subset \mathcal{A}_f \) be a regular generator of the l.f.p. \( \mathcal{A} \), and write \( \mathcal{G}^e \) for the set of objects which are coequalizers of pairs \( u, v: H \rightarrow K \) with \( H, K \in \mathcal{G}^{\sigma} \). Then \( \mathcal{A}_f \) consists of the retracts of the objects of \( \mathcal{G}^e \); and \( \mathcal{A}_f \) coincides with \( \mathcal{G}^e \) if all extremal epimorphisms in \( \mathcal{A} \) are regular.

PROOF. By (2.5), \( \mathcal{A}_f \) certainly contains all the retracts of the objects of
For the converse, let $A \in \mathcal{G}_f$, and express $A$ as in (8.8) as the coequalizer of maps $\phi, \psi : P \to Q$ with $P, Q \in \mathcal{G}_\Sigma$. In the functor category $[\mathcal{G}, \mathcal{G}]$ consider the canonical cone of vertex $(\phi, \psi)$ given by all maps $(i, j) : (u, v) \to (\phi, \psi)$ of the form

\[
\begin{array}{ccc}
H & \xrightarrow{u} & K \\
\downarrow{i} & & \downarrow{j} \\
\phi & \xleftarrow{v} & \psi \\
\end{array}
\]

where $H, K \in \mathcal{G}_\sigma$. It is easy to see that the comma category indexing this cone is filtered; for, since the coproduct $P$ is the filtered colimit of its finite sub-coproducts, every $f : B \to P$ with $B \in \mathcal{G}_f$ factorizes through one of these finite sub-coproducts, which lies in $\mathcal{G}_\sigma$; and similarly for maps $g : B \to Q$. The same reasoning shows that the $i : H \to P$ which occur in this cone include all the coprojections from finite sub-coproducts of $P$; while the possibility of taking $H = 0$ shows that the $j : K \to Q$ which occur include all the coprojections from finite sub-coproducts of $Q$; and from these observations it follows that this filtered cone is a colimit-cone. If $A_{u,v} \in \mathcal{G}_e$ denotes the coequalizer of $u, v$ in the diagram above, we conclude that the coequalizer $A$ of $\phi, \psi$ is the filtered colimit of the $A_{u,v}$; so that, by (2.12), the f.p. $A$ is a retract of some $A_{u,v}$. If the composite extremal epimorphism $K \to A_{u,v} \to A$ is in fact a regular epimorphism, we have by (8.3) that $K \in \mathcal{G}_e$.

(8.12) When $\mathcal{G}$ is the category of algebras for an $X$-sorted finitary algebraic theory, as in (2.8), it follows from (8.5) that extremal epimorphisms are regular and that the $\{F_x, l\}_{x \in X}$ constitute a regular generator. By (8.11), therefore, $\mathcal{G}_f$ consists of the coequalizers of the finite coproducts of the $F_x, l$, as stated in (2.7) and (2.8). When $\mathcal{G} = \mathcal{C}at_a$, the ordinary category of small categories, it is easy to verify (cf. [*] Section 5.3) that the ordered set $3 = \{0, 1, 2\}$ is dense in $\mathcal{G}$. A fortiori the set $\mathcal{G}$ of all free categories $F_g$ on some finite graph $g$ is dense, since $3$ is such a category. Hence $\mathcal{G}$ is certainly a regular generator, and $\mathcal{G} \subset \mathcal{G}_f$ by (2.9). Thus, by
(8.11), any \( A \in \mathcal{A}_f \) is a retract of an object of \( \mathcal{G}_e \); so that there is some extremal epimorphism \( p : F g \to A \). If \( h \) is the image in \( A \) of the finite graph \( g \) under \( p \), then \( h \) too is a finite graph, with the same objects as \( A \), which generates \( A \), the induced \( q : F h \to A \) being an extremal epimorphism. But such a \( q \), which is the identity on objects, is easily seen to be a regular epimorphism. Hence by (8.3) we conclude that \( A \in \mathcal{G}_e \) as asserted in (2.9).

Similar arguments apply to the category \( \mathcal{G}_{pd_0} \) of small groupoids, and to the subcategories of \( \text{Cat}_0 \) and \( \mathcal{G}_{pd_0} \) mentioned in (3.5).

9. FINITARY ESSENTIALLY-ALGEBRAIC \( \mathcal{G} \)-THEORIES

(9.1) Whenever \( \mathcal{V}_0 \) is l.f.p., the closed category \( \mathcal{V} \) is locally bounded in the sense of \([*]\) Section 6.1, as is pointed out there; so that all the results of \([*]\) Ch. 6 apply. Our aim now is to give more delicate results under our present stronger hypothesis of (5.12), that \( \mathcal{V} \) is l.f.p. as a closed category. In addition, some of the results of \([*]\) admit simpler proofs under this hypothesis.

(9.2) By a finitary essentially-algebraic \( \mathcal{G} \)-theory we mean a small finitely-complete \( \mathcal{G} \)-category \( \mathcal{F} \). If \( \mathcal{F} \) denotes the set of finite indexing types, this is the same thing as an \( \mathcal{F} \)-theory in the sense of \([*]\) Section 6.3. We shall henceforth omit, as understood, the adjective «essentially-algebraic», and call \( \mathcal{F} \) a finitary \( \mathcal{V} \)-theory, or just a finitary theory. A model of \( \mathcal{F} \) in \( \mathcal{B} \) is a left-exact functor \( \mathcal{F} \to \mathcal{B} \); and these form a \( \mathcal{V} \)-category

\[
\text{Mod}(\mathcal{F}, \mathcal{B}) = \text{Lex}(\mathcal{F}, \mathcal{B}) \subseteq [\mathcal{F}, \mathcal{B}].
\]

A model of \( \mathcal{F} \) in \( \mathcal{V} \) is called a \( \mathcal{F} \)-algebra; and these form a \( \mathcal{V} \)-category

\[
\text{F-Alg} = \text{Mod}(\mathcal{F}, \mathcal{V}) = \text{Lex}(\mathcal{F}, \mathcal{V}) \subseteq [\mathcal{F}, \mathcal{V}].
\]

Similarly, for a regular cardinal \( \alpha \), we define an \( \alpha \)-ary \( \mathcal{V} \)-theory to be a small \( \alpha \)-complete \( \mathcal{V} \)-category \( \mathcal{F} \); and now a model or an algebra is to be \( \alpha \)-left-exact. The results below for finitary theories extend at once to \( \alpha \)-ary ones.

(9.3) For any small \( \mathcal{V} \)-category \( K \), let \( \mathcal{K} : K \to \mathcal{K} \) be the full embedding into its free completion under finite limits; see \([*]\) Section 5.7. Then, by
(7.1) above, \( \overline{K} \) is a finitary theory. By [\*] Thm 5.35, restriction along \( K \) gives for every finitely-complete \( B \) an equivalence \( \text{Mod}[\overline{K}, B] \cong [K, B] \), whose equivalence-inverse is \( \text{Ran}_K \) (right Kan extension along \( K \)). When \( K \) is the unit \( \mathbb{O} \)-category \( \mathbb{1} \), it follows from [\*] Thm 5.35 together with (3.4) and (7.2) above that \( \overline{K} = \mathbb{O}_f^{op} \); and now, since \( \text{Mod}[\mathbb{O}_f^{op}, B] \cong B \) for any finitely-complete \( B \), we may call \( \mathbb{O}_f^{op} \) the finitary theory of an object.

(9.4) For any \( \mathbb{O} \)-theory \( \mathcal{T} \), the representable \( \mathcal{T}(t, -): \mathcal{T} \to \mathbb{O} \) preserves all limits and is therefore a \( \mathcal{T} \)-algebra. We conclude from [\*] Prop. 5.16 that the Yoneda embedding \( Y: \mathcal{T}^{op} \to [\mathcal{T}, \mathbb{O}] \) factorizes through \( \mathcal{T} \)-Alg, say via the fully-faithful \( Z: \mathcal{T}^{op} \to \mathcal{T} \)-Alg; the full inclusion \( \mathcal{T} \)-Alg \( \to [\mathcal{T}, \mathbb{O}] \) is isomorphic to \( \tilde{Z} \) where \( \tilde{Z} F = \mathcal{T} \)-Alg(\( \mathcal{T}^*, F \)); and \( Z \) is a dense embedding of \( \mathcal{T}^{op} \) in \( \mathcal{T} \)-Alg. It also follows from (4.12) that \( \mathcal{T} \)-Alg is closed in \( [\mathcal{T}, \mathbb{O}] \) under filtered colimits, so that \( \mathcal{T} \)-Alg admits filtered colimits and \( \tilde{Z} \) is finitary. (We know from [\*] Thm 6.11 that \( \mathcal{T} \)-Alg is in fact reflective in \([\mathcal{T}, \mathbb{O}]\), and hence complete and cocomplete; but in the present hypotheses we get a simpler proof of this in (9.7) below.)

(9.5) By a map \( M: \mathcal{T} \to \mathcal{T}' \) of finitary \( \mathbb{O} \)-theories we mean a left-exact functor. Since \( FM: \mathcal{T} \to \mathcal{O} \) is left exact when \( F: \mathcal{T}' \to \mathcal{O} \) is, the functor \( [M, 1]: [\mathcal{T}', \mathcal{O}] \to [\mathcal{T}, \mathcal{O}] \) restricts to a functor \( M^*: \mathcal{T}' \)-Alg \( \to \mathcal{T} \)-Alg; such a functor between categories of algebras induced by a map of theories is called an algebraic functor. Note that \( M^* \) is finitary; for \( [M, 1] \) preserves all colimits, while \( \mathcal{T}' \)-Alg and \( \mathcal{T} \)-Alg are closed under filtered colimits. Now by (6.12), the left adjoint \( \text{Lan}_M: [\mathcal{T}, \mathbb{O}] \to [\mathcal{T}', \mathbb{O}] \) of \( [M, 1] \) restricts to a functor \( M^*: \mathcal{T} \)-Alg \( \to \mathcal{T}' \)-Alg left adjoint to \( M^* \). Since

\[
\text{Lan}_M \mathcal{T}'(s, -) = \mathcal{T}^* \mathcal{T}'(s, -) \cong (\mathcal{T}^*)s = \mathcal{T}(Ms, -),
\]

we have commutativity to within isomorphism in each square of the diagram

\[
\begin{array}{ccc}
\mathcal{T}^{op} & \xrightarrow{Z} & \mathcal{T} \text{-Alg} & \xrightarrow{\tilde{Z}} & [\mathcal{T}, \mathbb{O}] \\
M^{op} \downarrow & & M^* \downarrow & & \text{Lan}_M \downarrow \\
\mathcal{T}'^{op} & \xrightarrow{Z'} & \mathcal{T}' \text{-Alg} & \xrightarrow{\tilde{Z}'} & [\mathcal{T}', \mathbb{O}].
\end{array}
\]

The commutativity of the left square here is, by [\*] Prop. 6.16, a general
fact about maps of \( \mathcal{F} \)-theories for any \( \mathcal{F} \) and any \( \mathcal{V} \); that of the right square, however, depending on (6.12), is special to l.f.p. \( \mathcal{V} \) and finitary (or more generally \( a \)-ary) theories.

(9.7) For the finitary theory \( \mathcal{F} \), let \( K: \mathcal{F} \to \mathcal{F} \) be, as in (9.3), the embedding into the free finite-limit completion of \( \mathcal{F} \). Since \( \mathcal{F} \) is finitely complete, there is a left-exact \( L: \mathcal{F} \to \mathcal{F} \), unique to within isomorphism, for which \( L K \cong 1: \mathcal{F} \to \mathcal{F} \). The composite of \( L^*: \mathcal{F}\text{-Alg} \to \mathcal{F}\text{-Alg} \) (which is just composition with \( L \)) and the equivalence \( \mathcal{F}\text{-Alg} \cong [\mathcal{F}, \mathcal{V}] \) of (9.3) (which is given by composition with \( K \)) is just the inclusion \( \tilde{Z}: \mathcal{F}\text{-Alg} \to [\mathcal{F}, \mathcal{V}] \). Since \( L^* \) has a left adjoint by (9.5), so has \( \tilde{Z} \); so that \( \mathcal{F}\text{-Alg} \) is a reflective full subcategory of \([\mathcal{F}, \mathcal{V}]\).

(9.8) **Theorem.** For a finitary theory \( \mathcal{F} \), the category \( \mathcal{F}\text{-Alg} \) is l.f.p.; and \( (\mathcal{F}\text{-Alg})_f \) is the replete image of the embedding \( Z: \mathcal{F}^{\text{op}} \to \mathcal{F}\text{-Alg} \), and is hence equivalent to \( \mathcal{F}^{\text{op}} \). Moreover a category \( \mathcal{G} \) is equivalent to \( \mathcal{F}\text{-Alg} \) for some finitary theory \( \mathcal{G} \) precisely when \( \mathcal{G} \) is l.f.p., whereupon \( Z \cong \mathcal{G}^{\text{op}} \).

**Proof.** \( \mathcal{F}\text{-Alg} \) is l.f.p. by (3.1) and (9.7), since \( \tilde{Z} \) is finitary by (9.4). The representables \( \mathcal{F}(t, -) \) being f.p. in \([\mathcal{F}, \mathcal{V}]\) by (2.2), their images under the left adjoint \( -^*Z \) of \( \tilde{Z} \) are f.p. in \( \mathcal{F}\text{-Alg} \) by (2.4). But they are their own images, since the representables already lie in \( \mathcal{F}\text{-Alg} \); whence \( \mathcal{F}^{\text{op}} \subseteq (\mathcal{F}\text{-Alg})_f \). Since \( \mathcal{F}^{\text{op}} \) is dense in \( \mathcal{F}\text{-Alg} \) by (9.4), it follows from (7.2) that \( (\mathcal{F}\text{-Alg})_f \) is the closure of \( \mathcal{F}^{\text{op}} \) in \( \mathcal{F}\text{-Alg} \) under finite colimits. This, however, is just the repletion of \( \mathcal{F}^{\text{op}} \); for \( Z: \mathcal{F}^{\text{op}} \to \mathcal{F}\text{-Alg} \) preserves finite colimits. To see this, it suffices to observe that \( \mathcal{F}\text{-Alg}(Z -, F): \mathcal{F} \to \mathcal{V} \) preserves finite limits for each \( F \in \mathcal{F}\text{-Alg} \); but \( \mathcal{F}\text{-Alg}(Z -, F) \) is just \( \tilde{Z} F \), or \( F \) seen as an object of \([\mathcal{F}, \mathcal{V}]\) - and the algebra \( F \) is by definition left exact. The final assertion now follows from (7.2).

(9.9) **Theorem.** Let \( \mathcal{F} \) be a finitary theory and let \( \mathcal{B} \) be cocomplete. A functor \( S: \mathcal{F}\text{-Alg} \to \mathcal{B} \) has a left adjoint \( T \) precisely when it is cocontinuous; and then its restriction \( G = SZ: \mathcal{F}^{\text{op}} \to \mathcal{B} \) is a comodel for \( \mathcal{F} \), in the sense that \( G^{\text{op}}: \mathcal{F} \to \mathcal{B}^{\text{op}} \) is a model. Moreover \( 1: \mathcal{B} \to SZ \) then expresses \( S \) as \( \text{Lang}_G \). On the other hand, a given functor \( G: \mathcal{F}^{\text{op}} \to \mathcal{B} \) is a comodel precisely when \( \tilde{G}: \mathcal{B} \to [\mathcal{F}, \mathcal{V}] \) factorizes through the full em-
bedding \( \tilde{Z} : \mathcal{J}\text{-Alg} \to [\mathcal{J}, \mathcal{C}] \), say as \( \tilde{G} \cong \tilde{Z} T \); then \( T \) has the left adjoint \( S = \text{Lan}_Z G \), and \( SZ \cong G \). In this way we obtain an equivalence of \( \mathcal{C} \)-categories

\[
\text{Cocets}[\mathcal{J}\text{-Alg}, \mathcal{B}] = \text{Ladj}[\mathcal{J}\text{-Alg}, \mathcal{B}] \cong \text{Com}[\mathcal{J}^{\text{op}}, \mathcal{B}] = \text{Mod}[\mathcal{J}, \mathcal{B}^{\text{op}}]^{\text{op}},
\]

the equivalence from left to right being composition with \( Z \) and that from right to left being \( \text{Lan}_Z \).

**Proof.** When account is taken of \([\ast]\) Thm 6.11, this is a special case of \([\ast]\) Thm 5.56. The latter result, however, is very general; accordingly, we give a simple direct proof in the present case. \( Z \) being fully faithful, so is \( \text{Lan}_Z \) by \([\ast]\) Thm 4.99, and \( (\text{Lan}_Z G)Z \cong G \) for any \( G \) by \([\ast]\) Prop. 4.23. We have therefore only to identify the image under \( \text{Lan}_Z \) of the comodels \( G \). Since \( SZ \) is trivially a comodel if \( S \) is cocontinuous, this image contains the cocontinuous \( S \) by \([\ast]\) Thm 5.29; and a cocontinuous \( S \) is left adjoint by (7.8) and (9.8). For the other direction, we observe that \( G \) is a comodel when \( G \) preserves finite colimits, which is to say that \( B(G,-, B) : \mathcal{J} \to \mathcal{C} \) preserves finite limits for each \( B \in \mathcal{B} \), or that \( \tilde{G} : \mathcal{B} \to [\mathcal{J}, \mathcal{C}] \) factorizes through \( \mathcal{J}\text{-Alg} \) as \( \tilde{G} \cong \tilde{Z} T \) for some \( T \). But then \( T \) has as left adjoint \( S \) the restriction \( (-*G)\tilde{Z} = \tilde{Z} -*G \) of the left adjoint \(-*G\) of \( \tilde{G} \); so that \( S = \text{Lan}_Z G \) by \([\ast]\) Section 4.1. Thus for each model \( G \), \( \text{Lan}_Z G \) is cocontinuous.

(9.10) **Theorem.** Let \( \mathcal{J} \) be a finitary theory, let \( \mathcal{B} \) be cocomplete, and let the adjunction \( S \dashv T : \mathcal{B} \to \mathcal{J}\text{-Alg} \) be related to the comodel \( G : \mathcal{J}^{\text{op}} \to \mathcal{B} \) as in (9.9), by \( S \cong \text{Lan}_Z G \) and \( G \cong SZ \). Then \( T \) is finitary if and only if \( G(\mathcal{J}^{\text{op}}) \subset \mathcal{B}_f \).

**Proof.** If \( T \) is finitary, (2.4) gives \( S((\mathcal{J}\text{-Alg})_f) \subset \mathcal{B}_f \), and hence \( G(\mathcal{J}^{\text{op}}) \subset \mathcal{B}_f \), by (9.9) and (9.8). For the converse, since \( \tilde{Z} \) is finitary and conservative, \( T \) is finitary by (1.3) if \( \tilde{G} \) is so. Since colimits in \( [\mathcal{J}, \mathcal{C}] \) are formed pointwise, \( \tilde{G} \) is finitary when each \( E_t \tilde{G} : \mathcal{B} \to \mathcal{C} \) is finitary. However \( E_t \tilde{G} = B(G_t, -) \), which is finitary since \( G_t \in \mathcal{B}_f \).

(9.11) **Proposition** (cf. [7] Satz 7.8). Suppose that the \( G : \mathcal{J}^{\text{op}} \to \mathcal{B} \) of (9.10) not only takes its values in \( \mathcal{B}_f \), but is fully faithful. Then \( S \) is
fully faithful, and its image is the closure of $G(\mathcal{T}^{op})$ in $\mathcal{B}$ under small colimits.

**Proof.** $S$ is fully faithful precisely when the unit $\eta : I \to TS$ is an isomorphism. Since both $T$ and $S$ are finitary, and since $\mathcal{T}$-Alg is by (6.11) the closure of $\mathcal{T}^{op}$ under filtered colimits, it suffices to prove that $\eta Z : Z \to TSZ$ is an isomorphism; or equivalently that $\tilde{Z}\eta Z : \tilde{Z}Z \to \tilde{Z}TSZ$ is an isomorphism in $[\mathcal{I}, \mathcal{V}]$. But $\tilde{Z}Z \cong Y$, and $\tilde{Z}TSZ \cong \tilde{G}G$ by (9.9); and $\tilde{Z}\eta Z$ is the canonical map $Y \to \tilde{G}G$, which is an isomorphism (see [*] Section 4.2) if $G$ is fully faithful. Since $\mathcal{T}$-Alg is the closure of $\mathcal{T}^{op}$ under filtered colimits, the image of $S$ is contained in the closure in $\mathcal{B}$ of $G(\mathcal{T}^{op})$ under small colimits, and therefore coincides with this closure.

(9.12) **Example** (cf. [7], loc. cit.). With $\mathcal{V} = \mathcal{S}$-set, let $\mathcal{B}$ be the dual of the category of compact Hausdorff spaces. By [7] Section 6.5, $\mathcal{B}_f$ consists of the finite spaces. Taking $\mathcal{I} = \mathcal{S}$-set $\cong (\mathcal{B}_f)^{op}$ and $G$ to be the inclusion, we get by (9.11) an identification of $(\mathcal{S}$-set-Alg$)^{op}$ with the small-limit closure in $\mathcal{B}^{op}$ of the finite spaces: that is, the category of totally-disconnected compact spaces. On the other hand, $\mathcal{S}$-set-Alg is easily seen to be the category of boolean algebras.

(9.13) **Proposition.** If $\mathcal{I}$ and $\mathcal{I}'$ are finitary theories, a functor $T : \mathcal{I}'$-Alg $\to \mathcal{I}$-Alg is algebraic - that is, of the form $M^*$ for some map $M : \mathcal{I}' \to \mathcal{I}$ of theories - precisely when it is finitary and has a left adjoint.

**Proof.** One direction comes from (9.5), and the other from (9.9) and (9.10) on taking $\mathcal{B} = \mathcal{I}'$-Alg.

(9.14) We can improve (9.13) by observing that every $\mathcal{V}$-natural transformation $M_1^* \to M_2^*$ between algebraic functors is induced by a unique $\mathcal{V}$-natural transformation $M_1 \to M_2$. In fact the situation is still richer than this. Denote by $\mathcal{V}$-Th the ($\mathcal{V}$-$\mathcal{C}$-Cat)-category whose objects are the finitary $\mathcal{V}$-theories and whose hom-object $\mathcal{V}$-Th($\mathcal{I}, \mathcal{I}'$) is the $\mathcal{V}$-category Lex$[\mathcal{I}, \mathcal{I}']$, a full subcategory of $[\mathcal{I}, \mathcal{I}']$. Denote by $\mathcal{V}$-$\mathcal{L}$fp the ($\mathcal{V}$-$\mathcal{C}$-Cat)-category whose objects are the l.f.p. $\mathcal{V}$-categories and whose hom-object $\mathcal{V}$-$\mathcal{L}$fp($\mathcal{A}, \mathcal{A}'$) is the full sub-$\mathcal{V}$-category of $[\mathcal{A}, \mathcal{A}']$ given by the finitary $\mathcal{V}$-functors with
left adjoints. Then (9.9) and (9.10) give an equivalence

\[(9.15)\quad \mathcal{V}\mathcal{F}(\mathcal{J}, \mathcal{J}^\prime) \simeq \mathcal{V}\mathcal{L}_f^p(\mathcal{J}^-\text{Alg}, \mathcal{J}^-\text{Alg}) ;\]

so that, by (9.8), we have in fact a biequivalence

\[(9.16)\quad (\mathcal{V}\mathcal{F})^{op} = \mathcal{V}\mathcal{L}_f^p.\]

The functor sending the \(\mathcal{V}\)-theory \(\mathcal{J}\) to \(\mathcal{J}^-\text{Alg}\) may be called the \textit{semantics}
functor \(\text{Sem}\), and that sending the l.f.p. \(\mathcal{A}\) to \(\mathcal{A}_f^{op}\) may be called the \textit{structure}
functor \(\text{Str}\). We can in fact define \(\text{Str}\) on a category \(\mathcal{V}\mathcal{F}\text{-Fin}\) bigger
than \(\mathcal{V}\mathcal{L}_f^p\); the objects are now all cocomplete \(\mathcal{A}\) with \(\mathcal{A}_f\) small, and
the hom-object is still the full subcategory of \([\mathcal{A}, \mathcal{A}^\prime]\) given by the finitary
functors with left adjoints; this hom-object is in general not a \(\mathcal{V}\)-category, but a \(\mathcal{V}^\prime\)-category for some larger \(\mathcal{V}\) in a higher universe. If \(\text{Str}\mathcal{A}\)
is still defined as \(\mathcal{A}_f^{op}\), we have a biadjunction

\[(9.17)\quad \text{Str} \dashv \text{Sem}: (\mathcal{V}\mathcal{F})^{op} \to \mathcal{V}\mathcal{F}\text{-Fin},\]

since (9.9) and (9.10) give \(\mathcal{V}\mathcal{F}(\mathcal{J}, \mathcal{A}_f^{op}) \simeq \mathcal{V}\mathcal{F}\text{-Fin}(\mathcal{A}, \mathcal{J}^-\text{Alg})\).

(9.18) It follows from (7.5) that if \(\mathcal{J}\) is a finitary \(\mathcal{V}\)-theory, then \(\mathcal{J}_o\) is
a finitary \(\text{Set}\)-theory and \((\mathcal{J}^-\text{Alg})_o \simeq \mathcal{J}_o^-\text{Alg}\). Using (6.11), (7.2), (7.5)
and (9.8), we can describe the equivalence in detail: the \(\mathcal{J}\)-algebra \(F: \mathcal{J} \to \mathcal{V}\) is sent to
the \(\mathcal{J}_o\)-algebra \(V F_o: \mathcal{J}_o \to \text{Set}\); while the equivalence-inverse sends
the \(\mathcal{J}_o\)-algebra \(H: \mathcal{J}_o \to \text{Set}\) to the colimit \(F\) in \([\mathcal{J}, \mathcal{V}]\) of
the canonical functor \(Y_o d^{op}: (el H)^{op} \to [\mathcal{J}, \mathcal{V}]_o\) of (6.5). Not every
finitary \(\text{Set}\)-theory need admit an enrichment to a finitary \(\mathcal{V}\)-theory, since
an l.f.p. ordinary category need admit no enrichment to a \(\mathcal{V}\)-category. Thus,
when \(\mathcal{V} = \mathcal{A}_f\), the l.f.p. category of all groups admits no additive
structure. When \(\mathcal{V} = \text{Cat}\) or \(\text{Gpd}\), however, every finitary \(\text{Set}\)-theory \(\mathcal{S}\) admits
at least one enrichment to a finitary \(\mathcal{V}\)-theory \(\mathcal{J}\); it suffices to take \(\mathcal{J}\) to
be \(\mathcal{S}\), made into a 2-category with only identity 2-cells; then \(\mathcal{J}\) trivially
admits the cotensor product \(J_h^o\), and is hence finitely complete as a 2-category
and \textit{a fortiori} as a \(\text{Gpd}\)-category. Such an \(\mathcal{S}\) may admit two different
enrichments to a finitary \(\mathcal{V}\)-theory: taking \(\mathcal{V} = \text{Cat}\) again, we have
the \(\mathcal{V}\)-theory \(\text{Cat}_f^{op}\) of an object; and if \(\mathcal{S}\) is the underlying category
\(\text{Cat}_o^{op}\) of this, the \(\mathcal{J}\) constructed above is not \(\text{Cat}_f^{op}\). In fact,
(Cat^{op})-Alg = \text{Cat}, while J-Alg is the 2-category of all small categories, all functors, but only identity 2-cells.

10. FINITARY \mathcal{V}-SKETCHES

(10.1) By a finitary \mathcal{V}-sketch \((\mathcal{S}, \Phi)\), or just \(\Phi\) for short, we mean a small \mathcal{V}-category \(\mathcal{S}\) together with a small set \(\Phi = \{\Phi_y : H_y : \mathcal{S}(A_y, P_y)\}_{y \in \Gamma}\), where the \(H_y : \mathcal{K}_y \to \mathcal{V}\) are finite indexing types, where \(P_y : \mathcal{K}_y \to \mathcal{S}\), and where \(A_y \in \mathcal{S}\). A model of \(\Phi\) in \(\mathcal{B}\) is a functor \(F : \mathcal{S} \to \mathcal{B}\) such that each cylinder

\[
H_y : \mathcal{S}(A_y, P_y) \xrightarrow{\Phi_y} \mathcal{S}(FA_y, FP_y) \xrightarrow{F} \mathcal{B}(FA_y, FP_y)
\]

is a limit-cylinder in \(\mathcal{B}\); and a model of \(\Phi\) in \(\mathcal{V}\) is called a \(\Phi\)-algebra; see \([\ast]\) Section 6.3. The results of \([\ast]\) Ch. 6 all apply; and in fact the proof of \([\ast]\) Thm 6.5 can be simplified in our present case, replacing the appeal to Theorem 10.2 of [9] by one to the simpler Theorem 6.2 of [9]. In particular, from \([\ast]\) Section 6.4, we get

(10.2) THEOREM. Consider the composite of the Yoneda embedding \(\mathcal{S}^{op} \to [\mathcal{S}, \mathcal{V}]\) and the reflexion of \([\mathcal{S}, \mathcal{V}]\) onto \(\Phi-\text{Alg}\), and let \(\mathcal{S}^{op}\) be the closure of its image in \(\Phi-\text{Alg}\) under finite colimits, so that we have a functor \(K : \mathcal{S} \to \mathcal{J}\). Then \(K\) is a model of \(\Phi\), and is fully faithful precisely when the cylinders \(\Phi_y\) are already limit-cylinders in \(\mathcal{S}\). The sketch \(\Phi\) and the finitary theory \(\mathcal{J}\) have the same models in any finitely-complete \(\mathcal{B}\), restriction along \(K\) giving an equivalence \(\text{Mod}[\mathcal{J}, \mathcal{B}] \cong \Phi-\text{Mod}[\mathcal{S}, \mathcal{B}]\), whose equivalence-inverse is given by right Kan extension along \(K\).

(10.3) If \(\mathcal{J}\) and \(\mathcal{J}'\) are finitary \(\mathcal{V}\)-theories, we have as in \([\ast]\) Section 6.5 the sketch \((\mathcal{J} \otimes \mathcal{J}', \Phi)\), where \(\Phi\) is so chosen that a \(\Phi\)-model in \(\mathcal{B}\) is a functor \(F : \mathcal{J} \otimes \mathcal{J}' \to \mathcal{B}\) with each \(F(t, -)\) and each \(F(-, s)\) left exact. The corresponding finitary theory in the sense of (10.2), which was denoted in \([\ast]\) by \(\mathcal{J} \otimes_{\text{lex}} \mathcal{J}'\) when the appropriate set of indexing types was \(\mathcal{J}\), may be denoted here by \(\mathcal{J} \otimes_{\text{lex}} \mathcal{J}'\).

(10.4) PROPOSITION. If \(\mathcal{A}\) is l.f.p. and \(\mathcal{J}\) is a finitary theory, then \(\text{Mod}[\mathcal{J}, \mathcal{A}]\) is l.f.p.
PROOF. $\mathcal{G}$ is equivalent by (9.8) to $\mathcal{F}'_{\text{Alg}}$ for some finitary theory $\mathcal{F}'$. Now $\text{Mod}[\mathcal{J}, \mathcal{G}] \cong \text{Mod}[\mathcal{J}, \text{Mod}[\mathcal{J}' \mathcal{V}]]$ is equivalent by [*] Section 6.5 to the l.f.p. category $(\mathcal{F} \Theta_{\text{lex}} \mathcal{F'})_{\text{Alg}}$.

(10.5) Let $\mathcal{J}$ and $\mathcal{J}'$ be finitary $\mathcal{V}$-theories as in (10.3), and let $K: \mathcal{J} \Theta \mathcal{J}' \to \mathcal{J} \Theta_{\text{lex}} \mathcal{J}'$ be the «generic model» of $\Phi$, as in (10.2). At the level of ordinary categories, let $(\mathcal{J}_o \times \mathcal{J}'_o, \Psi)$ be the sketch whose algebras in $C$ are the functors $\mathcal{J}_o \times \mathcal{J}'_o \to C$ left exact in each variable separately, and let $L: \mathcal{J}_o \times \mathcal{J}'_o \to \mathcal{J}_o \Theta_{\text{lex}} \mathcal{J}'_o$ be the generic model. Let $N: \mathcal{J}_o \times \mathcal{J}'_o \to (\mathcal{J} \Theta \mathcal{J'})_o$ be the canonical functor of [*] Section 1.4, and recall that, if $F: \mathcal{J} \Theta \mathcal{J}' \to \mathcal{A}$ is any $\mathcal{V}$-functor, the partial functors of $F_0 N$ are $(F(t, -))_o$ and $(F(-, s))_o$. We see at once that the composite $K_0 N: \mathcal{J}_o \times \mathcal{J}'_o \to (\mathcal{J} \Theta_{\text{lex}} \mathcal{J'})_o$ is left exact in each variable separately, and therefore factorizes (within isomorphism) through $L$ to give a map $M: \mathcal{J}_o \Theta_{\text{lex}} \mathcal{J}'_o \to (\mathcal{J} \Theta_{\text{lex}} \mathcal{J'})_o$ of theories. This map is not an equivalence of theories: the theory $\mathcal{V}_f^{op}$ of an object is the identity for the $\Theta_{\text{lex}}$ of $\mathcal{V}$-theories, so that

$$(\mathcal{V}_f^{op} \Theta_{\text{lex}} \mathcal{V}_f^{op})_o \cong \mathcal{V}_f^{op} = \mathcal{V}_f^{op},$$

which is different from $\mathcal{V}_f^{op} \Theta_{\text{lex}} \mathcal{V}_f^{op}$; when $\mathcal{V} = \text{Cat}$, the first is the theory of a category, while the second is the theory of a double category. More generally, we have for any finitary $\mathcal{V}$-theory $\mathcal{J}$ a map of theories

$$M: \mathcal{V}_f^{op} \Theta_{\text{lex}} \mathcal{J}_o \to (\mathcal{V}_f^{op} \Theta \mathcal{J})_o \cong \mathcal{J}_o;$$

the corresponding functor $\mathcal{V}_f^{op} \times \mathcal{J}_o \to \mathcal{J}_o$ left exact in each variable is that sending $(x, t)$ to $x \circ t$. The relation between theory-maps

$$\mathcal{V}_f^{op} \Theta_{\text{lex}} \mathcal{S} \to \mathcal{S}$$

and $\mathcal{V}$-enrichments of the finitary Set-theory $\mathcal{S}$ will be discussed in the second part of this article.
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