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A NOTE ON THE GENERALIZED REFLEXION OF GUITART AND LAIR

by G. M. KELLY *

By a weak reflexion of a locally-small category $\mathcal{A}$ onto a full subcategory $\mathcal{B}$ we mean the assigning to each $A \in \mathcal{A}$ of a small projective cone $\pi_A$, with vertex $A$ and with base in $\mathcal{B}$, such that $\mathcal{A}(\pi_A, B)$ is a colimit-cone in $\text{Set}$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. When each $\pi_A$ has its base indexed by a discrete category, $\pi$ is a multi-reflexion in the sense of Diers [1]; it is an actual reflexion if moreover each of these discrete categories is 1.

For example, let $\mathcal{A}$ be the category of commutative rings. When $\mathcal{B}$ consists of local rings, a weak reflexion is given by taking for $\pi_A$ the cone of localizations $A \rightarrow A_p$ of $A$; its base is indexed by the ordered set of prime ideals $p$ of $A$. When $\mathcal{B}$ consists of the fields, a multi-reflexion is given by the discrete cone $A \rightarrow A/m$ where $m$ runs through the maximal ideals of $A$. When $\mathcal{B}$ consists of the rings $A$ with $2A = 0$, an actual reflexion is given by $A \rightarrow A/2A$.

Guitart and Lair study in [4] the existence of weak reflexions when $\mathcal{B}$ is given as follows. We have a set $\Theta = \{ \theta_\beta \}$ of projective cones

$$\theta_\beta : \Delta N_\beta \rightarrow T_\beta : \mathcal{L}_\beta \rightarrow \mathcal{A}$$

in $\mathcal{A}$, where $\Delta N_\beta$ denotes the functor constant at $N_\beta$; and $\mathcal{B}$ consists of those $A \in \mathcal{A}$ for which each $\mathcal{A}(\theta_\beta, A)$ is a colimit-cone in $\text{Set}$. They further restrict themselves to the special case in which each generator of each cone $\theta_\beta$ is an epimorphism in $\mathcal{A}$.

Each of the examples above is of this kind. For local rings there are two cones $\theta_1$ and $\theta_2$ in $\mathcal{A}$; $\theta_1$ is the pushout diagram of the two (epi-morphic) maps

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\[ \mathbb{Z}[x, y]/(xy - 1) \to \mathbb{Z}[x] \to \mathbb{Z}[x, y]/((1 \cdot x)y - 1), \]

while 0₂ is the cone of vertex 0 over the empty diagram. For fields there are again two cones: 0₂ as above and 0₃ the discrete cone

\[ \mathbb{Z} \to \mathbb{Z}[x] \to \mathbb{Z}(x). \]

For rings with \( 2A = 0 \), there is a single cone 0₄ whose base is indexed by 1, namely \( \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \).

We suppose henceforth that 0₃ is given as above. We recall that, for a regular cardinal \( \alpha \), an object \( A \in \mathcal{A} \) is called \( \alpha \)-presentable if \( \mathcal{A}(A, -) : \mathcal{A} \to \text{Set} \) preserves \( \alpha \)-filtered colimits. Guitart and Lair sketch in [4] a rather complicated proof by transfinite induction of the following:

*There is a weak reflexion \( \pi \) of \( \mathcal{A} \) onto \( \mathcal{B} \) if \( \mathcal{A} \) is cocomplete, if each \( L^\beta \) is small, and if there is a regular cardinal \( \alpha \) such that each \( N^\beta \) and each \( T^\beta_\mathcal{L} \) is \( \alpha \)-presentable. Moreover \( \pi \) can be taken to be a multi-reflexion if each \( L^\beta \) is discrete.*

The \( \alpha \)-presentability hypothesis is a strong one; hardly any objects are \( \alpha \)-presentable in the category of topological spaces or in the dual of an algebraic category. By analogy with the case where each \( L^\beta \) is 1 - the «orthogonal subcategory problem» of [2] - this hypothesis should not be needed when the generators of the cones \( \theta^\beta \) are epimorphic: at least if \( \mathcal{A} \) is cowellpowered, which is not a grave restriction. By the same analogy, there should be a simple and direct proof in this case. We now verify that this is so, and that moreover the base of each cone \( \pi^\beta \) may then be taken to be an ordered set.

We refer to [5] for the notion of *strong monomorphism*, and for the fact that epimorphisms and strong monomorphisms constitute a factorization system (see [2]) on \( \mathcal{A} \) if \( \mathcal{A} \) admits finite limits and all intersections of strong monomorphisms, or if \( \mathcal{A} \) admits finite colimits and all cointersections of epimorphisms; certainly, therefore, if \( \mathcal{A} \) is complete and well-powered, or cocomplete and cowellpowered.

**Theorem 1.** *Let the full subcategory \( \mathcal{B} \) of the locally-small category \( \mathcal{A} \)
be determined as above by a set $\Theta$ (not necessarily small) of cones $\theta_B$ (not necessarily small), where each generator of each $\theta_B$ is epimorphic in $\mathfrak{A}$. Let epimorphisms and strong monomorphisms constitute a factorization system on $\mathfrak{A}$, and let $\mathfrak{A}$ be cowellpowered.

For each $A \in \mathfrak{A}$ denote by $S_A$ the small category whose objects are (a set of representatives of) the epimorphisms $p : A \to C$ in $\mathfrak{A}$ with domain $A$ and codomain in $\mathfrak{B}$, and whose maps $p \to p'$ are the maps $q : C \to C'$ with $qp = p'$; clearly $S_A$ is an ordered set. Let $d_A : S_A \to \mathfrak{B} \subseteq \mathfrak{A}$ be the projection functor sending $p : A \to C$ to $C$, and let

$$\pi_A : \Delta A \to d_A : S_A \to \mathfrak{A}$$

be the cone whose $p$-th component is $p$ itself.

Then an object $B$ of $\mathfrak{A}$ lies in $\mathfrak{B}$ if and only if each $\mathfrak{A}(\pi_A, B)$ is a colimit-cone in $\text{Set}$.

**Proof.** The essential observation is that $\mathfrak{B}$ is closed in $\mathfrak{A}$ under strong subobjects. To see this it suffices to consider a single cone $\theta : \Delta N \to T$ of $\Theta$, with epimorphic generators $\theta_i : N \to T_i$. Let $j : D \to B$ be a strong monomorphism in $\mathfrak{A}$, with $B \in \mathfrak{B}$. By the diagonal-fill-in property for epimorphisms and strong monomorphisms, the diagram

$$\begin{array}{ccc}
\mathfrak{A}(T_i, D) & \to & \mathfrak{A}(\theta_i, D) \\
\downarrow \mathfrak{A}(T_i, j) & & \downarrow \mathfrak{A}(\theta_i, j) \\
\mathfrak{A}(T_i, B) & \to & \mathfrak{A}(\theta_i, B) \\
\end{array}$$

is a pullback in $\text{Set}$. Since colimits are universal in $\text{Set}$, and since $\mathfrak{A}(\theta_i, B)$ is a colimit-cone in $\text{Set}$, so is $\mathfrak{A}(\theta_i, D)$; so that $D \in \mathfrak{B}$.

It is now easy to see that $\mathfrak{A}(\pi_A, B)$ is a colimit-cone for $B \in \mathfrak{B}$. For let $f : A \to B$, and let $f$ factorize as an epimorphism $p : A \to C$ followed by a strong monomorphism $j : C \to B$. Since $C \in \mathfrak{B}$ by the above, $p$ is a generator of $\pi_A$ through which $f$ factorizes. If $f$ also factorizes as $g p'$ through another generator $p' : A \to C'$ of $\pi_A$, the diagonal-fill-in property applied to $gp' = j p$ gives a $q : C' \to C$ with $qp' = p$ and $j q = g$. Hence $\mathfrak{A}(\pi_A, B)$ is a colimit-cone.
Conversely, if \( \mathcal{A}(\pi_A, B) \) is a colimit-cone for each \( A \), then \( \mathcal{A}(\pi_B, B) \) is a colimit-cone; so that \( I : B \to B \) factorizes as \( I = \eta p \) for some epimorphism \( p : B \to C \) with \( C \in \mathcal{B} \). But then the epimorphism \( p \), being a coretraction, is invertible; and \( B \in \mathcal{B} \).

**THEOREM 2.** Add to the hypotheses of Theorem 1 the completeness of \( \mathcal{A} \), and suppose each cone \( \theta_B \) to have a discrete base \( \mathcal{L}_B \). Then the restriction of \( \pi_A \) to a suitable full subcategory of \( S_A \) gives a multi-reflexion of \( \mathcal{A} \) onto \( \mathcal{B} \).

**PROOF.** Since connected limits commute with discrete colimits in \( \text{Set} \), we have \( \mathcal{B} \) closed in \( \mathcal{A} \) under connected limits. For each connected component \( \delta \) of \( S_A \), therefore, the limit of \( d_A|\delta : \delta \to S_A \to \mathcal{A} \) is an object \( E_\delta \) of \( \mathcal{B} \); and the \( \rho : A \to C \) of \( S_A \) induce a map \( r_\delta : A \to E_\delta \). Let this factorize as the epimorphism \( s_\delta : A \to K_\delta \) followed by the strong monomorphism \( k_\delta : K_\delta \to E_\delta \). Then \( K_\delta \in \mathcal{B} \), and \( s_\delta \) is an object of \( S_A \); clearly, the greatest object of the ordered set \( S_A \) which belongs to \( \delta \). It is now evident that any \( f : A \to B \) with \( B \in \mathcal{B} \) factorizes uniquely through some \( s_\delta \), and through one only. \( \square \)

We include for completeness the classical:

**THEOREM 3.** If each \( \mathcal{L}_B = 1 \) in Theorem 2, \( \mathcal{B} \) is closed under limits in \( \mathcal{A} \), and we get an actual reflexion \( \rho_A \) of \( \mathcal{A} \) onto \( \mathcal{B} \), where \( \rho_A \) is the epimorphic part of the factorization of \( A \to \lim d_A \) into an epimorphism followed by a strong monomorphism. \( \square \)

We end by observing that the cowellpoweredness hypothesis of Theorem 1 does hold in the example to which Guitart and Lair give most prominence - that of the algebras for a mixed sketch \( S \). By this is meant a small category \( \mathcal{S} \) in which are given a small set \( \Phi = \{ \phi_a \} \) of small projective cones and a small set \( \Psi = \{ \psi_\beta \} \) of small inductive cones; unlike Guitart and Lair, we do not ask the \( \phi_a \) to be limit-cones nor the \( \psi_\beta \) to be colimit-cones. The category \( S-\text{Alg} \) of \( S \)-algebras is the full subcategory of \( [\mathcal{S}, \text{Set}] \) given by those \( A : \mathcal{S} \to \text{Set} \) for which each \( A\phi_a \) is a limit-cone and each \( A\psi_\beta \) is a colimit-cone. The sketch \( S \) is projective when
the set $\Psi$ is empty; write $S_0$ for the projective sketch obtained from $S$ by discarding $\Psi$. It is classical that categories of the form $S_0$-$\mathbf{Alg}$ are the locally presentable ones of Gabriel-Ulmer [3]; and that such a category is reflective in $[\mathbf{S}, \mathbf{Set}]$, and is therefore complete and cocomplete.

Let $Z : S^{op} \to S_0$-$\mathbf{Alg}$ be the composite of the Yoneda embedding $Y : S^{op} \to [\mathbf{S}, \mathbf{Set}]$ and the reflexion $R : [\mathbf{S}, \mathbf{Set}] \to S_0$-$\mathbf{Alg}$. Clearly $B = S$-$\mathbf{Alg}$ is the full subcategory of $\mathbf{A} = S_0$-$\mathbf{Alg}$ consisting of those objects $A$ such that $\mathbf{A}(\cdot, A)$ sends the projective cone $\theta_B = Z\psi_B$ of $\mathbf{A}$ to a co-limit-cone in $\mathbf{Set}$ for each $B$. Note that each generator of $\theta_B$ is epimorphic if each generator of $\psi_B$ is monomorphic.

Finally, observe that $\mathbf{A}$ is cowellpowered by Satz 7.14 of [3], an account of which in English can be found in Section 8.6 of [6].

BIBLIOGRAPHY.