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IN WHICH CATEGORIES ARE FIRST-ORDER AXIOMATIZABLE HULLS CHARACTERIZABLE BY ULTRAPRODUCTS?

by Bui Huy HIEN and I. SAIN

In Andreka-Nemeti [1] the class $STr(C)$ of all small trees over $C$ is defined for an arbitrary category $C$. Throughout the present paper $C$ denotes an arbitrary category. In Definition 4 of [1] on page 367 the injectivity relation $\models \subseteq (Ob C) \times (STr(C))$ is defined. Intuitively, the members of $STr(C)$ represent the formulas, and $\models$ represents the validity relation between objects of $C$ considered as models and small trees of $C$ considered as formulas. If $\phi \in STr(C)$ and $a \in Ob C$ then the statement $a \models \phi$ is associated to the model theoretic statement «the formula $\phi$ is valid in the model $a$». It is proved there that the Los Lemma is true in every category $C$ if we use the above quoted concepts. To this the notion of an ultraproduct $\prod_{i \in I} a_i/U$ of objects $\langle a_i \mid i \in I \rangle \in Ob C$ of $C$ is defined in [1], in [2] and in [7] Definition 12. Then the problem was asked in [1] («Open Problem 1» on page 375) «for which categories is the characterization theorem of axiomatizable hulls of classes of models Mod Th $K = Uf Up K$ true?», where the operators $Uf$ and $Up$ on classes of models is defined on page 319 of the book [3], but here we recall them in Definition 6 of the present paper. Of course, here in the definition of $Uf$ and $Up$ on classes $K \subseteq Ob C$ of objects of $C$ we have to replace the standard notion of ultraproducts of models by the above quoted category theoretic ultraproduct $\prod_{i \in I} a_i/U$ of objects of $C$, see Definitions 4 and 6 in the present paper.

For the definitions of the class $STr(C)$ and the injectivity relation $\models$ the reader is referred to [1]. We note that the relation $\models$ is defined between objects of $C$ and elements of $STr(C)$.
DEFINITION 1. Let $C$ be an arbitrary category and let $K \subseteq \text{Ob } C$ and $T \subseteq STr(C)$ be arbitrary classes. Let $a \in \text{Ob } C$ and $\phi \in STr(C)$. Then we define:

(i) $K \models T$ iff $(\forall b \in K)(\forall \psi \in T) \; b \models \psi$.

(ii) $K \models \phi$ iff $K \models \{ \phi \}$, and $a \models T$ iff $\{ a \} \models T$.

(iii) $\text{Mod } T \overset{d}{=} \{ b \in \text{Ob } C \mid b \models T \}$.

(iv) $\text{Th } K \overset{d}{=} \{ \psi \in STr(C) \mid K \models \psi \}$.

(v) $a \equiv_{ee} b$ iff $\text{Th } \{ a \} = \text{Th } \{ b \}$.

(vi) $EeK \overset{d}{=} \{ b \in \text{Ob } C \mid (\exists \text{ a } K) \; b \equiv_{ee} a \}$.

In the present paper we characterize those categories in which $\text{Mod } \text{Th } K = Ee \text{Up } K$ holds for all $K \subseteq \text{Ob } C$.

Note that the above introduced notations $\text{Mod } T$ and $\text{Th } K$ are sloppy since the precise notation would be $\text{Mod } C T$ and $\text{Th } C K$ since e.g. $\text{Mod } C T$ is a function of both $C$ and $T$. We hope that context will help.

Strongly small objects of $C$ were defined in [1], [7] Definition 13 and [2]. We shall use this notion. We note that in the textbook [4] in item 22E there on page 155 strongly small objects were defined under the name strongly finitary objects.

Let $(I, \leq)$ be an arbitrary preordered set, i.e. a small category in which there are no parallel arrows. Diagrams indexed by $(I, \leq)$ will be denoted by

$$<a_i \xrightarrow{b_j} a_j \mid i, j \in I, i \leq j> \quad \text{or shortly} \quad <b_j^i \mid i \leq j \in I>.$$ 

I.e., let $F : (I, \leq) \to C$ be a functor. Now,

$$F \overset{d}{=} <F(i), F(i) \xrightarrow{b_{ij}} F(j) \mid i, j \in I, i \leq j>.$$ 

The colimit of this diagram $F$ is denoted by $<b^i_j : F(i) \to b_{i, j}>_{i, j \in I}$, where $<F(i) \xrightarrow{b_i} >_{i \in I}$ is the cocone part and $b$ the object part of the colimit.

DEFINITION 2 (Nemeti-Sain [7], page 556). An object $a$ is strongly small (for short s. small) if the functor $\text{Hom}(a, \cdot)$ is continuous (i.e. preserves direct limits).

NOTATION. s. small objects will be denoted by $\Theta$-s. $\Theta \xrightarrow{\text{L}}$ means that
dom( f ) is s. small and we use \( f \rightarrow \emptyset \) similarly.

REMARK. From the above definition it follows that the object \( a \) is s. small iff for any directed diagram \( \langle b^i \rangle \mid i \leq j \in I \rangle \) with colimiting cocone
\[ c \cong \langle \langle b^i \rangle_{i \in I}, b \rangle, \]
conditions (i) and (ii) below are satisfied:

(i) Every morphism \( f: a \rightarrow b \) cofactors through the cocone \( c \).

(ii) To any pair
\[ a \xrightarrow{p/q} \]
such that \( b^i \cdot p = b^i \cdot q \) for some \( i \in I \),
there exists a \( j \in I \) such that \( b^i_j \cdot p = b^i_j \cdot q \).

We note that limits and colimits are always small in this paper. E. g., \( Hom(\emptyset, -) \) does not necessarily preserve large direct limits.

An object is called small if it satisfies (i) of the above remark.

DEFINITION 3. Let \( C \) be an arbitrary category. We say that \( C \) has only set-many nonisomorphic strongly small objects iff there is a set \( B \subset Ob C \) such that every strongly small object of \( C \) is isomorphic to some element of \( B \).

NOTATIONS connected to products: The product \( P_{i \in I} a_i \) of a family of objects \( \langle a_i \rangle_{i \in I} \) will also be (ambiguously) denoted by \( P_I \). We use the notation \( \pi^I_i \) for the \( i \)-th member of the cone of projections belonging to the product \( P_I \). I. e., the «product cone» is \( \langle P_I, \langle \pi^I_i \rangle_{i \in I} \rangle \). By the definition
of a product, a cone \( <f_i : d \to a_i>_{i \in I} \) induces a unique morphism \( f : d \to P_I \), such that the diagrams

\[
\begin{array}{ccc}
  d & \xrightarrow{f} & P \prod_{i \in I} a_i \\
  f_i & \downarrow & \pi_i \\
  a_i & & \\
\end{array}
\]

commute for each \( i \in I \) (provided that the product exists). We shall denote this induced morphism \( f : d \to P_I \) by \( \prod_{i \in I} f_i \). Sometimes, though, it is better to write \( \prod c = \prod d, f_i >_{i \in I} \). E.g. \( \prod d, \emptyset \) is the unique element of \( \text{Hom}(d, e) \) where \( e \) is the terminal object \( P_{i \in I} a_i \).

**Definition 4** ([1, 2, 7, 8]). Let \( a_i >_{i \in I} \) be a family of objects. Let \( U \) be a set of subsets of \( I \) (i.e., \( U \subseteq \mathcal{P}I \) is arbitrary). Now, consider all the products \( P_X \left( \prod_{i \in X} a_i \right) \) for the sets \( X \in U \). If \( X, Y \in U \) and \( Y \supset X \) then the morphism induced by the cone of projections of \( P_Y \) into the product \( P_X \) is denoted by \( \pi^Y_X \). I.e. \( \pi^Y_X = \prod_{i \in X} \pi^Y_i \). By this we have defined a diagram of products and projections:

\[
< \pi^Y_X : P_Y \to P_X \mid X, Y \in U, Y \supset X>.
\]

Note that this diagram is indexed by the poset \( (U, \supset) \). (This poset consists of \( U \) ordered by the inverse \( \supset \) of the inclusion relation \( \subseteq \).) The colimit of the above diagram is denoted by

\[
< \pi^Y : P_Y \to (\prod_{i \in I} a_i / U) >_{Y \in U}.
\]

If \( U \) is a filter, then \( \prod_{i \in I} a_i / U \) is called a reduced product of \( <a_i>_{i \in I} \).

If \( U \) is an ultrafilter, then \( \prod_{i \in I} a_i / U \) is called an ultraproduct.

The next figure illustrates the definition.
DEFINITION 5. We say that ultraproducts exist in C iff for every set I and for all \( a_i \in \{\text{Ob } C\} \) and for every ultrafilter \( U \) on I the ultraproduct \( P_{i \in I} a_i / U \) exists in C.

DEFINITION 6. Let \( K \subseteq \text{Ob } C \) be arbitrary. Then

(i) \( U p K \triangleq \{ P_{i \in I} a_i / U \mid I \text{ is a set, } \{ a_i \}_{i \in I} \subseteq K, \ U \text{ is an ultrafilter on } I \text{ and the ultraproduct } P_{i \in I} a_i / U \text{ exists in } C \} \).

(ii) \( U f K \triangleq \{ b \in \text{Ob } C \mid U p \{ b \cap K \neq 0 \} \} \).

THEOREM 1. Let C be an arbitrary category. Assume that conditions (i)-(iii) below hold. Let \( K \subseteq \text{Ob } C \) be an arbitrary class. Then

\( \text{Mod Th } K = E e \ U p K. \) (That is: \( \text{Mod Th } = \text{Ee Up } \) on C.)

(i) C has only set-many nonisomorphic strongly small objects.

(ii) Ultraproducts exist in C (the small ones only, Definition 3).

(iii) C has an initial object.

PROOF is that of Theorem 1 in [5]. \( \square \)

Theorems 2, 3 below state that both conditions (i) and (ii) are needed in Theorem 1 above.

THEOREM 2 (necessity of (i) in Theorem 1). There exists a category C in which all ultraproducts exist and C has an initial object, but

\[ \text{Mod Th } K \neq E e \ U p K \text{ for some } K \subseteq \text{Ob } C. \]

That is, C satisfies (ii) and (iii) of Theorem 1 but not its conclusion.

PROOF. Let \( \omega \triangleq \{ 1 \} \). Let Ord be the class of all ordinals. Then we have \( \omega \subseteq \text{Ord} \). Let \( \text{Ord} + 1 \triangleq \text{Ord } \cup \{ \omega \} \). Let \( \leq \subseteq \{ \langle \beta, \omega \rangle \mid \beta \in \text{Ord } + 1 \} \). Let \( \leq \subseteq \{ \langle \beta, a \rangle \mid a \in \text{Ord } \text{ and } (\beta \in \omega \text{ or } \beta = a) \} \).

Then \( P \triangleq \langle \text{Ord } + 1, \leq \rangle \) is an ordered class. Hence P may be considered as a category with \( \text{Ob } P = \text{Ord } + 1 \).

FACT 1. The s. small objects of P are exactly the successor ordinals and 0. Hence there is a proper class of nonisomorphic s. small objects. The initial object of P is 0.

LEMMA 2. Let \( \phi \in S T r (P) \). Then \[ \text{Ord } \models \phi \Rightarrow \omega \models \phi. \]
**PROOF of Lemma 2.** Assume $\text{Ord} \models \phi$. By $\phi \in STr(P)$, all objects occurring in $\phi$ are s. small, hence $\omega$ does not occur in $\phi$. Since only set-many objects can occur in $\phi$ we conclude that

$$(\exists \kappa \in \text{Ord})( \text{for every object } a \text{ occurring in } \phi \text{ we have } a < \kappa).$$

Then $\phi$ is related to $\kappa$ exactly the same way as it is related to $\omega$. Hence $\kappa \models \phi$ implies $\omega \models \phi$. But $\text{Ord} \models \phi$ implies $\kappa \models \phi$.

**COROLLARY 3.** $\omega \in \text{Mod Th (Ord)}$.

**LEMMA 4.** Let $\alpha, \beta \in \text{Ord} + 1$. Then $\text{Th}(\alpha) = \text{Th}(\beta)$ iff $\alpha = \beta$.

**PROOF of Lemma 4.** Assume $\alpha \neq \beta$. Then $\alpha < \beta$ or $\beta < \alpha$, assume $\alpha < \beta$. Clearly $\langle a + 1, \emptyset \rangle \in STr(P)$ since $a + 1$ is s. small. Then $a \models \langle a + 1, \emptyset \rangle$ while $\beta \models \langle a + 1, \emptyset \rangle$ since $\text{Hom}(a + 1, \beta) \neq 0$ by $a + 1 \leq \beta$.

Clearly, all reduced products exist in $P$ since suprema and infima of subsets of $(\text{Ord} + 1)$ do exist in $(\text{Ord} + 1, \leq)$. Obviously, $\text{Up Ord} = \text{Ord}$, in $P$ since by ultraproducts we understand ultraproducts of sets of objects only. Hence by Lemma 4 we have $Ee \text{Up Ord} = \text{Ord}$ in $P$. Thus

$$\text{Mod Th Ord} = \text{Ord} + 1 \neq \text{Ord} = Ee \text{Up Ord}$$

is proved to hold in $P$. QED of Theorem 2.

**THEOREM 3.** There is a category $C$ and a class $K$ of objects of $C$ such that (i) and (iii) of Theorem 1 hold as well as (I) and (II) below:

(I) $\text{Mod Th } K \supset Ee \text{Up } K$.

(II) Let $\text{Up}^\omega$ denote the formation of weak ultraproducts which were introduced in [11] under the name «universal ultraproducts». Then

$$\text{Mod Th } K \supset Ee \text{Up}^\omega K.$$
Let \( K = \{ A \in \text{Ob} C \mid |A| < \omega \} \). We claim that

\[ \text{Up}^w K = K \quad \text{and} \quad \text{Ee} K = K, \quad \text{hence} \quad \text{Ee Up}^w K = K. \]

But an object \( A \) is s. small in \( C \) iff \( |A| < \omega \). Since the formula \( <A, O> \) is not valid in \( K \) and since there are no other non-trivial formulas, we have \( \text{Mod} Th K = \text{Ob} C \). Obviously \((i)\) and \((ii)\) of Theorem 1 hold in \( C \).

If \( C \) is an arbitrary category and \( K \subseteq \text{Ob} C \), then

\[ \text{Mod} Th K \supset \text{Ee Up} K \supset \text{Uf Up} K \quad \text{(by [1])}. \]

**Proposition 4.** The conditions of Theorem 1 are not the best possible, namely: There exists a category \( C \) such that all three conditions \((i), (ii)\) and \((iii)\) of Theorem 1 fail but the conclusion of Theorem 1 is true.

**Proof.** Let \( C \) be a large discrete category. That is \( \text{Ob} C \) is a proper class (not a set) and the only morphisms are identities. Then every object of \( C \) is s. small. Thus there is a proper class of nonisomorphic s. small objects. Further ultraproducts do not exist in \( C \), since there are no non-identity morphisms. Let \( K \subseteq \text{Ob} C \). We claim that \( \text{Mod} Th K = K \). Let \( a \in \text{Ob} C \). Assume \( a \nless K \). Then \( <a, 0> \in \text{S Tr}(C) \), namely \( <a, 0> \) is the one-element tree with root \( a \) and no branches. Clearly

\[ a \nless <a, 0> \quad \text{and} \quad (\forall b \in \text{Ob} C)(b \neq a \Rightarrow b \lessdot <a, 0>). \]

Thus \( K \lessdot <a, 0> \) proving that \( a \nless \text{Mod} Th K \). \( \Box \)

**Problems.** (i) Improve Theorem 1. Find a sharper characterization of those categories in which \( \text{Mod} Th = \text{Ee Up} \).

(ii) Under what conditions is \( \text{Mod} Th = \text{Uf Up} \) true?

(iii) Is there a category \( C \) satisfying \((i)\) and \((ii)\) of Theorem 1 in which \( \text{Mod} Th K \neq \text{Ee Up} K \) for some \( K \subseteq \text{Ob} C \)? This is solved by I. Sain affirmatively, see [5] Theorem 2.

For the category \( \text{Lf}_a \) of locally finite cylindric algebras, see [6] or in the textbook on representable cylindric algebras [3] page 321. The following is a corollary of results in [6] and Theorem 1 above. For a motivation we note that \( \text{Lf}_a \) is exactly the class of algebras obtained from classical first-order theories, as it was proved in Proposition 1 of [6].

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COROLLARY 5. Let $a$ be any ordinal and $K \subseteq L_f a$ be arbitrary. Then in the category $L_f a$ we have $\text{Mod Th } K = \bigcup K$.

PROBLEM. Is $\text{Mod Th } K = \bigcup K$ true in $L_f a$?

For a comprehensive study of our subject see [9]. The fact that $S Tr(C)$ corresponds exactly to the class of first-order formulas is proved in [10].

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