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TOPOI OVER GRAPHS

by John MACDONALD and Arthur STONE

In presenting these results on the monadicity of *Topoi* and several other categories over *Graphs* we wish to acknowledge the independent investigations of Burroni [1] on this topic and of Lambek [3] on the related question of *Topoi* over *Cat*. We hope that the difference between our approach and that of Burroni, who was first in proving results on the monadicity of *Topoi* over *Graphs*, will help further clarify the area of monadic and essentially monadic adjunctions (cf. Freyd [2] and MacDonald-Stone [4]).

By *Topoi* we will mean the category of small topoi and logical morphisms. Our topoi will have particular (= chosen) finite limits and colimits that are preserved by the morphisms (so our morphisms are what are sometimes called *strict* logical morphisms).

We will give presentations by giving lists of operations and axioms for the category $\underline{\underline{A}}$ appearing in the adjunction diagram

$$Graphs \Rightarrow \rightleftarrows \Rightarrow \underline{\underline{A}}$$

in which the algebras that are the objects of $\underline{\underline{A}}$ are built up successively in four stages, that is, $\underline{\underline{A}}$ takes on four values:

1. Categories with equalizers,
2. Finitely complete categories,
3. Cartesian closed categories, and
4. Topoi.

Algebras here means *essentially algebraic* structures, in the sense of Freyd [2]: operations may be *partial* - with domains determined by equations in lower operations - here, by equations in the operations of *Graphs*.

In the final stage, giving a list of operations and axioms for $\underline{\underline{A}}$ is equivalent to giving a description of the free topos over a graph; so the list is long. $\underline{\underline{A}}$ will be given by 21 operations and 67 axioms.

By *Graphs* we mean the category of one-sorted universal algebras for the two unary operations $Sce x$ and $Tgt x$, with four axioms:

$$\begin{aligned} Sce(Sce x) &= Sce x, & Sce(Tgt x) &= Tgt x, \\ Tgt(Sce x) &= Sce x, & Tgt(Tgt x) &= Tgt x. \end{aligned}$$

These algebras are equivalent to what graph theorists call *directed multigraphs*. Here we will speak of a *morphism* (= element) x as an *object* if it satisfies the equation $Sce x = x$. Upper case letters will usually be used to begin expressions that denote such objects. A two-sorted exposition, with disjoint classes of objects and morphisms, would be slightly longer (with $Sets \times Sets$ for the base category).

In equations involving partial operations, as in our axioms, no assumption is implied concerning the existence of the element denoted by either side.

The bottom line of all this is of course that each of the four categories listed are monadic over *Graphs* and that the free structures in each (over *Graphs*) may be described in terms of the operations and axioms given for that category.

1. CATEGORIES WITH EQUALIZERS.

The operations are the following:

01. Composition $c(x, y) = y \cdot x$ is a partially defined binary operation defined when $Tgt x = Sce y$.

02. Equalizer $e(x, y)$ is defined when

$$Sce x = Sce y \quad \text{and} \quad Tgt x = Tgt y.$$

This is the equalizer *morphism*. The equalizer *object* is its source.

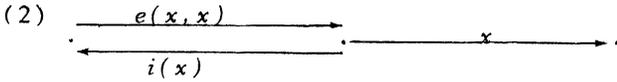
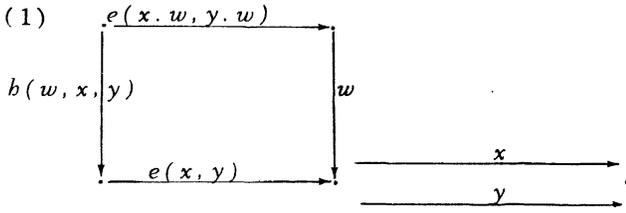
03. The universal morphism for equalizers $b(w, x, y)$ is defined when

$$Tgt w = Sce x = Sce y \quad \text{and} \quad Tgt x = Tgt y.$$

This will be slightly different from the usual universal morphism which is defined when $x \cdot w = y \cdot w$ (using language that is not available to us at this point).

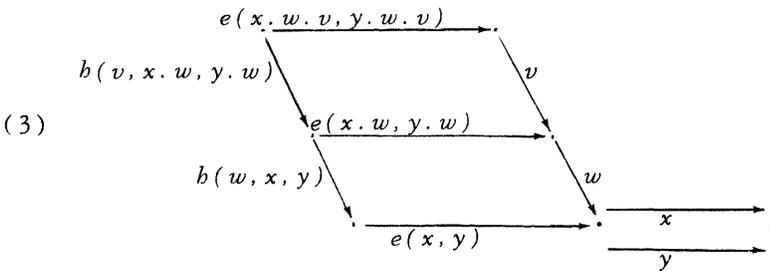
04. The equalizer inverse $i(x)$ is defined for all x . This will be an inverse for $e(x, x)$.

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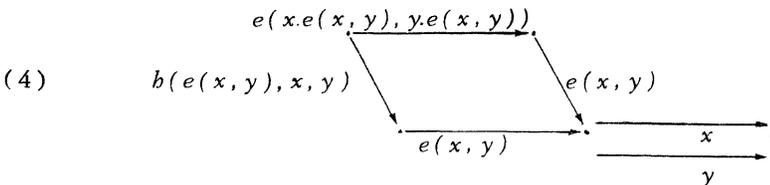


The following axioms are needed:

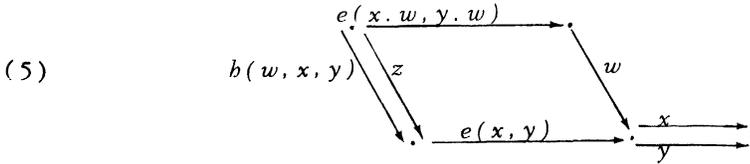
- A1, 2 $Sce y. x = Sce x, Tgt y. x = Tgt y.$
- A3. Associativity: $(z. y). x = z. (y. x).$
- A4, 5. Unit axioms: $x. Sce x = x, (Tgt x). x = x.$
- A6. $Tgt(e(x, y)) = Sce x.$
- A7. $x. e(x, y) = y. e(x, y).$
- A8. $Tgt(b(w, x, y)) = Sce(e(x, y)).$
- A9. $e(x, y). b(w, x, y) = w. e(x, w, y. w)$ (so
 $Sce(b(w, x, y)) = Sce(e(x, w, y. w))$).
- A10, 11 $Sce(i(x)) = Sce x, Tgt(i(x)) = Sce(e(x, x)).$
- A12, 13 $e(x, x). i(x) = Sce x, i(x). e(x, x) = Sce e(x, x).$
- A14. $b(w. v, x, y) = b(w, x, y). b(v, x. w, y. w).$



- A15. $e(x. e(x, y), y. e(x, y)) = b(e(x, y), x, y).$

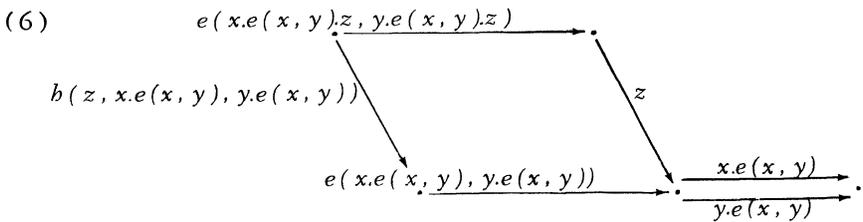


PROPOSITION. *Given*



with $e(x, y).z = w.e(x.w, y.w)$, then $z = h(w, x, y)$.

PROOF.



commutes by A9. But

$$h(w.e(x.w, y.w), x, y) \stackrel{\text{By A14}}{=} h(w, x, y).h(e(x.w, y.w), x.w, y.w)$$

$$\stackrel{\text{By A15}}{=} h(w, x, y).e(x.w.e(x.w, y.w), y.w.e(x.w, y.w))$$

$$\stackrel{\text{By diagram (5)}}{=} h(w, x, y).e(x.e(x, y).z, y.e(x, y).z).$$

However

$$h(w.e(x.w, y.w), x, y) \stackrel{\text{By diagram (5)}}{=} h(e(x, y).z, x, y)$$

$$\stackrel{\text{By A14}}{=} h(e(x, y), x, y).h(z, x.e(x, y), y.e(x, y))$$

$$\stackrel{\text{By A15}}{=} e(x.e(x, y), y.e(x, y)).h(z, x.e(x, y), y.e(x, y)).$$

Combining the last two statements we have

$$\begin{aligned} h(w, x, y).e(x.e(x, y).z, y.e(x, y).z) &= \\ &= e(x.e(x, y), y.e(x, y)).h(z, x.e(x, y), y.e(x, y)). \end{aligned}$$

Thus from diagram (6) and A12, 13, it follows that $z = h(w, x, y)$.

2. CATEGORIES WITH FINITE LIMITS.

Further operations :

Here $-_{\pi} Y$ is of course the extension of the function $P(\cdot, Y)$, defined on objects, to a functor; where $p = p(\text{Sce } x, Y)$ and $q = q(\text{Sce } x, Y)$.

$$(9) \quad \begin{array}{ccc} P(W, Y) & \xrightarrow{x \pi Y = b_p(x \cdot p, q)} & P(X, Y) \\ \swarrow p & & \swarrow \\ W & \xrightarrow{x} & X \\ \searrow q & & \searrow \\ & Y & \xrightarrow{Y} & Y \end{array}$$

For the other components of the adjunction we need new operations and axioms. But there are no uniqueness questions.

011. Internal hom $-^Y$ defined when $\text{Sce } Y = Y$.

012, 13. Diagonal and evaluation maps $dgl(X, Y)$ and $evl(X, Y)$ are defined when $\text{Sce } X = X$ and $\text{Sce } Y = Y$.

A29-32. Source and target of $dgl(X, Y)$ and $evl(X, Y)$.

A33-35. $-^Y$ preserves Sce , Tgt and composition :

$$\text{Sce } x^Y = (\text{Sce } x)^Y, \text{ Tgt } x^Y = (\text{Tgt } x)^Y \text{ and } (x \cdot w)^Y = x^Y \cdot w^Y.$$

A36, 37. The naturality of $dgl(\cdot, Y)$ and $evl(\cdot, Y)$.

A38, 39. The adjunction equations

$$\begin{aligned} evl(X \pi Y, Y) \cdot dgl(X, Y) \pi Y &= X \pi Y, \\ (evl(X, Y))^Y \cdot dgl(X^Y, Y) &= X^Y. \end{aligned}$$

4. TOP01.

The approach we will use to the presentation of the operations associated with a subobject classifier (using nothing more than the operations of *Graphs* for the defining of domains) calls for the concurrent presentation of coequalizers. It is of course well known that a cartesian closed category with finite limits and a subobject classifier has coequalizers (Paré [5]), but we cannot use this since we need coequalizers on our way to the subobject classifier.

014, 15, 16. The duals of the operations e , b and i .

017, 18. Constants denoted by Ω and $true$.

019. The characteristic map $c(x)$ defined for all x . Usually the char-

acteristic map for x is defined only if x is a monomorphism. But, «mono- morphism» cannot be expressed in the language of *Graphs*. We will get around this by requiring that x and its monic component in its epic-monic factorization have the same characteristic map, so that $c(x)$ is in effect superfluous when x is non-monic.

020, 21. Unary operations $i_1(x)$ and $i_2(x)$ defined for all x .

These operations will be used to make certain kernel pairs and pullbacks behave. They could be omitted if we were to ask for more coherence in our Topoi. They will be isomorphisms. In the more coherent situation, they would be identities.

A40-49. These are the duals of A6-15 and give us coequalizers.

A50-53. These give the sources and targets of *true* and $c(x)$.

$$(10) \quad \begin{array}{ccc} \bullet & \xrightarrow{x} & \bullet \\ & & \downarrow c(x) \\ 1 & \xrightarrow{true} & \Omega \end{array}$$

We may now form coequalizers of kernel pairs and we define $im(x)$ to be the unique morphism with $x = im(x) \cdot c$ where c is the coequalizer of the kernel pair of x .

A54. $c(x) = c(im(x))$.

All that remains is to compel $im(x)$ to be monic and the familiar square

$$(12) \quad \begin{array}{ccc} \bullet & \xrightarrow{im(x)} & \bullet \\ \downarrow b_1(Sce(im(x))) & & \downarrow c(x) \\ 1 & \xrightarrow{true} & \Omega \end{array}$$

to be a pullback as well as to show the uniqueness property. We can do this with thirteen more axioms A55-67 involving previously defined operations. With four axioms we give $i_1(x)$ the source, target and composites that will make it an inverse for one of the morphisms in the kernel pair of $im(x)$.

$$(12) \quad \begin{array}{c} \xleftarrow{\quad i_1(x) \quad} \\ \xrightarrow{\quad im(x) \quad} \\ \xrightarrow{\quad} \end{array} \cdot$$

Now let $Pb(x)$ be the chosen pullback of $true$ and $c(x)$. With the next five axioms we make (11) commute and cause $i_2(x)$ to be an inverse for the resulting morphism.

$$(13) \quad Sce(im(x)) \longrightarrow Pb(x).$$

Let t_1 and t_2 be operations defined in terms of the equalizer e and the product projections p and q as follows:

$$\begin{aligned} t_1(x, y) &= q(Sce\ x, Sce\ y). e(x.p(Sce\ x, Sce\ y), y. q(Sce\ x, Sce\ y)), \\ t_2(x, y) &= p(Sce\ x, Sce\ y). e(x.p(Sce\ x, Sce\ y), y. q(Sce\ x, Sce\ y)). \end{aligned}$$

The canonical pullback of x and y is defined to be

$$(14) \quad \begin{array}{ccc} & \xrightarrow{t_1(x, y)} & \\ t_2(x, y) \downarrow & & \downarrow y \\ & \xrightarrow{x} & \end{array}$$

With the last four axioms we ensure that if there is a pullback diagram of the form (11) with $c(x)$ replaced by y , then $c(x) = y$. These axioms are as follows:

- A64. $c(y. e(z, z)) = c(y)$.
- A65. $c(t_1(x, y)) = c(x). y$.
- A66. $c(true) = 1$.
- A67. $c(t_1(x, y). e(z. t_1(x, y), x. t_2(x, y))) =$
 $= c(t_1(x, z). e(y. t_1(x, z), x. t_2(x, z)))$.

We next show explicitly how the preceding axioms can be used to prove the Ω axiom, namely that for each monic $m: A \rightarrow V$ there is one and only one morphism $y: V \rightarrow \Omega$ such that

$$(15) \quad \begin{array}{ccc} A & \xrightarrow{m} & V \\ \downarrow b_1(Scem) & & \downarrow y \\ 1 & \xrightarrow{true} & \Omega \end{array}$$

is a pullback square.

From diagram (11) and associated axioms it is clear that $y = c(m)$ is such a morphism. To show there is only one such y is more subtle and uses axioms A64-A67.

PROPOSITION 16. *Suppose*

$$(17) \quad \begin{array}{ccc} A & \xrightarrow{m} & V \\ v \downarrow & & \downarrow y \\ F & \xrightarrow{x} & E \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{m} & V \\ v \downarrow & & \downarrow z \\ F & \xrightarrow{x} & E \end{array}$$

are pullbacks, then $c(x) \cdot y = c(x) \cdot z$.

PROOF. Let

$$\begin{array}{ccc} C & \xrightarrow{t_1(x, y)} & V \\ t_2(x, y) \downarrow & & \downarrow y \\ F & \xrightarrow{x} & E \end{array} \quad \begin{array}{ccc} D & \xrightarrow{t_1(x, z)} & V \\ t_2(x, z) \downarrow & & \downarrow z \\ F & \xrightarrow{x} & E \end{array} \quad (18)$$

be the canonical pullbacks. Then there is a unique $\theta: C \rightarrow A$ with

$$m \cdot \theta = t_1(x, y) \quad \text{and} \quad v \cdot \theta = t_2(x, y).$$

Similarly there is a unique $\bar{\theta}: D \rightarrow A$ with

$$m \cdot \bar{\theta} = t_1(x, z) \quad \text{and} \quad v \cdot \bar{\theta} = t_2(x, z)$$

and both θ and $\bar{\theta}$ are isomorphisms. From the preceding it follows that

$$(19) \quad y \cdot t_1(x, z) = y \cdot m \cdot \bar{\theta} = x \cdot v \cdot \bar{\theta} = x \cdot t_2(x, z), \quad \text{and}$$

$$(20) \quad z \cdot t_1(x, y) = z \cdot m \cdot \theta = x \cdot v \cdot \theta = x \cdot t_2(x, y).$$

Thus

$$\begin{aligned} c(x) \cdot z &\stackrel{(A65)}{=} c(t_1(x, z)) \stackrel{(A64)}{=} c(t_1(x, z) \cdot e(y, t_1(x, z), y \cdot t_1(x, z))) \\ &\stackrel{(19)}{=} c(t_1(x, z) \cdot e(y \cdot t_1(x, z), x \cdot t_2(x, z))) \stackrel{(A67)}{=} \\ &\stackrel{(A67)}{=} c(t_1(x, y) \cdot e(z \cdot t_1(x, y), x \cdot t_2(x, y))) \stackrel{(20)}{=} \\ &c(t_1(x, y) \cdot e(z \cdot t_1(x, y), z \cdot t_1(x, y))) \stackrel{(A64)}{=} c(t_1(x, y)) \stackrel{(A65)}{=} c(x) \cdot y. \end{aligned}$$

COROLLARY 21. *Suppose*

$$(22) \quad \begin{array}{ccc} A & \xrightarrow{m} & V \\ b_1(\text{Scem}) \downarrow & & \downarrow \\ 1 & \xrightarrow{\text{true}} & \Omega \end{array}, \quad \begin{array}{ccc} A & \xrightarrow{m} & V \\ b_1(\text{Scem}) \downarrow & & \downarrow z \\ 1 & \xrightarrow{\text{true}} & \Omega \end{array}$$

are pullbacks, then $y = z$.

PROOF. By Proposition 16,

$$c(\text{true}) \cdot y = c(\text{true}) \cdot z.$$

By A66, $c(\text{true}) = 1$, hence $y = z$.

This completes the proof of the monadicity of *Topoi* over *Graphs*.

The desired Corollary 21 can be proved using the following simpler (nonalgebraic) axioms A68 and A69 plus axioms and operations up to and including A64.

We first give A68 and A69 as propositions provable from the axioms through A67.

Let s be the operation defined by $s(y) = t_1(\text{true}, y)$.

PROPOSITION 23. (A68) $c(s(y)) = y$.

PROOF. $c(s(y)) = c(t_1(\text{true}, y)) \stackrel{(A65)}{=} c(\text{true}) \cdot y \stackrel{(A66)}{=} 1 \cdot y = y$. Note that if

$$(24) \quad \begin{array}{ccc} & \xrightarrow{t_1(\text{true}, y)} & \\ t_2(\text{true}, y) \downarrow & & \downarrow y \\ 1 & \xrightarrow{\text{true}} & \Omega \end{array}$$

is the canonical pullback of true and y , then clearly

$$t_2(\text{true}, y) = b_1(\text{Sce } t_1(\text{true}, y))$$

since 1 is terminal.

PROPOSITION 25.

$$(A69) \quad c(s(y) \cdot e(z \cdot s(y), \text{true} \cdot b_1)) = c(s(z) \cdot e(y \cdot s(z), \text{true} \cdot b_1)).$$

PROOF. $c(s(y) \cdot e(z \cdot s(y), \text{true} \cdot b_1)) =$

$$= c(t_1(\text{true}, y) \cdot e(z \cdot t_1(\text{true}, y), \text{true} \cdot t_2(\text{true}, y))) \stackrel{(A67)}{=}$$

$$\begin{aligned}
 &= c(t_1(\text{true}, z). e(y. t_1(\text{true}, z), \text{true}. t_2(\text{true}, z))) \\
 &= c(s(z). e(y. s(z), \text{true}. b_1)).
 \end{aligned}$$

We now prove Corollary 21 using A68 and A69 instead of the more general axioms A65-67. Namely,

$$\begin{aligned}
 z &\stackrel{(A68)}{=} c(s(z)) \stackrel{(A64)}{=} c(s(z). e(y. s(z), y. s(z))) \stackrel{(19)}{=} \\
 &c(s(z). e(y. s(z), \text{true}. t_2(\text{true}, z))) = \\
 c(s(z). e(y. s(z), \text{true}. b_1)) &\stackrel{(A69)}{=} c(s(y). e(z. s(y), \text{true}. b_1)) \stackrel{(24)}{=} \\
 c(s(y). e(z. s(y), \text{true}. t_2(\text{true}, y))) &\stackrel{(20)}{=} \\
 c(s(y). e(z. s(y), z. s(y))) &\stackrel{(A64)}{=} c(s(y)) \stackrel{(A68)}{=} y.
 \end{aligned}$$

Finally since Axiom (A67) is more complicated in its formulation than the earlier axioms we end with the following

PROPOSITION 26. (A67) holds for Topoi.

PROOF. Let

$$e_1 = e(z. t_1(x, y), x. t_2(x, y)), \quad e_2 = e(y. t_1(x, z), x. t_2(x, z)).$$

Then we must show that

$$c(t_1(x, y). e_1) = c(t_1(x, z). e_2).$$

We prove this from the Ω -axiom by showing that

$$t_1(x, y). e_1 \quad \text{and} \quad t_1(x, z). e_2$$

differ by an isomorphism of domain. Starting with canonical pullback diagrams (18) we build the diagrams

$$(27) \quad \begin{array}{ccc}
 G_1 \dashrightarrow t_1(x, y). e_2 & \dashrightarrow & G_2 \dashrightarrow t_1(\bar{x}, y). e_1 \\
 \downarrow \theta_1 & \dashrightarrow & \downarrow \theta_2 \\
 C \xrightarrow{t_1(x, y)} V & \dashrightarrow & D \xrightarrow{t_1(x, \bar{z})} V \\
 \downarrow t_2(x, y) & \dashrightarrow & \downarrow t_2(x, z) \\
 F \xrightarrow{x} E & \dashrightarrow & F \xrightarrow{x} E
 \end{array}$$

The outer part of the diagrams commute by definition of e_1 and e_2 . Hence there is a unique θ_1 with

$$t_1(x, z) \cdot e_2 = t_1(x, y) \cdot \theta_1 \quad \text{and} \quad t_2(x, z) \cdot e_2 = t_2(x, y) \cdot \theta_1$$

and there is a unique θ_2 with

$$t_1(x, y) \cdot e_1 = t_1(x, z) \cdot \theta_2 \quad \text{and} \quad t_2(x, y) \cdot e_1 = t_2(x, z) \cdot \theta_2.$$

But

$$z \cdot t_1(x, y) \cdot \theta_1 = z \cdot t_1(x, z) \cdot e_2 = x \cdot t_2(x, z) \cdot e_2 = x \cdot t_2(x, y) \cdot \theta_1.$$

Thus θ_1 factors uniquely through e_1 as $\theta_1 = e_1 \cdot \phi_1$. Similarly, $\theta_2 = e_2 \cdot \phi_2$, for unique ϕ_2 . But

$$\begin{aligned} t_1(x, y) \cdot e_1 \cdot \phi_1 \cdot \phi_2 &= t_1(x, y) \cdot \theta_1 \cdot \phi_2 = t_1(x, z) \cdot e_2 \cdot \phi_2 = \\ &= t_1(x, z) \cdot \theta_2 = t_1(x, y) \cdot e_1. \end{aligned}$$

Similarly,

$$\begin{aligned} t_2(x, y) \cdot e_1 \cdot \phi_1 \cdot \phi_2 &= t_2(x, y) \cdot \theta_1 \cdot \phi_2 = t_2(x, z) \cdot e_2 \cdot \phi_2 = \\ &= t_2(x, z) \cdot \theta_2 = t_2(x, y) \cdot e_1. \end{aligned}$$

Thus $e_1 \cdot \phi_1 \cdot \phi_2 = e_1$ and $\phi_1 \cdot \phi_2 = 1$ since e_1 is monic. Similarly $\phi_2 \cdot \phi_1 = 1$.

Finally,

$$t_1(x, y) \cdot e_1 = t_1(x, z) \cdot \theta_2 = t_1(x, z) \cdot e_2 \cdot \phi_2,$$

with ϕ_2 an isomorphism.

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