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MODELS FOR ACTIONS OF CERTAIN GROUPOIDS

by Peter GREENBERG

Résumé. Soit $B^n$ l'espace classifiant pour les feuilletages $C^\infty$ de codimension $n$, et soit $M^n$ le monoïde discret des plongements $C^\infty$ de $\mathbb{R}^n$ dans $\mathbb{R}^n$. G. Segal a montré qu'il existe une équivalence d'homotopie faible $BM^n \to B^n$, et D. McDuff a obtenu des résultats analogues pour les espaces classifiants de feuilletages $C^\infty$ avec une forme volume transverse, de codimension au moins 3. Cet résultat généralise ici ces résultats.

1. INTRODUCTION.

Let $B^n$ be the classifying space for $C^\infty$ codimension $n$ foliations, and let $M^n$ be the discrete monoid of $C^\infty$ embeddings from $\mathbb{R}^n$ to $\mathbb{R}^n$. G. Segal showed [13] that there is a weak homotopy equivalence $BM^n \to B^n$, and D. McDuff [10] obtained similar results for the classifying spaces for $C^\infty$ foliations with transverse volume form, with codimension at least 3. This paper generalizes these results.

Much of this work was done at M.I.T. and is my doctoral thesis; I'd like to thank my advisors Daniel Kan and Sol Jekel. Sol Jekel has obtained similar results with other methods [7].

1.1. Classifying spaces for Haefliger structures.

A pseudogroup $G$ of transformations of a space $X$ is a collection, closed under composition and inverse, of homeomorphisms between open subsets of $X$. There are various models for a classifying space $BG$; such a classifying space is of interest in the homotopy theory of foliations whose transverse geometric structure is modelled on $X$. In this paper we start with a topological category $\Gamma$ called a groupoid of homeomorphisms (2.1) of $X$; the classifying space (1.7) $|\Gamma|$ of this category is a standard model for $BG$.

1.2. Monoids of immersions.

We associate to a groupoid of homeomorphisms $\Gamma$ of a space $X$ a discrete monoid of immersions $M$ (2.3), which acts on $X$ by maps $m : X \to X$ which are locally one-to-one. Let $|M\backslash X|$ denote the homotopy quotient of the action (sometimes denoted $EM \times_M X$); if $X$ is contractible there is an (homotopy) equivalence $|M\backslash X| \to BM$.

1.3. Theorem. If the images $mX$, $m \in M$ form a basis for the topology of $X$, then there is a weak (homotopy) equivalence $|M\backslash X| \to |\Gamma|$.
For example, let \( \Gamma \) be the groupoid of all area preserving diffeomorphisms between open subsets of the plane; \( |\Gamma| \) is the classifying space for codimension 2 foliations with a transverse area form. Let \( M \) be the monoid of area preserving immersions of the plane. Then there is a weak equivalence \( BM \to |\Gamma| \).

1.4. Modelling actions.

Let \( Y \) be a space with a map \( p : Y \to X \), and suppose that the homeomorphisms between open subsets of \( X \) in the pseudogroup \( G \) lift naturally to homeomorphisms between the inverse image of the open sets in \( Y \). Such extra data may correspond to some facet of the geometry of \( X \). For example, consider the tangent bundle \( p : T_x \mathbb{R}^n \to \mathbb{R}^n \). If \( f : U \to V \) is a diffeomorphism between open subsets of \( \mathbb{R}^n \), there is a natural induced isomorphism \( f^* : T^*_U \to T^*_V \) of tangent bundles.

We define \((2.2)\) an action of a groupoid of homeomorphisms \( \Gamma \) on a space \( Y \) over \( X \). There is an appropriate definition of the homotopy quotient \( |\Gamma \backslash Y| \) of the action; in \([5]\) we show that there is a spectral sequence abutting to \( H_* |\Gamma \backslash Y| \), with

\[
E^2_{pq} = H_p(\Gamma; H_q Y),
\]

where the latter expression is defined in analogy with group homology. In some cases the \( E^2 \) term has some geometrical significance \([4]\).

If \( \Gamma \) is a groupoid of homeomorphisms of \( X \), and \( p : Y \to X \) is a map, and \( \Gamma \) acts on \( Y \) over \( X \), then the monoid of immersions \( M \) acts on \( Y \).

1.5. Theorem. If the images \( mX, m \in M, \) form a basis for the topology of \( X \), then there is a weak equivalence \( |M \backslash Y| \to |\Gamma \backslash Y| \).

It turns out that 1.3 is a corollary to 1.5.

1.6. Organization. In Section 2, definitions are given, along with re-statements of the main results and some examples. Section 3 contains the main body of proof, with a major Lemma 3.4 proved in Sections 4 and 5.

1.7. Notation.

We assume familiarity with semisimplicial notations; references are \([9]\) and \([14]\).

Recall \([2]\) that a topological category is a small category in which the sets of objects and morphisms are given topologies, such that the structure maps of the category are continuous. The nerve of a (topological) category \( C \) is a simplicial (space) set \( C_* \); a functor \( F : C \to D \) induces a simplicial map \( F_* : C_* \to D_* \). If \( C \) is a (topological) category, we denote by \( D, R : C_n \to C_0 \) the domain and range maps

\[
D = d_1 \ldots d_n, \quad R = d_{n-1} \ldots d_1 \ d_0.
\]
We employ the geometric realization functor $|.| : \text{simplicial spaces} \to \text{spaces}$
defined by Segal ([12], App. A; Segal calls the functor $\|\|_1$). If $A_\ast$ is a simplicial space, $|A_\ast|$ is defined as
$$|A_\ast| = \bigsqcup_{n} A_n \times \Delta^n$$
where $\Delta^n$ is the standard $n$-simplex, and we set
$$(F_{a_n}, t_{k}) \sim (a_n, F^*t_{k})$$
for every composite of face maps $F : A_n \to A_k$; here $F^* : \Delta^k \to \Delta^n$ is the inclusion induced by $F$. Segal's realization is useful to us because of the proposition:

1.8. Proposition. (Segal, [12], App. A.1.ii) If $f_\ast : A_\ast \to B_\ast$ is a map of simplicial spaces such that every $f_n : A_n \to B_n$ is a weak equivalence, then $|f_\ast| : |A_\ast| \to |B_\ast|$ is a weak equivalence.

2. MAIN RESULTS.

2.1. Definition. A groupoid of homeomorphisms $\Gamma$ of a space $X$ is a topological category $\Gamma$ with space of objects $X$, such that (denoting by $\Gamma_1$ the space of morphisms of $\Gamma$):

i) every morphism in $\Gamma_1$ has an inverse.

ii) the domain and range maps $D, R : \Gamma_1 \to X$ are locally homeomorphisms.

Let $\Gamma$ be a groupoid of homeomorphisms of a space $X$. If $U \subseteq X$ is open, and $s : U \to \Gamma_1$ is a section of the domain map, then $Rs : U \to X$ is locally a homeomorphism. The sections such that $Rs$ is a homeomorphism form a pseudogroup of transformations of $X$ in the sense of Ehresmann [11] or Haefliger [6].

2.2. Definition [2]. Let $\Gamma$ be a groupoid of homeomorphisms of $X$. Let $p : Y \to X$ be a continuous map. An action of $\Gamma$ on $Y$ over $X$ is a groupoid of homeomorphisms $\Gamma \setminus Y$ of $Y$, and a functor $p : \Gamma \setminus Y \to \Gamma$, which on objects, is the map $p : Y \to X$, such that:

i) the diagram

\[
\begin{array}{ccc}
(\Gamma \setminus Y)_1 & \xrightarrow{D} & Y \\
\downarrow p & & \downarrow p \\
\Gamma_1 & \xrightarrow{D} & X
\end{array}
\]
is a pullback, so that we can write elements of \((\Gamma\backslash Y)_1\) as pairs \((g, y)\) such that \(Dg = py\);

ii) \(pR(g, y) = Rg\);

iii) if \(f, g \in \Gamma_1\) with \(Df = Rg\), and \(y \in Y\) with \(Dg = py\), then

\[(f \circ g, y) = (f, R(g, y)) \circ (g, y).\]

The range map \(R : (\Gamma\backslash Y)_1 \to Y\) of \(\Gamma\backslash Y\) is called the range map for the action. Of course, \(\Gamma\) itself is an action of \(\Gamma\) on \(X\).

2.3. Definition. Let \(\Gamma\) be a groupoid of homeomorphisms of a space \(X\). Define the monoid of immersions \(M\) of \(X\) to be the set

\[M = \{ m : X \to \Gamma_1 \mid Dm = \text{id} \}, \]

with the composition

\[m \circ n(x) = m(R nx) \circ n(x),\]

where the composition in the latter expression is in \(\Gamma_1\). (Note that if \(m \in M\), \(m(x) \in \Gamma_1\).)

2.4. Definition. Let \(\Gamma\) be a groupoid of homeomorphisms of a space \(X\), \(p : Y \to X\) a map, and \(\Gamma\backslash Y\) an action of \(\Gamma\) on \(Y\) over \(X\). To every \(m \in M\) define a section \(m : Y \to (\Gamma\backslash Y)_1\) of the domain map by

\[m(y) = (m(py), y).\]

Let \(M\backslash Y\) denote the topological category with objects \(Y\), morphisms \(M \times Y\), and domain and range maps \(D, R : M \times Y \to Y\) given by

\[D(m, y) = y, \quad R(m, y) = R m(y).\]

We define a functor \(i_Y : M \backslash Y \to \Gamma \backslash Y\) to be the identity on objects, with \(i_Y(m, y) = m(y) \in (\Gamma\backslash Y)_1\). In particular, there is a functor \(i: M \backslash X \to \Gamma\), realizing \(M \backslash X\) as the category of global sections of the domain map [3].

We can now restate Theorems 1.3 and 1.5.

2.5. Theorem. Let \(\Gamma\) be a groupoid of homeomorphisms of a space \(X\), and let \(M\) be the monoid of immersions of \(X\). Suppose that the open sets \(R m(X), m \in M\), form a basis for the topology of \(X\). Then:

i) for any action \(\Gamma \backslash Y\) of \(\Gamma\) on a space \(Y\), the functor \(M \backslash Y \to \Gamma \backslash Y\) induces a weak equivalence \(|M\backslash Y| \to |\Gamma\backslash Y|\);

ii) in particular, there is a weak equivalence \(|M \backslash X| \to |\Gamma|\).

Our applications use Corollary 2.8 below.
2.6. Definition. Let $\Gamma$ be a groupoid of homeomorphisms of $X$ and let $U \subseteq X$ be open. The stabilizer $\Gamma^U$ of $U$ is the groupoid of homeomorphisms of $U$ whose space of morphisms is

$$\Gamma^U = \{ g \in \Gamma_1 \mid Dg, Rg \in U \}.$$

The monoid of immersions of $U$ is

$$M^U = \{ m : U \to \Gamma_1 \mid Dm = \text{id} \text{ and } RmU \subseteq U \}.$$

2.7. Proposition [5]. Suppose the open sets $R \subseteq U$, $s : U \to \Gamma_1$ a section of the domain map, form a basis for the topology of $X$. Then the inclusion functor $\Gamma^U \to \Gamma$ induces a weak equivalence $|\Gamma^U| \to |\Gamma|$.

2.8. Corollary. Let $\Gamma$ be a groupoid of homeomorphisms of $X$, let $U \subseteq X$, and suppose the open sets $R \subseteq U$, $s : U \to \Gamma_1$ a section of the domain map, form a basis for the topology of $X$. Then there is a weak equivalence $|M^{U}\setminus U| \to |\Gamma^U|$. 

Proof. By 2.7 there is a weak equivalence $|\Gamma^U| \to |\Gamma|$ and by 2.5 there is a weak equivalence $|M^{U}\setminus U| \to |\Gamma^U|$.

2.9. Examples. i) Let $G$ be a discrete group acting on a space $X$. Define a groupoid of homeomorphisms $G \backslash X$ of $X$ whose space of morphisms is $G \times X$, with $D, R : G \times X \to X$ defined by

$$D(g, x) = x, \quad R(g, x) = gx.$$ 

There is a weak equivalence $|G \backslash X| \to EG \times X$. 

Let $U \subseteq X$ be an open set, and suppose that the images $\{gU\}; g \in G$ form a basis for the topology of $X$. Let $M^U$ be the monoid of elements of $g$ taking $U$ into $U$. Then there is a weak equivalence $|M^U \setminus U| \to |G \backslash X|$. If $U$ is contractible there is a weak equivalence $BM \to |G \backslash X|$.

ii) Let $N^K$ be the "final $k$-dimensional submanifold of $R^n$" defined as $N^K = \bigsqcup U_f/\sim$ where we take one copy $U_f$ for every open subset $U$ of $R^K$ and every $C^\infty$ immersion $f : U \to R^n$ where if $u \in U_f, v \in V_g$ we set $u - v$ if there is a neighborhood $W$ of $u$ in $U_f$ and a $C^\infty$ embedding $h : W \to V_g$ such that $g \cdot h = f$ on $W$. $N^K$ is a non-Hausdorff $k$-dimensional manifold with an immersion $N^K \to R^n$.

Let $\Gamma^n$ be the groupoid of diffeomorphisms of $R^n$; $\Gamma^n_1$ is the space of germs of diffeomorphisms of $R^n$ with the sheaf topology. There is an obvious action $\Gamma \backslash N^K$, Picking a standard embedding $R^K \to R^n$, we can regard $R^K$ as a submanifold of $N^K$. The stabilizer $(\Gamma \backslash N^K)_R^K$ is the groupoid of diffeomorphisms of $R^K$ with germs of extension to $R^n$; by (2.7) there is a weak equivalence $|(\Gamma \backslash N^K)_R^K| \to |\Gamma \backslash N^K|$.
The monoid $M^{R^k}$ is the monoid of immersions of $R^k$ with germs of extensions to $R^k$. By (2.9), there is a weak equivalence $BM^{R^k} \rightarrow |\Gamma \setminus N^k|$. This example is exploited in [4].

3. PROOF OF 2.5.

From now on, we assume the conditions of 2.5: $\Gamma$ is a groupoid of homeomorphisms of a space $X$, such that the images $RmX$, $m \in M$, form a basis for the topology of $X$, $p : Y \rightarrow X$ is a map, and $\Gamma \setminus Y$ is an action of $\Gamma$ on $Y$ over $X$.

3.1. Definition [8]. Let $M$ be a monoid acting on a set $S$ on the right, and on the space $W$ on the left. Define a topological category $S/M \setminus W$ with objects $S x W$, topologized as a disjoint union of copies $s x W$ of $W$, and morphisms $S x N x W$, topologized as a disjoint union of copies $s x n x W$ of $W$. The structure maps are given by

$$D(s, n, w) = (sn, w) \quad \text{and} \quad R(s, n, w) = (s, nw).$$

If $S$ is a set, let $\Delta_{S}$ denote the "simplex on $S$", the simplicial set whose set of $n$-simplices is $S^{n+1}$, with

$$d_i(x_0, ..., x_n) = (x_0, ..., \hat{x_i}, ..., x_n) \quad \text{and} \quad s_i(x_0, ..., x_n) = (x_0, ..., x_i, \hat{x_i}, ..., x_n).$$

If a monoid $N$ acts on $S$, then $N$ acts on $S^{n+1}$ by the diagonal action, and in fact $N$ acts on $\Delta_{S}$ by simplicial maps.

3.2. Definition. Let $N$ be a monoid acting on a set $S$ on the right and on a space $W$ on the left. We define a simplicial topological category $\Delta_{S}/M \setminus W$ by

$$(\Delta_{S}/M \setminus W)_n = S^{n+1}/M \setminus W,$$

with functors

$$d_i : S^{n+1}/M \setminus W \rightarrow S^n/M \setminus W \quad \text{and} \quad s_i : S^n/M \setminus W \rightarrow S^{n+1}/M \setminus W$$

induced by the simplicial structure of $\Delta_{S}$.

Since $\Delta_{S}$ is contractible, there is a homotopy equivalence

$$|\Delta_{S}/M \setminus W| \rightarrow |M \setminus W|.$$

3.3. Definition. We define a simplicial topological category $\Gamma \setminus Y$ with a homotopy equivalence $|\Gamma \setminus Y| \rightarrow \Gamma \setminus Y$. Let $|\Gamma \setminus Y|_n$ be the topological category with space of objects $(\Gamma \setminus Y)_n$, space of morphisms $(\Gamma \setminus Y)_n$ and all structure maps the identity. The simplicial maps between the $(\Gamma \setminus Y)_n$ define the functors between the $(\Gamma \setminus Y)_n$.

Since $(\Gamma \setminus Y)_n$ is just $|\Gamma \setminus Y|_n$ crossed with a simplex $|\Delta_{S}|$. 


on $\mathbb{N} = \{0, 1, 2, \ldots\}$, there is an equivalence $|\Gamma \backslash \mathcal{V}| \rightarrow |\Gamma \backslash \mathcal{V}|$.

3.4. **Lemma.** Let $M$ (the monoid of immersions of $\Gamma$) act on itself on the right, by composition. There is a simplicial functor $F : \Delta M / M \backslash Y \rightarrow \Gamma \backslash \mathcal{V}$ such that each $F_n : M^{n+1} / M \backslash Y \rightarrow (\Gamma \backslash \mathcal{V})_n$ induces a weak equivalence.

Lemma 3.4 is proved in Sections 4 and 5.

3.5. **Proof of 2.5.** By the remark after 3.2 there is a homotopy equivalence $|\Delta M / M \backslash Y| \rightarrow |\Delta \Gamma \backslash \mathcal{V}|$. By 3.3 there is a weak equivalence

$|\Delta M / M \backslash Y| \rightarrow |\Gamma \backslash \mathcal{V}|$.

Therefore, there is a weak equivalence $|M \backslash Y| \rightarrow |\Gamma \backslash \mathcal{V}|$.

With more work one can show that

$\begin{array}{c}
|\Delta M / M \backslash Y| \\
|\Gamma \backslash \mathcal{V}|
\end{array}$

commutes up to weak homotopy.

4. **DEFINITION OF $F_n$.**

We define the functors $F_n : M^{n+1} / M \backslash Y \rightarrow (\Gamma \backslash \mathcal{V})_n$ and prove (4.2) that $F_n$ is "onto" in a certain sense. We write elements of $(\Gamma \backslash \mathcal{V})_n$ as $(f_n, \ldots, f_1, y)$ where

$f_i \in \Gamma_1$ and $D f_i = R f_{i-1}$, $i > 1$, and $D f_1 = p y$.

Recall that if $m \in M$, $m(x)$ is an element of $\Gamma_1$ for $x \in X$.

4.1. **Definition.** On objects, $F_n$ is the map $F_{n,0} : M^{n+1} x Y \rightarrow (\Gamma \backslash \mathcal{V})_n$ given by

$F_{n,0} (m_n, \ldots, m_0, y) = (m_n p y) \circ m_{n-1} (p y)^{-1}, \ldots, m_1 (p y) \circ m_0 (p y)^{-1}, R m_0 (y)).$

On morphisms, $F_{n,1} : M^{n+1} x M x Y \rightarrow (\Gamma \backslash \mathcal{V})_n$ is defined by

$F_{n,1} (m_n, \ldots, m_0, k, y) = F_{n,0} (m_n k, \ldots, m_0 k, y)$.

It’s not hard to verify that the $F_n$ define a simplicial functor

$F_* : \Delta M / M \backslash Y \rightarrow \Gamma \backslash \mathcal{V}$.

4.2. **Lemma.** Let $(f_n, \ldots, f_1, y) \in (\Gamma \backslash \mathcal{V})_n$. Then there exists
Proof. We prove the result for \( n = 1 \). For \( n > 1 \) the result follows similarly, using induction. Let \((f_1, y) \in (\Gamma \setminus \mathcal{Y})_1\). Let \( x = Df_1 = py \). There is a section \( f_1 : U \to \Gamma_1 \) of \( D \) on a neighborhood \( U \) of \( x \), such that \( f_1(x) = f_1 \) and \( Rf_1 \) is one-to-one on \( U \). Let \( m_0 \in \mathcal{M} \) such that \( x \in Rm_0X \subset U \);

such a \( m_0 \) exists because the \( RmX, m \in \mathcal{M} \), form a basis for the topology of \( X \). Let \( x' \in X \) such that \( Rm_0(x') = x \). Define \( m_1 \in \mathcal{M} \) by

\[
m_1(x) = f_1(Rm_0x) \circ m_0(x).
\]

Since \( x = py \), \( y \in Rm_0Y \). Pick \( z \in Y \) so that \( Rm_0(z) = y \). Then

\[
F_{1,0}(m_1, m_0, z) = (f_1, y).
\]

5. PROOF OF LEMMA 3.4.

We have defined functors \( F_n : \mathcal{M}^{n+1}/\mathcal{M} \setminus \mathcal{Y} \to (\Gamma \setminus \mathcal{Y})_n \). There is a projection map \( (\mathcal{M} \setminus \mathcal{Y})_n \to (\Gamma \setminus \mathcal{Y})_n \). Let \( T \) be the composition

\[
T_n : \mathcal{M}^{n+1}/\mathcal{M} \setminus \mathcal{Y} \to (\Gamma \setminus \mathcal{Y})_n.
\]

To prove 3.4, it is enough that each \( T_n \) be a weak equivalence by (1.8).

First we show that the maps \( T_n \) are almost locally trivial ([10], Appendix). By ([13], A.1) it then suffices to prove that \( T^{-1}_n(x) \) is contractible for every \( x \in (\Gamma \setminus \mathcal{Y})_n \).

5.1. Definition [13]. A map \( f : B \to A \) of spaces is almost locally trivial if for every \( a \in A \) there is a neighborhood of \( f^{-1}(a) \) in \( B \) which is homeomorphic to a neighborhood of \( f^{-1}(a) \times a \) in \( f^{-1}(a) \times A \).

5.2. The space \( T^{-1}_n(x) \). Let \( x \in (\Gamma \setminus \mathcal{Y})_n \). We describe \( T^{-1}_n(x) \) as the geometric realization \( |C_X| \) of a discrete category \( C_X \). The objects of \( C_X \) are pairs \((s, y)\) with

\[
s \in \mathcal{M}^{n+1}, y \in Y \quad \text{such that} \quad F_{n,0}(s, y) = x.
\]

The morphisms of \( C_X \) are triples \((s, m, y)\) with

\[
s \in \mathcal{M}^{n+1}, m \in \mathcal{M}, y \in Y \quad \text{so that} \quad F_{n,0}(s, m, y) = F_{n,0}(s, Rm(y)) = x.
\]

The structure maps \( D, R \) of \( C_X \) are defined as

\[
D(s, m, y) = (sm, y) \quad \text{and} \quad R(s, m, y) = (s, Rm(y)).
\]
5.3. Proposition. \( T_n : |M^{n+1}/M \setminus Y| \to (\Gamma \setminus Y)_n \) is almost locally trivial.

Proof. If \((s, y) \in C_x\) let \(V(s, y)\) be a neighborhood of \((s, y)\) in \(s \times Y\) such that \(F_{n,0}\) is one-to-one restricted to \(V(s, y)\). Denote by \(N(x)\) the subcategory of \(M^{n+1}/M \setminus Y\) generated by the points of the \(V(s, y)\);
\[
|N(x)| \text{ is a neighborhood of } |C_x| \text{ in } |M^{n+1}/M \setminus Y|.
\]

Let \(M(x)\) be the subcategory of \(C_x \times (\Gamma \setminus Y)_n\) generated by objects
\[
( (s, y), F_{n,0}(s, y') ) \quad \text{where } y' \in V(s, y).
\]

Then \(|M(x)|\) is a neighborhood of \(|C_x\times |x|\) in \(|C_x\times |(\Gamma \setminus Y)_n|\).

The functor \(G : N(x) \to M(x)\) given on objects by
\[
G(s, y') = ( (s, y), F_{n,0}(s, y') )
\]
is an isomorphism of categories. Thus, \(T_n\) is almost locally trivial.

To complete the proof of Lemma 3.4 we show that the categories \(C_x\) have contractible realization.

5.4. Definition. A category \(C\) is codirected if:

i) for any objects \(A_1, A_2\) of \(C\) there is an object \(B\) of \(C\), and maps \(f_i : B \to A_i\).

ii) If \(f_i : B \to A, i = 1, 2\) are maps in \(C\) there is an object \(E\) in \(C\) and a map \(g : E \to B\) in \(C\) such that \(f_1 \circ g = f_2 \circ g\).

After Quillen [11], codirected categories have contractible realizations.

5.5. Proof of 3.4. Since the maps \(T_n\) are almost locally trivial, we need only show that the \(C_x\) have contractible realizations. We will prove that the \(C_x\) are codirected. Note that by 4.2, the \(C_x\) are nonempty.

Condition ii of 5.4 follows for \(C_x\) from the fact that there can be at most one morphism between any two objects in \(C_x\). To verify i we need to show that for every \((s_1, y_1), (s_2, y_2) \in C_x\) there are \(y \in Y, m_1, m_2 \in M\) such that

\[
\begin{align*}
\text{(i)} & \quad s_2m_2 = s_1m_1 \quad \text{and} \quad \text{(ii)} \quad Rm_1(y) = y_1, \quad Rm_2(y) = y_2.
\end{align*}
\]

Write
\[
\begin{align*}
s_2 &= (s_2^0, \ldots, s_2^0) \quad \text{and} \quad s_1 = (s_1^0, \ldots, s_1^0),
\end{align*}
\]

where each \(s_j^j \in M\). Let \(U_1\) be a neighborhood of \(y_1\) on which each \(R^j s_2^j\) is one-to-one; define \(U_2\) similarly. Let
\[
m_1 \in M \quad \text{so that} \quad \rho y_1 \in Rm_1 \times C \quad \rho U_1,
\]

and define \(m_2 \in M\) by
It is not hard to verify that \( s_2 \circ m_2 = s_1 \circ m_1 \), and then, by induction, that \( s_{2j} \circ m_2 = s_{1j} \circ m_1 \). Therefore, \( s_1 m_1 = s_2 m_2 \).

Now let \( y \in Rm X \), so there is some \( y \in Y \) such that \( Rm_1(y) = y_1 \). Then it follows that

\[
Rs_2 \circ (Rm_2 y) = Rs_1 \circ (y_1).
\]

But \( s_2 \) is one-to-one on \( U_2 \), so \( Rm_2 y = y_2 \).

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