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Models for actions of certain groupoids

Cahiers de topologie et géométrie différentielle catégoriques, tome 26, no 1 (1985), p. 33-42

<http://www.numdam.org/item?id=CTGDC_1985__26_1_33_0>
MODELS FOR ACTIONS OF CERTAIN GROUPOIDS
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Résumé. Soit $B\Gamma^n$ l'espace classifiant pour les feuilletages $C^\infty$ de codimension $n$, et soit $M^n$ le monoïde discret des plongements $C^\infty$ de $\mathbb{R}^n$ dans $\mathbb{R}^n$. G. Segal a montré qu'il existe une équivalence d'homotopie faible $BM^n \to B\Gamma^n$, et D. McDuff a obtenu des résultats analogues pour les espaces classifiants de feuilletages $C^\infty$ avec une forme volume transverse, de codimension au moins 3. Cn généralise ici ces résultats.

1. INTRODUCTION.

Let $B\Gamma^n$ be the classifying space for $C^\infty$ codimension $n$ foliations, and let $M^n$ be the discrete monoid of $C^\infty$ embeddings from $\mathbb{R}^n$ to $\mathbb{R}^n$. G. Segal showed [13] that there is a weak homotopy equivalence $BM^n \to B\Gamma^n$, and D. McDuff [10] obtained similar results for the classifying spaces for $C^\infty$ foliations with transverse volume form, with codimension at least 3. This paper generalizes these results.

Much of this work was done at M.I.T. and is my doctoral thesis; I'd like to thank my advisors Daniel Kan and Sol Jekel. Sol Jekel has obtained similar results with other methods [7].

1.1. Classifying spaces for Haefliger structures.

A pseudogroup $G$ of transformations of a space $X$ is a collection, closed under composition and inverse, of homeomorphisms between open subsets of $X$. There are various models for a classifying space $BG$; such a classifying space is of interest in the homotopy theory of foliations whose transverse geometric structure is modelled on $X$. In this paper we start with a topological category $\Gamma$ called a groupoid of homeomorphisms (2.1) of $X$; the classifying space $(1.7) \mid \Gamma \mid$ of this category is a standard model for $BG$.

1.2. Monoids of immersions.

We associate to a groupoid of homeomorphisms $\Gamma$ of a space $X$ a discrete monoid of immersions $M$ (2.3), which acts on $X$ by maps $m : X \to X$ which are locally one-to-one. Let $|M\backslash X|$ denote the homotopy quotient of the action (sometimes denoted $EM_{\mathcal{M}}X$); if $X$ is contractible there is an (homotopy) equivalence $|M\backslash X| \to BM$.

1.3. Theorem. If the images $mX$, $m \in M$ form a basis for the topology of $X$, then there is a weak (homotopy) equivalence $|M\backslash X| \to |\Gamma|$.
For example, let $\Gamma$ be the groupoid of all area preserving diffeomorphisms between open subsets of the plane; $|\Gamma|$ is the classifying space for codimension 2 foliations with a transverse area form. Let $M$ be the monoid of area preserving immersions of the plane. Then there is a weak equivalence $BM \to |\Gamma|$.

### 1.4. Modelling actions.

Let $Y$ be a space with a map $p : Y \to X$, and suppose that the homeomorphisms between open subsets of $X$ in the pseudogroup $G$ lift naturally to homeomorphisms between the inverse image of the open sets in $Y$. Such extra data may correspond to some facet of the geometry of $X$. For example, consider the tangent bundle $p : T^*_R \to R^n$. If $f : U \to V$ is a diffeomorphism between open subsets of $R^n$, there is a natural induced isomorphism $f^* : T^*_U \to T^*_V$ of tangent bundles.

We define (2.2) an action of a groupoid of homeomorphisms $\Gamma$ on a space $Y$ over $X$. There is an appropriate definition of the homotopy quotient $\l TBY l$, *of the action; in [5] we show that there is a spectral sequence abutting to $H_*|\Gamma Y|$, with

$$E^2_{pq} = H_p(\Gamma; H_q Y),$$

where the latter expression is defined in analogy with group homology. In some cases the $E^2$ term has some geometrical significance [4].

If $\Gamma$ is a groupoid of homeomorphisms of $X$, and $p : Y \to X$ is a map, and $\Gamma$ acts on $Y$ over $X$, then the monoid of immersions $M$ acts on $Y$.

### 1.5. Theorem. If the images $mX$, $m \in M$, form a basis for the topology of $X$, then there is a weak equivalence $|M Y| \to |\Gamma Y|$.

It turns out that 1.3 is a corollary to 1.5.

### 1.6. Organization. In Section 2, definitions are given, along with restatements of the main results and some examples. Section 3 contains the main body of proof, with a major Lemma 3.4 proved in Sections 4 and 5.

### 1.7. Notation.

We assume familiarity with semisimplicial notations; references are [9] and [14].

Recall [2] that a topological category is a small category in which the sets of objects and morphisms are given topologies, such that the structure maps of the category are continuous. The nerve of a (topological) category $C$ is a simplicial (space) set $C_*; a$ functor $F : C \to D$ induces a simplicial map $F_* : C_* \to D_*$. If $C$ is a (topological) category, we denote by $D, R : C_n \to C_0$ the domain and range maps

$$D = d_1 \ldots d_n, \quad R = d_{n-1} \ldots d_1 d_0.$$
We employ the geometric realization functor

\[ | \cdot | : \text{simplicial spaces} \to \text{spaces} \]

defined by Segal (\cite{12}, App. A; Segal calls the functor \( \| \cdot \| \)). If \( A_* \) is a simplicial space, \( |A_*| \) is defined as

\[ |A_*| = \bigsqcup A_n \times \Delta^n / \sim \]

where \( \Delta^n \) is the standard \( n \)-simplex, and we set

\[ (F \alpha_n; t_k) \sim (a_n, F^* t_k) \]

for every composite of face maps \( F : A_n \to A_k \); here \( F^* : \Delta^k \to \Delta^n \) is the inclusion induced by \( F \). Segal's realization is useful to us because of the proposition:

1.8. Proposition. (Segal, 12, App. A.1.ii) If \( f_* : A_* \to B_* \) is a map of simplicial spaces such that every \( f_n : A_n \to B_n \) is a weak equivalence, then \( |f_*| : |A_*| \to |B_*| \) is a weak equivalence.

2. MAIN RESULTS.

2.1. Definition. A groupoid of homeomorphisms \( \Gamma \) of a space \( X \) is a topological category \( \Gamma \) with space of objects \( X \), such that (denoting by \( \Gamma_1 \) the space of morphisms of \( \Gamma \)):

i) every morphism in \( \Gamma_1 \) has an inverse.

ii) the domain and range maps \( D, R : \Gamma_1 \to X \) are locally homeomorphisms.

Let \( \Gamma \) be a groupoid of homeomorphisms of a space \( X \). If \( U \subset X \) is open, and \( s : U \to \Gamma_1 \) is a section of the domain map, then \( R s : U \to X \) is locally a homeomorphism. The sections such that \( R s \) is a homeomorphism form a pseudogroup of transformations of \( X \) in the sense of Ehresmann [1], or Haefliger [6].

2.2. Definition [2]. Let \( \Gamma \) be a groupoid of homeomorphisms of \( X \). Let \( p : Y \to X \) be a continuous map. An action of \( \Gamma \) on \( Y \) over \( X \) is a groupoid of homeomorphisms \( \Gamma \setminus Y \) of \( Y \), and a functor \( p : \Gamma \setminus Y \to \Gamma \), which on objects, is the map \( p : Y \to X \), such that:

i) the diagram

\[ \begin{array}{ccc}
(\Gamma \setminus Y)_1 & \xrightarrow{D} & Y \\
\downarrow p & & \downarrow p \\
\Gamma_1 & \xrightarrow{D} & X 
\end{array} \]
is a pullback, so that we can write elements of \((TBY)_1\) as pairs \((g, y)\) such that \(Dg = py\);

ii) \(pR(g, y) = Rg;\)

iii) if \(f, g \in \Gamma_1\) with \(Df = Rg\), and \(y \in Y\) with \(Dg = py\), then

\[(f \circ g, y) = (f, Rg, y) \circ (g, y).\]

The range map \(R : (\Gamma \backslash Y)_1 \to Y\) of \(\Gamma \backslash Y\) is called the range map for the action. Of course, \(\Gamma\) itself is an action of \(\Gamma\) on \(X\).

2.3. Definition. Let \(\Gamma\) be a groupoid of homeomorphisms of a space \(X\). Define the monoid of immersions \(M\) of \(X\) to be the set

\[M = \{m : X \to \Gamma_1 \mid Dm = \text{id}\},\]

with the composition

\[m \circ n(x) = m(Rn x) \circ n(x),\]

where the composition in the latter expression is in \(\Gamma_1\). (Note that if \(m \in M\), \(m(x) \in \Gamma_1\).)

2.4. Definition. Let \(\Gamma\) be a groupoid of homeomorphisms of a space \(X\), \(p : Y \to X\) a map, and \(\Gamma \backslash Y\) an action of \(\Gamma\) on \(Y\) over \(X\). To every \(m \in M\) define a section \(m : Y \to (TBY)_1\) of the domain map by

\[m(y) = (m(py), y).\]

Let \(M\backslash Y\) denote the topological category with objects \(Y\), morphisms \(MxY\), and domain and range maps \(D, R : MxY \to Y\) given by

\[D(m, y) = y, \quad R(m, y) = R m(y).\]

We define a functor \(i_Y : M\backslash Y \to \Gamma \backslash Y\) to be the identity on objects, . . with \(i_Y(m, y) = m(y) \in (\Gamma \backslash Y)_1\). In particular, there is a functor \(i : M\backslash X \to \Gamma\), realizing \(M\backslash X\) as the category of global sections of the domain map \([3]\).

We can now restate Theorems 1.3 and 1.5.

2.5. Theorem. Let \(\Gamma\) be a groupoid of homeomorphisms of a space \(X\), and let \(M\) be the monoid of immersions of \(X\). Suppose that the open sets \(Rm(X), m \in M\), form a basis for the topology of \(X\). Then :

i) for any action \(\Gamma \backslash Y\) of \(\Gamma\) on a space \(Y\), the functor \(M\backslash Y \to \Gamma \backslash Y\) induces a weak equivalence \(|M\backslash Y| \to |\Gamma \backslash Y| ;\)

ii) in particular, there is a weak equivalence \(|M\backslash X| \to |\Gamma| .\)

Our applications use Corollary 2.8 below.
2.6. Definition. Let \( \Gamma \) be a groupoid of homeomorphisms of \( X \) and let \( U \subset X \) be open. The stabilizer \( \Gamma^U \) of \( U \) is the groupoid of homeomorphisms of \( U \) whose space of morphisms is
\[
\Gamma^U = \{ g \in \Gamma \mid \text{Dg}, \text{Rg} \in U \}.
\]
The monoid of immersions of \( U \) is
\[
M^U = \{ m : U \to \Gamma \mid Dm = \text{id} \text{ and } \text{RmU} \subset U \}.
\]

2.7. Proposition [5]. Suppose the open sets \( R \times U, s : U \to \Gamma \) a section of the domain map, form a basis for the topology of \( X \). Then the inclusion functor \( \Gamma^U \to \Gamma \) induces a weak equivalence \( |\Gamma^U| \to |\Gamma| \).

2.8. Corollary. Let \( \Gamma \) be a groupoid of homeomorphisms of \( X \), let \( U \subset X \), and suppose the open sets \( R \times U, s : U \to \Gamma \) a section of the domain map, form a basis for the topology of \( X \). Then there is a weak equivalence \( |M^U \setminus U| \to |\Gamma^U| \).

Proof. By 2.7 there is a weak equivalence \( |\Gamma^U| \to |\Gamma| \) and by 2.5 there is a weak equivalence \( |M^U \setminus U| \to |\Gamma^U| \).

2.9. Examples. i) Let \( G \) be a discrete group acting on a space \( X \). Define a groupoid of homeomorphisms \( GBX \) of \( X \) whose space of morphisms is \( G \times X \), with \( D, R : G \times X \to X \) defined by
\[
D(g, x) = x, \quad R(g, x) = gx.
\]
There is a weak equivalence \( |GBX| \to |EG \times X| \).

Let \( U \subset X \) be an open set, and suppose that the images \( \{ gU \}, g \in G \) form a basis for the topology of \( X \). Let \( M^U \) be the monoid of elements of \( g \) taking \( U \) into \( U \). Then there is a weak equivalence \( |M^U \setminus U| \to |GBX| \).
If \( U \) is contractible there is a weak equivalence \( BM \to |GBX| \).

ii) Let \( N^K \) be the "final \( k \)-dimensional submanifold of \( R^n \)" defined as \( N^K = \bigsqcup U_f \sim \) where we take one copy \( U_f \) for every open subset \( U \) of \( R^K \) and every \( C^\infty \) immersion \( f : U \to R^k \) where if \( u \in U_f, \text{ and } v \in V_g \) we set \( u \sim v \) if there is a neighborhood \( W \) of \( u \) in \( U_f \) and a \( C^\infty \) embedding \( h : W \to V_g \) such that \( g \circ h = f \) on \( W \). \( N^K \) is a non-Hausdorff \( k \)-dimensional manifold with an immersion \( N^K \to R^n \).

Let \( \Gamma^N \) be the groupoid of diffeomorphisms of \( R^n \); \( \Gamma^N \) is the space of germs of diffeomorphisms of \( R^n \) with the sheaf topology. There is an obvious action \( \Gamma \backslash N^K \). Picking a standard embedding \( R^K \to R^n \), we can regard \( R^K \) as a submanifold of \( N^K \). The stabilizer \( (\Gamma \backslash N^K)^{R^K} \) is the groupoid of diffeomorphisms of \( R^K \) with germs of extension to \( R^n \); by (2.7) there is a weak equivalence
\[
| (\Gamma \backslash N^K)^{R^K} | \to | \Gamma \backslash N^K |.
\]
The monoid $M^R_k$ is the monoid of immersions of $R^k$ with germs of extensions to $R^k$. By (2.9), there is a weak equivalence $BM^R_k \to |\Gamma \backslash W^k|$. This example is exploited in [4].

3. PROOF OF 2.5.

From now on, we assume the conditions of 2.5 : $\Gamma$ is a groupoid of homeomorphisms of a space $X$, such that the images $RmX$, $m \in M$, form a basis for the topology of $X$, $p : Y \to X$ is a map, and $\Gamma \backslash Y$ is an action of $\Gamma$ on $Y$ over $X$.

3.1. Definition [8]. Let $M$ be a monoid acting on a set $S$ on the right, and on the space $W$ on the left. Define a topological category $S/M \backslash W$ with objects $S \times W$, topologized as a disjoint union of copies $S \times W$ of $W$, and morphisms $S \times N \times W$, topologized as a disjoint union of copies $S \times N \times W$ of $W$. The structure maps are given by

$$D(s, n, w) = (sn, w) \quad \text{and} \quad R(s, n, w) = (s, nw).$$

If $S$ is a set, let $\Delta_S$ denote the "simplex on $S"$, the simplicial set whose set of $n$-simplices is $S^{n+1}$, with

$$d_i(x_1, \ldots, x_o) = (x_1, \ldots, \hat{x}_i, \ldots, x_o) \quad \text{and} \quad s_i(x_1, \ldots, x_o) = (x_1, \ldots, x_i, \ldots, x_o).$$

If a monoid $N$ acts on $S$, then $N$ acts on $S^{n+1}$ by the diagonal action, and in fact $N$ acts on $\Delta_S$ by simplicial maps.

3.2. Definition. Let $N$ be a monoid acting on a set $S$ on the right and on a space $W$ on the left. We define a simplicial topological category $\Delta_S/M \backslash W$ by

$$(\Delta_S/M \backslash W)_n = S^{n+1}/M \backslash W,$$

with functors

$$d_i : S^{n+1}/M \backslash W \to S^n/M \backslash W \quad \text{and} \quad s_i : S^n/M \backslash W \to S^{n+1}/M \backslash W$$

induced by the simplicial structure of $\Delta_S$.

Since $\Delta_S$ is contractible, there is a homotopy equivalence

$$|\Delta_S/M \backslash W| \to |M \backslash W|.$$

3.3. Definition. We define a simplicial topological category $\Gamma \backslash Y$ with a homotopy equivalence $|\Gamma \backslash Y| \to |\Gamma \backslash W|$. Let $(\Gamma \backslash Y)_n$ be the topological category with space of objects $(\Gamma \backslash Y)_n$, space of morphisms $(\Gamma \backslash Y)_n$ and all structure maps the identity. The simplicial maps between the $(\Gamma \backslash Y)_n$ define the functors between the $(\Gamma \backslash Y)_n$.

Since $(\Gamma \backslash Y)_n$ is just $(\Gamma \backslash Y)_n$ crossed with a simplex $|\Delta_S|$.
on $\mathbb{N} = \{0, 1, 2, \ldots\}$, there is an equivalence $|\Gamma\setminus Y| \rightarrow |\Gamma\setminus Y|$. 

3.4. Lemma. Let $M$ (the monoid of immersions of $\Gamma$) act on itself on the right, by composition. There is a simplicial functor $F : M/\Gamma \rightarrow \Gamma\setminus Y$ such that each $F_n : M^n/\Gamma \rightarrow (\Gamma\setminus Y)_n$ induces a weak equivalence. Lemma 3.4 is proved in Sections 4 and 5.

3.5. Proof of 2.5. By the remark after 3.2 there is a homotopy equivalence $|\Delta M/\Gamma\setminus Y| \rightarrow |\Delta\setminus Y|$. By 3.3 there is a weak equivalence $|\Delta M/\Gamma\setminus Y| \rightarrow |\Gamma\setminus Y|$. Therefore, there is a weak equivalence $|\Delta M/\Gamma\setminus Y| \rightarrow |\Gamma\setminus Y|$. With more work one can show that

\[
\begin{array}{ccc}
|\Delta M/\Gamma\setminus Y| & \longrightarrow & |\Gamma\setminus Y| \\
|\Delta M/\Gamma\setminus Y| & \longrightarrow & |\Gamma\setminus Y| \\
\end{array}
\]

commutes up to weak homotopy.

4. DEFINITION OF $F_n$. 

We define the functors $F_n : M^n/\Gamma\setminus Y \rightarrow (\Gamma\setminus Y)_n$ and prove (4.2) that $F_n$ is "onto" in a certain sense. We write elements of $(\Gamma\setminus Y)_n$ as $(f_n, \ldots, f_1, y)$ where

$$f_i \in \Gamma_1 \quad \text{and} \quad D f_i = R f_{i-1}, \ i > 1, \ \text{and} \ D f_1 = py.$$ 

Recall that if $m \in M$, $m(x)$ is an element of $\Gamma_1$ for $x \in X$.

4.1. Definition. On objects, $F_n$ is the map $F_{n,0} : M^{n+1}xY \rightarrow (\Gamma\setminus Y)_n$ given by

$$F_{n,0} (m_n, \ldots, m_o, y) = (m_n(py) \circ m_{n-1}(py)^{-1}, \ldots, m_1(py) \circ m_0(py)^{-1}, R m_0(y)).$$

On morphisms, $F_{n,1} : M^{n+1}xMxY \rightarrow (\Gamma\setminus Y)_n$ is defined by

$$F_{n,1} (m_n, \ldots, m_o, k, y) = F_{n,0} (m_nk, \ldots, m_0k, y).$$

It's not hard to verify that the $F_n$ define a simplicial functor

$$F_* : \Delta M/\Gamma\setminus Y \rightarrow \Gamma\setminus Y.$$ 

4.2. Lemma. Let $(f_n, \ldots, f_1, y) \in (\Gamma\setminus Y)_n$. Then there exists
Proof. We prove the result for \( n = 1 \). For \( n > 1 \) the result follows similarly, using induction. Let \((f_1, y) \in (\Gamma \setminus Y)_1\). Let \( x = Df_1 = py \). There is a section \( f_1 : U \to \Gamma_1 \) of \( D \) on a neighborhood \( U \) of \( x \), such that \( f_1(x) = f_1 \) and \( Rf_1 \) is one-to-one on \( U \). Let

\[
\begin{align*}
\forall \sigma \in (\sigma) \quad \text{such that } \quad x \in Rm_0 X \subset U ;
\end{align*}
\]
such a \( m_0 \) exists because the \( Rm_0 X, m \in M \), form a basis for the topology of \( X \). Let \( x' \in X \) such that \( Rm_0 (x') = x \). Define \( m_1 \in M \) by

\[
\begin{align*}
\forall \sigma \in (\sigma) \quad \text{such that } \quad x \in Rm_0 X \subset U ;
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\]
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such a \( m_0 \) exists because the \( Rm_0 X, m \in M \), form a basis for the topology of \( X \). Let \( x' \in X \) such that \( Rm_0 (x') = x \). Define \( m_1 \in M \) by

\[
\begin{align*}
\forall \sigma \in (\sigma) \quad \text{such that } \quad x \in Rm_0 X \subset U ;
\end{align*}
\]
5.3. Proposition. \( T_n : |M^{n+1}/M \setminus Y| \to (\Gamma \setminus Y)_n \) is almost locally trivial.

Proof. If \( (s, y) \in C_x \) let \( V(s, y) \) be a neighborhood of \((s, y)\) in \( s \times Y\) such that \( F_{n,0} \) is one-to-one restricted to \( V(s, y) \). Denote by \( N(x) \) the subcategory of \( M^{n+1}/M \setminus Y \) generated by the points of the \( V(s, y) \); \( |N(x)| \) is a neighborhood of \( |C_X| \) in \( |M^{n+1}/M \setminus Y| \).

Let \( M(x) \) be the subcategory of \( C_x \times (\Gamma \setminus Y)_n \) generated by objects

\[
\big((s, y), \ F_{n,0}(s, y')\big) \quad \text{where} \quad y' \in V(s, y).
\]

Then \( |M(x)| \) is a neighborhood of \( |C_X| \times |x| \) in \( |C_X| \times |(\Gamma \setminus Y)_n| \).

The functor \( G : N(x) \to M(x) \) given on objects by

\[
G(s, y') = \big((s, y), \ F_{n,0}(s, y')\big)
\]

is an isomorphism of categories. Thus, \( T_n \) is almost locally trivial.

To complete the proof of Lemma 3.4 we show that the categories \( C_x \) have contractible realization.

5.4. Definition. A category \( C \) is codirected if:

i) for any objects \( A_1, A_2 \) of \( C \) there is an object \( B \) of \( C \), and maps \( f_i : B \to A_i \).

ii) If \( f_i : B \to A_i \), \( i = 1, 2 \) are maps in \( C \) there is an object \( E \) in \( C \) and a map \( g : E \to B \) in \( C \) such that \( f_1 \circ g = f_2 \circ g \).

After Quillen \([11]\), codirected categories have contractible realizations.

5.5. Proof of 3.4. Since the maps \( T_n \) are almost locally trivial, we need only show that the \( C_x \) have contractible realizations. We will prove that the \( C_x \) are codirected. Note that by 4.2, the \( C_x \) are nonempty.

Condition ii of 5.4 follows for \( C_x \) from the fact that there can be at most one morphism between any two objects in \( C_x \). To verify i we need to show that for every \((s_1, y_1), (s_2, y_2) \in C_x \) there are \( y \in Y, \ m_1, m_2 \in M \) such that

\[
(i) \ s_2m_2 = s_1m_1 \quad \text{and} \quad (ii) \ Rm_1(y) = y_1, \ Rm_2(y) = y_2.
\]

Write

\[
s_2 = \big(s_2^n, \ldots, s_2^0\big) \quad \text{and} \quad s_1 = \big(s_1^n, \ldots, s_1^0\big),
\]

where each \( s_j^n \in M \). Let \( U_1 \) be a neighborhood of \( y_1 \) on which each \( R_{s_2^n} \) is one-to-one; define \( U_2 \) similarly. Let

\[
m_1 \in M \quad \text{so that} \quad p_{y_1} \in Rm_1 \times \mathcal{C} \cap pU_1,
\]

and define \( m_2 \in M \) by
It is not hard to verify that \( s_2^0 \circ m_2 = s_1^0 \circ m_1 \), and then, by induction, that \( s_{2j}^0 \circ m_2 = s_{1j}^0 \circ m_1 \). Therefore, \( s_1^0 m_1 = s_2^0 m_2 \).

Now \( py_1 \in RmX \), so there is some \( y \in Y \) such that \( Rm_1(y) = y_1 \).

Then it follows that

\[
R s_2^0(Rm_2y) = R s_1^0(y_1).
\]

But \( s_2^0 \) is one-to-one on \( U_2 \), so \( Rm_2y = y_2 \).

BIBLIOGRAPHY.

5. P. GREENBERG, Actions of pseudogroups, Preprint.