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# CONVENIENT VECTOR SPACES EMBED INTO THE CAHIERS TOPOS

by Anders KOCK

**RÉSUMÉ.** Nous construisons un plongement plein de la catégorie des applications lisses entre espaces vectoriels convenables (Frölicher - Kriegl) dans l'un des topos connu comme un modèle de la Géométrie Différentielle Synthétique. L'étape essentielle consiste à étendre les foncteurs "points proches" de Weil du cas de dimension finie au cas convenable.

We construct a full embedding with good preservation properties of the Frölicher-Kriegl category $\underline{F}$ (cf. [2, 3, 7, 9]) of "convenient" vector spaces, with all smooth maps, into the fully well-adapted model $\underline{C}$ for synthetic differential geometry considered by Dubuc in [1], the so-called Cahiers topos (cf. also [4]). Each convenient vector space will, after the embedding, satisfy the vector form of the Axiom $1^W$ (Kock-Lawvere axiom, cf. [4]) for each Weil algebra W, and so the rich calculus of smooth maps in $\underline{F}$ can be dealt with synthetically in $\underline{C}$.

The idea of the construction is this : to construct a site of definition for the Cahiers topos, one utilizes that for each Weil algebra W, the endofunctor $-\otimes W$ on the category of finite dimensional vector spaces with linear maps extends to an endofunctor on the category $\underline{f}$ of finite-dimensional vector spaces and *smooth* maps, a construction which goes back to Weil [10] ; the site is then the "semidirect product" $\underline{f} \ltimes \underline{W}$ of $\underline{f}$ and $\underline{W}$ ($\underline{W}$ being the category of Weil algebras). We then prove that $-\otimes W$ can also be defined as an endofunctor on the category $\underline{F}$ of convenient vector spaces and smooth maps. The semidirect product $\underline{F} \ltimes \underline{W}$ contains $\underline{f} \ltimes \underline{W}$ as well as $\underline{F}$, and the desired embedding $J : \underline{F} \to \underline{C}$ is then simply by "representing from the outside", i.e., utilizing the hom functor of $\underline{F} \ltimes \underline{W}$.

## 1. SOME CALCULUS IN CONVENIENT VECTOR SPACES.

We recall some facts about these, from [2, 3, 7, 8], cf. also [9] and [5].

A *convenient* vector space is a vector space over R equipped with a linear subspace X' of the full algebraic dual X*, such that X' separates points, and with the following two completeness properties :

1. The bornology induced on X by X' is a complete bornology ;

2. any linear $X \to \mathbf{R}$ which is bounded with respect to this bornology belongs to X'.

In the following X, Y, Z, etc. always denote convenient vector spaces, X = (X, X') etc. The vector space $\mathbf{R}^n$ carries a unique convenient structure, namely the full linear dual.

We recall that a map $c : \mathbf{R}^n \to X$ is called *smooth* (or a smooth *plot* on X) if for any $\varphi \in X'$, $\varphi \circ c : \mathbf{R}^n \to \mathbf{R}$ is smooth (= $C^\infty$). And a map $f : X \to Y$ is called *smooth,* if $f \circ c$ is smooth for any smooth plot c on X.

The smooth linear maps $X \to \mathbf{R}$ turn out to be exactly the elements of X'.

A main motivation for the notion of convenient vector space is that the vector space $C^\infty(X, Y)$ of smooth maps from X to Y itself carries a canonical convenient structure, making the category of convenient vector spaces and their smooth maps into a cartesian closed category.

A map $f : X \to Y$ is said to have *order* $\geqq k$ if there exists a smooth $f^* : X \times \mathbf{R} \to Y$ with

$$f(\lambda . x) = \lambda^k . f^*(x, \lambda) \qquad \forall\ x \in X \qquad \forall\ \lambda \in \mathbf{R} .$$

In [5] (Theorem 2.13), we prove that $f$ is of order $\geqq k$ iff for any $x \in X$ and $\varphi \in Y'$, the map

$$\mathbf{R} \to \mathbf{R} \qquad \text{given by} \qquad \lambda \mapsto \varphi(f(\lambda \cdot x))$$

is of order $\geqq k$ .

A map $f : X \to Y$ is *homogeneous* of degree $i$ if

$$f(\lambda . x) = \lambda^i . f(x) \qquad \forall\ x \in X \qquad \forall\ \lambda \in \mathbf{R} ,$$

and *polynomial* of degree $< k$ if it can be written as a sum

$$f = \Sigma f_i \qquad (i = 0 , \ldots, k-1 )$$

with $f_i$ homogeneous of degree $i$ . Since Y' separates points, a map $f : X \to Y$ is homogeneous (resp. polynomial) with given degree iff for all $\varphi \in Y'$, $\varphi \circ f$ has the corresponding property.

One has the following results :

***Theorem 1.1.*** *Any smooth* $g : X \to Y$ *can uniquely be written as a sum of a polynomial map of degree* $< k$, *and a map of order* $\geqq k$.
*In particular,* $g$ *is of order* $\geqq 1$ *iff* $g(0) = 0$.

In the light of the above mentioned equivalence of the two definitions of order, this is Corollary 1.3 of [5].

The polynomial map in the theorem should be viewed as an approximating Taylor polynomial.

***Theorem 1.2.*** *Any smooth i-homogeneous map* $h : X \to Y$ *is of form*

$$h(x) = H(x, \dots, x)$$

*for some unique symmetric i-linear map* $H : X^i \to Y$.

This is Corollary 1.4 in [5].

***Theorem 1.3.*** *Let* $f : \mathbb{R}^n \to X$ *be smooth. Let* $k \geq 0$ *be an integer. There exist smooth functions* $g_\alpha : \mathbb{R}^n \to X$ *and elements* $x_\alpha \in X$ *such that, for all* $\underline{t} \in \mathbb{R}^n$,

$$f(\underline{t}) = \sum_{|\alpha|<k} \underline{t}^\alpha . x_\alpha + \sum_{|\alpha|=k} \underline{t}^\alpha . g_\alpha(\underline{t})$$

*(with standard conventions about multi-indices* $\alpha$*). The* $x_\alpha$ *'s are uniquely determined.*

Except for the uniqueness assertion, this follows immediately from [5], Theorem 2.12. The uniqueness of the $x_\alpha$ 's follows easily from the corresponding result for the case $X = \mathbb{R}$ using that $X'$ separates points.

The $x_\alpha$ 's in Theorem 1.3 are of course the "Taylor coefficients"

$$x_\alpha = \frac{1}{|\alpha|!} \cdot \frac{\partial^{|\alpha|} f}{\partial \underline{t}^\alpha}(\underline{0}) ;$$

however, they do not appear explicitely in the present article.

For any smooth $f : X \to Y$ and $x \in X$, the map

$$x_1 \mapsto f(x+x_1) - f(x)$$

can, by Theorems 1.1 and 1.2, be written as a sum of a smooth linear map $df_x$ and a map or order $\geq 2$. The map

$$X \times X \to Y \qquad \text{given by} \qquad (x, x_1) \mapsto df_x(x_1)$$

is smooth, and linear in the second variable, cf. e.g. [3]. Thus, it defines a map

$$Df : X \to L(X, Y)$$

where $L(X, Y)$ is the vector space of smooth linear maps $X \to Y$. There is a canonical structure of convenient vector space on $L(X, Y)$ making all the evaluation maps $L(X, Y) \to Y$ smooth and such that $Df$ is smooth.

## 2. JET CALCULUS AND WEIL PROLONGATIONS.

Let $I \subset C^\infty(\mathbf{R}^n)$ be an ideal. For any convenient vector space X, we let I(X) be the set of those smooth $f : \mathbf{R}^n \to X$ such that for all $\varphi \in X'$, $\varphi \circ f \in I$. We say that

$$f_1 \equiv f_2 \quad \text{mod } I \qquad \text{if} \qquad f_1 - f_2 \in I(X).$$

This is an equivalence relation. An equivalence class is called a *mod* I *jet into* X. This notion will be proved to have good properties if I is large enough : Let $M \subset C^\infty(\mathbf{R}^n)$ denote the (maximal) ideal of functions

$$h : \mathbf{R}^n \to \mathbf{R} \quad \text{with} \quad h(0) = 0,$$

i.e., functions of order $\geq 1$. Then $M^r$ is the ideal of functions of order $\geq r$. It is of finite codimension. We shall say that an ideal $I \subset C^\infty(\mathbf{R}^n)$ is a Weil ideal if, for some $r$, $M^r \subset I \subset M$. The residue ring $C^\infty(\mathbf{R}^n)/I$ is then a Weil algebra (cf. e.g. [4] or [1] for the notion), and any Weil algebra comes about in this way. We shall use the letter W to denote any Weil algebra, but *with* a given presentation by a Weil ideal I, and use "mod-I-jet" and "W-jet" synonymously.

We denote by $X \boxtimes W$ or ${}^W X$ the set of all W-jets into X. Since $M^r \subset I$, we may choose a finite set of polynomials

$$h_1, \dots, h_m \in \mathbf{R}[t_1, \dots, t_n]$$

of degree $< r$ which form a basis in $C^\infty(\mathbf{R}^n)$ mod I. It then follows from Theorem 1.3 that any W-jet into X has a representative of the form

$$(t_1, \dots, t_n) \mapsto \sum^m h_i(\underline{t}).x_i$$

for unique $x_i \in X$, and thus $X \boxtimes W \simeq X^m$. This also justifies the $\boxtimes$ notation, since $W \simeq \mathbf{R}^m$. Likewise, if $f : X \to Y$ is linear, $f \boxtimes W : X \boxtimes W \to Y \boxtimes W$ may of course be defined. Our aim is to define $f \boxtimes W$ for any *smooth* $f : X \to Y$.

**Proposition 2.1.** *If* $f_1 \equiv f_2$ *mod* I *(where* $f_i : \mathbf{R}^n \to X$ *), then we have* $g \circ f_1 \equiv g \circ f_2$ *mod* I *; for any smooth* $g : X \to Y$.

**Proof.** We have $f_1(0) = f_2(0)$ ( $= x_o$, say) since $f_1 \equiv f_2$ mod $M$. Since

$$g \circ (f_i - x_o) = \tilde{g} \circ f_i \quad \text{for} \quad \tilde{g}(x) := g(x + x_o),$$

it suffices to prove the result in the case

$$f_1(0) = f_2(0) = 0.$$

So $f_1$ and $f_2$ may both be assumed to have order $\geq 1$.

To prove $g \circ f_1 \equiv g \circ f_2 \bmod I$ means by definition to prove

$$\varphi \circ g \circ f_1 - \varphi \circ g \circ f_2 \in I,$$

for any smooth linear $\varphi : Y \to \mathbf{R}$, so let such $\varphi$ be given. Change notation and write $g$ for $\varphi \circ g$. Then $g : X \to \mathbf{R}$ may by Theorem 1.1 be written as a sum

$$\sum_{q=0}^{r-1} h_q + G$$

with $h_q : X \to \mathbf{R}$ smooth homogeneous of degree $q$, and G of order $\geqq r$. It suffices to prove that

(2.1) $$h_q \circ f_1 \equiv h_q \circ f_2 \quad \bmod I \qquad \forall q = 0, \ldots, r-1$$

and that

(2.2) $$G \circ f_1 \equiv G \circ f_2 \bmod I.$$

For (2.2), this is trivial ; in fact each $G \circ f_i$ $(i = 1, 2)$ has itself order $\geqq r$ since

$$\text{order}(f_i) \geqq 1 \quad \text{and} \quad \text{order}(G) \geqq r .$$

So

$$G \circ f_i \in M^r \subset I, \quad i = 1, 2.$$

For (2.1), we write, by Theorem 1.2 $h_q$ in the form

$$h_q(x) = H(x, \ldots, x),$$

where $H : X^q \to \mathbf{R}$ is smooth $q$-linear. For simplicity, let $q = 2$. Then

$$H(f_1(\underline{t}), f_1(\underline{t})) - H(f_2(\underline{t}), f_2(\underline{t})) =$$

$$= H(f_1(\underline{t}), f_1(\underline{t})) - H(f_2(\underline{t}), f_1(\underline{t})) + H(f_2(\underline{t}), f_1(\underline{t})) - H(f_2(\underline{t}), f_2(\underline{t}))$$

$$= H(f_1(\underline{t}) - f_2(\underline{t}), f_1(\underline{t})) - H(f_2(\underline{t}), f_1(\underline{t}) - f_2(\underline{t})),$$

and the result follows from

**Lemma.** *Let* $H : X^q \to \mathbf{R}$ *be* $q$*-linear smooth, and let* $I \supset M^r$ *be an ideal in* $C^\infty(\mathbf{R}^n)$. *If* $k : \mathbf{R}^n \to X$ *belongs to* $I(X)$ *then, for any smooth* $\ell_i : \mathbf{R}^n \to X$ $(i = 2, \ldots, q)$,

(2.3) $$H(k(\underline{t}), \ell_2(\underline{t}), \ldots, \ell_q(\underline{t})) \in I .$$

**Proof.** Again, let $q = 2$ and write

$$\ell_2(\underline{t}) = \sum_{|\alpha| < r} \underline{t}^\alpha . x_\alpha + L(\underline{t})$$

with $L(\underline{t})$ or order $\geqq r$ . Then the function of $\underline{t}$ displayed in (2.3) can be written

$$\sum_{\alpha} \underline{t}^{\alpha}.H(k(\underline{t}), x_{\alpha}) + H(k(\underline{t}), L(\underline{t})).$$

The last term here clearly is a function of order $\geqq r$ , since L is, and so is in I. But also each $H(k(\underline{t}), x_{\alpha}) \in I$ since they are of form $\varphi \circ k$ , $\varphi \in X'$ (namely with $\varphi = H(-, x_{\alpha})$), so is in I since $k \in I(X)$. The Lemma, and thus the proposition, is proved. ◊

For $g : X \to Y$ smooth there is thus an evident way of defining $g \boxtimes W : X \boxtimes W \to Y \boxtimes W$ so as to make $- \boxtimes W$ a functor, namely composing with $g$. If $j \in X \boxtimes W$ is a W-jet represented by $f : \mathbf{R}^n \to X$, we let $(g \boxtimes W)(j)$ be the W-jet represented by $g \circ f : \mathbf{R}^n \to Y$. If $g$ is smooth linear, $g \boxtimes W$ will then be the usual map with this notation.

Our next task is to make $- \boxtimes W$ into a functor which also takes values in $\underline{F}$. Since $X \boxtimes W \simeq X^m$, $X \boxtimes W$ inherits a structure of convenient vector space from that of $X^m$. The isomorphism $X \boxtimes W \simeq X^m$ depends on a choice of basis mod I, but any other choice will define an invertible real $m \times m$ matrix, which then defines also a smooth linear isomorphism $X^m \to X^m$, so the convenient vector space structure on $X \boxtimes W$ is well defined.

***Proposition 2.2.*** *For $g : X \to Y$ smooth, the map $g \boxtimes W : X \boxtimes W \to Y \boxtimes W$ is smooth.*

**Proof.** We first do the special case where $I = M^r \subset C^{\infty}(\mathbf{R}^n)$. As basis mod I, we may choose all monomials in $t_1, \dots, t_n$ of degree $< r$. The statement is then just the fact that, for $g$ fixed, the $r$ degree partial derivatives $\partial^{\alpha}(g \circ f)/\partial t^{\alpha}(0)$ depend in a smooth (in fact polynomial) way on the partial derivatives $\partial^{\alpha} f / \partial t^{\alpha}(0)$ ("higher order chain rule"). Since I could not find a reference*, not even an exact statement, of this "evident" fact, I shall be more explicit. Write $g$ in the form

$$\sum^{r-1} h_q + G$$

with $h_q : X \to Y$ smooth homogeneous of degree $q$ and G of order $\geqq r$ . It suffices to prove the result for each $h_q$ separately, and for G. Now, since a jet is represented by a function $f : \mathbf{R} \to X$ or order $\geqq 1$, $G \circ f$ has order $\geqq r$ , so its partial derivatives of order $< r$ vanish, so depend smoothly on those of $f$. Now consider $h_q$ . Write $h_q(x) = H(x, \dots, x)$ where $H : X^q \to Y$ is smooth symmetric $q$-linear (Theorem 1.2). Since the partial derivatives of any $k : \mathbf{R}^n \to Z$ can be obtained from the $D^q k$ 's, by evaluation at the canonical basis vectors in $\mathbf{R}^n$, the result

*ADDED IN PROOF. I thank the referee for providing the following two references : A. Bastiani, Applications différentiables et variétés différentiables de dimension infinie, J. Analyse Math. Jérusalem XIII (1964), 2-113 ; and P. Ver Eecke, Fondements du Calcul Différentiel, P.U.F., Paris 1984.

can be obtained from the following Lemma (when writing $R^n$ for X, X for Y and Y for Z).

***Lemma 2.3.*** *Let* $H : Y^q \to Z$ *be symmetric smooth q-linear. Then there is a fixed formula*

$$D^p(H(f, \ldots, f)) = \Sigma H(D^{k_1} f, \ldots, D^{k_s} f)$$

*valid for all smooth* $f : X \to Y$.

**Proof** and more precise statement. Let

$$k(x) := H(f(x), \ldots, f(x)) .$$

Then $D^p k(x ; x_1, \ldots, x_p)$ equals the following finite sum (2.4), whose index set is the set of partitionings of $\underline{p} = \{1, 2, \ldots, p\}$ into $\leq q$ disjoint subsets $\pi(1), \ldots, \pi(s(\pi))$

$$(2.4) \quad \sum_{\pi} [q]_{s(\pi)} . H(D^{|\pi(1)|} f(x; x_{\pi(1)}), \ldots, D^{|\pi(s(\pi))|} f(x; x_{\pi(s(\pi))}), f(x), \ldots, f(x))$$

$(q - s(\pi)$ $f(x)$'s) ; here

$$[q]_r \quad \text{denotes} \quad q.(q-1). \ldots .(q-r+1),$$

and if $B \subset \underline{p}$ is a subset, with $b$ elements $i_1, \ldots, i_b$, then we have put

$$Df^{|B|}(x ; x_B) := D^b f(x ; x_{i_1}, \ldots, x_{i_b}).$$

This formula is easily verified by induction, and the Lemma is proved.

Now let $I \supset M^r$ be a general Weil ideal. Choosing a basis $h_1, \ldots, h_m$ mod I amounts to an **R**-linear splitting $\sigma$ of the projection

$$C^\infty(\mathbf{R}^n)/M^r \to C^\infty(\mathbf{R}^n)/I = W .$$

It induces a smooth linear splitting $X \boxtimes \sigma$ of

$$X^{m'} \simeq X \boxtimes (C^\infty(\mathbf{R}^n)/M^r) \xrightarrow{\pi_X} X \boxtimes W \simeq X^m .$$

By the well-definedness result (Proposition 2.1), for $g : X \to Y$ smooth, $g \boxtimes W$ equals the composite

$$X \boxtimes W \xrightarrow{X \boxtimes \sigma} X \boxtimes (C^\infty(\mathbf{R}^n)/M^r) \xrightarrow{g \boxtimes \ldots} Y \boxtimes (C^\infty(\mathbf{R}^n)/M^r) \xrightarrow{\pi_Y} Y \boxtimes W ,$$

where the middle map is smooth by the special case already proved. Thus, the composite is smooth.

This proves the Proposition. Thus each Weil algebra W defines an endofunctor $- \boxtimes W : \underline{F} \to \underline{F}$.

## 3. TRANSITIVITY OF PROLONGATIONS.

For any vector space X and Weil algebras $W_1$, $W_2$ we have of course

$$X \otimes (W_1 \otimes W_2) \simeq (X \otimes W_1) \otimes W_2 \tag{3.1}$$

naturally in X with respect to linear maps. Our aim in this section is to prove that for convenient vector spaces X, this isomorphism is natural in X with respect to smooth maps.

Recall that we may consider as a subring

$$R[t_1, \ldots, t_n] \subset C^\infty(\mathbf{R}^n).$$

Let $I \subset C^\infty(\mathbf{R}^n)$ be a Weil ideal representing the Weil algebra W. In the following commutative diagram with exact rows, $I^\bullet$ is defined as intersection (pullback) :

$$\begin{array}{ccccccccc}
0 & \to & I & \longrightarrow & C^\infty(\mathbf{R}^n) & \longrightarrow & C^\infty(\mathbf{R}^n)/ = W & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \alpha} & & \\
0 & \to & I^\bullet & \longrightarrow & R[t_1, \ldots, t_n] & \longrightarrow & R[t_1, \ldots, t_n]/I^\bullet & \longrightarrow & 0
\end{array} .$$

Since there is a basis mod I consisting of polynomials, it follows that

$$C^\infty(\mathbf{R}^n) = R[t_1, \ldots, t_n] + I ;$$

thus from the Noether isomorphism

$$P/P \cap I \simeq (P+I)/I ,$$

it follows that $\alpha$ is an isomorphism. More generally, if X is a convenient vector space, the subspace of $C^\infty(\mathbf{R}^n, X)$ consisting of smooth polynomial functions may be identified with $X \otimes R[t_1, \ldots, t_n]$ (Theorem 1.3). So if we denote by I(X) the subspace of functions $\mathbf{R}^n \to X$ which are $\equiv 0$ mod I, and $I^\bullet(X)$ the polynomial functions among them, we have a commutative diagram with exact rows and with the left hand square a pullback :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(X) & \longrightarrow & C^\infty(\mathbf{R}^n, X) & \longrightarrow & C^\infty(\mathbf{R}^n, X)/I(X) = X \otimes W & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^\bullet(X) = X \otimes I^\bullet & \longrightarrow & X \otimes R[t_1, \ldots, t_n] & \longrightarrow & X \otimes W & \longrightarrow & 0
\end{array}$$

Henceforth, we shall write I instead of I(X) when the context (diagram) will inform us about X.

For the proof of naturality of (3.1) with respect to smooth maps, we shall make essential use of the cartesian closedness of the category $\underline{F}$ of convenient vector spaces with smooth maps : for X, Y convenient vector spaces, the vector space $C^{\infty}(X, Y)$ of smooth maps $X \to Y$ carries a natural structure of convenient vector space making it the exponential object $Y^X$ in $\underline{F}$. In particular

$$(3.2) \qquad C^{\infty}(R^{n+m}, X) \simeq C^{\infty}(R^m, C^{\infty}(R^n, X)),$$

natural in $X \in \underline{F}$, and this will be the essence in the proof. Let $W_1$, $W_2$ be Weil algebras with presentation $C^{\infty}(R^n)/I_1$ and $C^{\infty}(R^m)/I_2$, respectively. Then $W_1 \otimes W_2$ has presentation $C^{\infty}(R^{n+m})/(I_1, I_2)$, where $(I_1, I_2)$ is the ideal generated by functions $h(\underline{s}).g(\underline{s}, \underline{t})$ with $h \in I_1$ and functions $h(\underline{s}, \underline{t}).g(\underline{t})$ with $g \in I_2$ (where $\underline{s} = (s_1, ..., s_n)$ etc.). Consider the following commutative diagram (in which the two bottom corners represent the two sides of (3.1)) :

$$(3.3) \qquad \begin{array}{ccccc} & & \cong & & \\ R[\underline{s}, \underline{t}]/(I_1', I_2')\otimes X & \xleftarrow{a\otimes X} & R[\underline{s}, \underline{t}] \otimes X & \xrightarrow{b\otimes X} & R[\underline{t}]/I_2 \otimes (R[\underline{s}]/I_1 \otimes X) \\ \simeq \downarrow & & \downarrow & & \downarrow \\ C^{\infty}(R^{n+m}, X)/(I_1, I_2) & \xleftarrow{\alpha_X} & C^{\infty}(R^{n+m}, X) & \xrightarrow{\beta_X} & C^{\infty}(R^m, C^{\infty}(R^n, X)/I_1)/I_2 \end{array}$$

Here $\alpha_X$ and $a\otimes X$ are evident, whereas $\beta_X$ utilizes (3.2) and $b\otimes X$ utilizes a mimicking of (3.2) on the level of polynomials, namely the linear isomorphism

$$R[\underline{s}, \underline{t}] \simeq R[\underline{t}] \otimes R[\underline{s}] \ ;$$

$\alpha_X$ and $\beta_X$ are surjective. The top isomorphism comes about purely algebraically by applying $- \otimes X$ to isomorphisms, well-known from algebra,

$$R[\underline{s}, \underline{t}]/(J_1, J_2) \simeq R[\underline{s}]/J_1 \otimes R[\underline{t}]/J_2 \ .$$

The maps $\alpha_X$ and $\beta_X$ are evidently natural in X with respect to smooth maps ; for the maps $a\otimes X$ and $b\otimes X$ such naturality does not make sense, since $R[\underline{s}, \underline{t}] \otimes X$ is not functorial in X with respect to smooth maps. However, this does not matter ; the smooth natural isomorphism of the two bottom corners in (3.3) now follows from a piece of diagram chasing, namely the following Lemma whose proof we leave to the reader.

***Lemma.*** *Let C, D and E be functors $\underline{A} \to \underline{B}$, and assume for each $X \in \underline{A}$ a commutative triangle*

$$\begin{array}{ccccc} & & \gamma_X & & \\ D(X) & \xleftarrow[\alpha_X]{} & C(X) & \xrightarrow[\beta_X]{} & E(X) \end{array}$$

*If all $\alpha_X$ are epic, and $\alpha$ and $\beta$ are natural in X, then so is $\gamma$ .*

We have thus proved the first statement in the following theorem (the second assertion being trivial) :

***Theorem 3.1.*** *The isomorphism (3.1) is natural with respect to smooth maps. Also* $X \otimes R \simeq X$, *naturally with respect to smooth maps.*

We end this section by remarking that the construction $X \otimes W$ is also functorial in W. A homomorphism F of Weil algebras

$$W_1 = C^\infty(R^n)/I \xrightarrow{F} C^\infty(R^m)/J = W_2$$

can be represented by a smooth map

$$\tilde{F} : R^m \to R^n \qquad \text{with} \qquad \tilde{F}(\underline{0}) = \underline{0},$$

and with $\varphi \circ \tilde{F} \in J$ whenever $\varphi \in I$. Then, for $f : R^n \to R$ representing an element $\{f\}$ of $W_1$, $f \circ \tilde{F}$ represents $F(\{f\}) \in W_2$. And if $f : R^n \to X$ represents an element of $X \otimes W_1$, $f \circ \tilde{F}$ represents $(X \otimes F)(\{f\})$.

All said, $\otimes$ defines a bifunctor

$$\underline{F} \times \underline{W} \to \underline{F} \qquad (3.4)$$

where $\underline{W}$ is the category of Weil algebras. In fact, by Theorem 3.1, the monoidal category $(\underline{W}, \otimes, R)$ acts on F in an associative unitary way (up to coherent isomorphisms). - Note that $\otimes$ is the coproduct in $\underline{W}$, R the initial object. (Actually, R is also terminal object in $\underline{W}$.)

## 4. SEMIDIRECT PRODUCT OF CATEGORIES.

Let $\underline{W}$ be any category with finite coproducts, denoted $\otimes$, and with initial object denoted R, and let $\underline{G}$ be a category on which $\underline{W}$ acts (from the right, say), i.e., there is given a functor $\otimes : \underline{G} \times \underline{W} \to \underline{G}$, and there are given natural isomorphisms (for $X \in \underline{G}$, $W_i \in \underline{W}$) :

$$(X \otimes W_1) \otimes W_2 \simeq X \otimes (W_1 \otimes W_2), \qquad X \simeq X \otimes R$$

which fit coherently with the associativity - and unit - isomorphisms of the monoidal category $(\underline{W}, \otimes, R)$.

We construct a new category $\underline{G} \ltimes \underline{W}$ as follows : the objects are pairs (X, W) with $X \in \underline{G}$, $W \in \underline{W}$. An arrow $(X_1, W_1) \to (X_2, W_2)$ is a pair of arrows in $\underline{G}$ and $\underline{W}$,

$$(X_1 \xrightarrow{f} X_2 \otimes W_1, \quad W_2 \xrightarrow{\varphi} W_1), \qquad (4.1)$$

and the composite of this pair with

$$(X_2 \xrightarrow{g} X_3 \otimes W_2, \quad W_3 \xrightarrow{\gamma} W_2)$$

is the pair (associativity isomorphisms omitted, by coherence) :

$$(X \xrightarrow{f} X_2 \otimes W_1 \xrightarrow{g \otimes W_1} X_3 \otimes W_2 \otimes W_1 \xrightarrow{X_3 \otimes (\varphi, id)} X_3 \otimes W_1 \ , \ W_3 \xrightarrow{\varphi \circ \gamma} W_1 )..$$

Identity arrow is

$$(X \simeq X \otimes R \xrightarrow{X \otimes i} X \otimes W, \ id_W) .$$

There is a full embedding $j : \underline{G} \to \underline{G} \times \underline{W}$ given by $X \mapsto (X, R)$ and

$$(X_1 \xrightarrow{f} X_2) \mapsto (X_1 \xrightarrow{f} X_2 \simeq X_2 \otimes R, \ id_R ) .$$

**Proposition 4.1.** *The inclusion $j : \underline{G} \to \underline{G} \times \underline{W}$ preserves all those inverse limits which are preserved by all $- \otimes W$.*

**Proof.** We prove the case of binary products only (which is all we need for what follows). We have in fact more generally

$$(4.2) \qquad (Z_1, W_1) \times (Z_2, W_2) \simeq (Z_1 \times Z_2, W_1 \otimes W_2)$$

due to the string of conversions

$$\frac{\dfrac{\dfrac{(Y, W) \longrightarrow (Z_1 \times Z_2, W_1 \otimes W_2)}{Y \to (Z_1 \times Z_2) \otimes W = (Z_1 \otimes W) \times (Z_2 \otimes W), \quad W_1 \otimes W_2 \to W}}{(Y \to Z_i \otimes W, \ W_i \to W)_{i=1,2}}}{((Y, W) \to (Z_i, W_i))_{i=1,2}}$$

**Proposition 4.2.** *If $\underline{G}$ has exponential objects $Y^X$ which are preserved by each $- \otimes W$ in the sense $Y^X \otimes W \simeq (Y \otimes W)^X$ and if each $- \otimes W$ preserves finite products, then $j$ preserves exponential objects.*

**Proof.** We have bijective correspondences

$$\frac{\dfrac{\dfrac{\dfrac{(Z, W) \to (Y^X, R)}{Z \to Y^X \otimes W = (Y \otimes W)^X}}{Z \times X \to Y \otimes W}}{(Z \times X, W) \to (Y, R)}}{(Z, W) \times (X, R) \to (Y, R)}$$

where we for the last conversion utilized (4.2), which we may by the second assumption made.

If the initial object $\mathbf{R}$ of $\underline{W}$ is also terminal, we have a canonical functor $\pi : \underline{G} \ltimes \underline{W} \to \underline{G}$, given on objects by $\pi(X, W) = X$ and with $\pi$ applied to the arrow (4.1) given as

$$X_1 \longrightarrow X_2 \otimes W \xrightarrow{X_2 \otimes !} X_2 \otimes \mathbf{R} \simeq X_2 \ .$$

Clearly $\pi \circ j = \mathrm{id}_{\underline{G}}$ , and there is a natural map making $j(\pi(X, W))$ a retract of $(X, W)$. (In fact, if each $- \otimes W$ preserves finite products, it follows from (4.2) that

(4.3) $$(Z, W) \simeq (Z, \mathbf{R}) \times (\mathbf{1}, W),$$

and $(\mathbf{1}, W)$ is an object in $\underline{G} \ltimes \underline{R}$ which has a unique point (= map from the terminal object).)

## 5. THE EMBEDDING.

We consider now the category $\underline{F}$, with the "action" $\otimes$ of $\underline{W}$, the category of Weil algebras, as described in §2 and §3, and we form $\underline{F} \ltimes \underline{W}$. The full subcategory $\underline{f} \subset \underline{F}$ of finite dimensional vector spaces is stable under the action, so that we get $\underline{f} \ltimes \underline{W}$ as a full subcategory of $\underline{F} \ltimes \underline{W}$.

We describe (essentially following [1]) a Grothendieck topology on $f \ltimes W$ which will make it a site of definition for the Cahiers topos [1]. We declare the following families to be covering :

(5.1) $$(X_i, W) \xrightarrow{a_i = (f_i, id)} (X, W)\ , \qquad i \in I$$

if $\pi(a_i) : X_i \to X$ form an open covering.

Let $i$ and $j$ denote the following full inclusions

$$\underline{f} \ltimes \underline{W} \xhookrightarrow{i} \underline{F} \ltimes W \xhookleftarrow{j} \underline{F}$$

Any $Y \in \underline{F}$ defines a functor $J(Y) : (\underline{f} \ltimes \underline{W})^{op} \to \underline{\text{Sets}}$, namely

$$J(Y) = \hom_{\underline{F} \ltimes \underline{W}} (i(-), j(Y)).$$

So $J(Y)$ is "representable from the outside". We may omit $i$ and $j$ from notation.

***Proposition 5.1.*** *$J(Y)$ is a sheaf.*

**Proof.** Let $\{a_i\}$ be a covering, as in (5.1), in $\underline{f} \ltimes \underline{W}$, and let

$$b_i : (X_i, W) \to Y$$

be a compatible family ($Y \in \underline{F}$). We should construct a map

$$c : (X, W) \to Y \qquad \text{with} \qquad c \circ a_i = b_i \quad \forall i.$$

The data of the $b_i$ 's amount to $\overline{b}_i : X \to Y \otimes W$ and the compatibility condition for the $b_i$ 's implies one for the $\overline{b}_i$'s. The required map c amounts to a map $\overline{c} : X \to Y \otimes W$. Also $\pi(a_i) : X_i \to X$ form an open covering. So the crux is to observe that any convenient vector space Z (in our case $Z = Y \otimes W$) represents (from the outside) a sheaf on the site $\underline{f}$ (with open coverings as its topology). This follows from concreteness of the categories $\underline{f}$ and $\underline{F}$, and the fact that smoothness of a set theoretic map $X \to Y$ between convenient vector spaces may be tested by smooth plots on an open covering of X and with finite dimensional domains.

We leave the full details to the reader. At this point, it would have been an advantage to consider the categories $\underline{\tilde{f}}$ and $\underline{\tilde{F}}$ consisting of *open subsets* of finite dimensional, resp. convenient vector spaces, with $\underline{W}$ acting on them (which it does by the same construction as the one of §2.3) because the open coverings in $\underline{\tilde{f}}$ and $\underline{\tilde{F}}$ admit pullbacks which are furthermore preserved by $-\otimes W$.

We can now state our main theorem ; $\underline{C}$ denotes the Cahiers topos (= sheaves on $\underline{f} \ltimes \underline{W}$) :

***Theorem 5.2.*** *The functor* $J : \underline{F} \to \underline{C}$ *is full and faithful. It preserves finite products, and it preserves exponentials* $Y^X$ *provided* X *is finite dimensional.*

**Remark.** By the remarks just before the statement of the theorem it follows that the embedding J may be extended to the category $\underline{\tilde{F}}$ of open subsets of convenient vector spaces, and their smooth maps, and thus possibly also to some category of "manifolds modelled on convenient vector spaces".

**Proof.** When J is composed with the global-sections functor $\Gamma$ : $\underline{C} \to \underline{\text{Sets}}$, we get the faithful underlying-set functor $|.| : \underline{F} \to \underline{\text{Sets}}$, so J is faithful. To test fulness, let $f : J(X) \to J(Y)$ be a map in $\underline{C}$. We get a set theoretic map $|f| : X \to Y$, which we have to test is smooth. But again, smoothness may be tested by checking with smooth plots $c : \mathbf{R}^n \to X$ (in fact $n = 1$ suffices), and since

$$\mathbf{R}^n \in \underline{f} \subset \underline{f} \ltimes \underline{W},$$

smoothness of $|f|$ follows. To see $J(|f|) = f$, just apply the faithful $|.|$.

Next we argue that J preserves finite products. It is clear from the construction that $-\otimes W : \underline{F} \to \underline{F}$ preserves finite products for each $W \in \underline{W}$. Hence, by Proposition 4.1, $j : \underline{F} \to \underline{F} \ltimes \underline{W}$ preserves finite products, and hence so does J, for standard categorical reasons (essentially, "Yoneda embedding preserves limits").

Finally, to argue for exponentials, we note that the functors $-\otimes W : \underline{F} \to \underline{F}$ satisfy

$$Y^X \boxtimes W \simeq (Y \boxtimes W)^X .$$

In fact, if W is $m$-dimensional as a vector space, both sides are isomorphic, by smooth linear isomorphisms, to

$$(Y^X)^m \simeq (Y^m)^X .$$

This isomorphism is in fact natural with respect to smooth maps, because if $h_1, ..., h_m \in C^\infty(\mathbf{R}^n)$ is a basis mod I, an element of $Y^X \boxtimes W$ has a unique representative of form

$$\underline{t} \mapsto \sum^m h_j(\underline{t}) . \xi_j \quad (\xi_j \in Y^X),$$

and under the isomorphism, this element goes to

$$x \mapsto [\underline{t} \mapsto \Sigma h_j(\underline{t}) . \xi_j (x) ],$$

the square bracket here representing an element of $Y \boxtimes W$. The passage thus described is clearly natural. So $-\boxtimes W$ satisfies the conditions of Proposition 4.2, so that $j : \underline{F} \to \underline{F} \ltimes \underline{W}$ preserves exponentiation. The rest of the argument is now purely categorical ; let $A \in \underline{f} \ltimes \underline{W}$, and let $\overline{A}$ be the object of $\underline{C}$ which it represents. For $X \in \underline{f}$ and $Y \in \underline{F}$, we then have

$$\hom_{\underline{C}} (\overline{A}, J(Y^X)) = \hom_{\underline{F} \ltimes \underline{W}} (A, j(Y^X)) = \hom_{\underline{F} \ltimes \underline{W}} (A, j(Y)^{j(X)})$$
$$= \hom_{\underline{F} \ltimes \underline{W}} (A \times j(X), j(Y)) = \hom_{\underline{C}} (\overline{A} \times J(X), J(Y)),$$

the last equality provided $A \times j(X) \in \underline{f} \ltimes \underline{W}$, which will be the case since $X \in \underline{f}$. The theorem is proved.

## 6. RETROSPECT.

Having Theorem 5.2, as well as the full power of synthetic reasoning in $\underline{C}$, many of the constructions and comparisons that we worked hard to get, become very transparent. For a Weil algebra W, let $\overline{W}$ denote the ("infinitesimal") object in $\underline{C}$ which it represents. Then $\underline{F} \times \underline{W}$ becomes the full subcategory of $\underline{C}$ of objects of form $J(X) \times \overline{W}$ ($X \in \underline{F}$, $W \in \underline{W}$), this being identified with $(X, W) \in \underline{F} \ltimes \underline{W}$. A W-jet into X becomes simply a map $\overline{W} \to J(X)$, explaining the functoriality of the jet notion. Also, $X \boxtimes W$ goes by J to $J(X)^{\overline{W}}$, explaining the properties of the functor $-\boxtimes W$, e.g. the transitivity

$$(X \boxtimes W_1) \boxtimes W_2 \simeq X \boxtimes (W_1 \boxtimes W_2)$$

is simply the categorical law $(A^B)^C \simeq A^{B \times C}$ .

Let us finally remark that each J(X) evidently will be an R-module object (R = J(**R**)), and that it will satisfy the "vector form of Axiom $1^W$" (cf. [4]), in the sense that, if $m$ is the linear dimension of W, we have an isomorphism $J(X)^m \to J(X)^W$ constructed out of a linear basis $h_1, \ldots, h_m$ for $\mathbf{R}[t_1, \ldots, t_n]$ mod I (where $W = R[\underline{t}]/I$) as the map with synthetic description

$$(6.1) \qquad (x_1, \ldots, x_m) \mapsto [(t_1, \ldots, t_n)] \mapsto \Sigma h_i(\underline{t}).x_i$$

($\bar{W}$ being identified with a sub"set" of $R^n$, namely the "zero-set of I "). This follows essentially from the fact that in $\underline{F}$ we have an isomorphism $X^m \simeq X \otimes W$ given by the same formula (6.1).

From the validity of Axiom $1^W$ for J(X) it follows, in turn, that J(X) is infinitesimally linear in the strong (Bergeron-) sense, cf. [6] ; the argument is as in [6], Proposition 1.2, with R replaced by J(X).

## REFERENCES.


1. E. DUBUC, Sur les modèles de la géométrie différentielle synthétique, Cahiers Top. et Géom. Diff. **XX**-3 (1979), 231-279.
2. A. FRÖLICHER, Smooth structures, Lecture Notes in Math. **962**, Springer (1982).
3. A. FRÖLICHER, B. GISIN & A. KRIEGL, General differentiation theory, in Category Theoretic Methods in Geometry, Aarhus Var. Publ. Series **35** (1983).
4. A. KOCK, Synthetic differential geometry, London Math. Soc. Lecture Notes Series **51**, Cambridge Univ. Press, 1981.
5. A. KOCK, Calculus of smooth functions between convenient vector spaces, Aarhus Preprint Series **18** (1984-85).
6. A. KOCK & R. LAVENDHOMME, Strong infinitesimal linearity, with applications to strong difference and affine connections, Cahiers Top. et Géom. Diff. **XXV**-3 (1984), 311-324.
7. A. KRIEGL, Die richtigen Räume für Analysis im Unendlichdimensionalen, Monatsh. f. Math. **94** (1982), 109-124.
8. A. KRIEGL, Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokallonvexen Vektorräumen, Monatsh. f. Math. **95** (1983), p. 287-309.
9. P. MICHOR, A convenient setting for differential geometry and global analysis, Cahiers Top. et Géom. Diff. **XXV**-1 (1984), 63-109.
10. A. WEIL, Théorie des points proches sur les variétés différentiables, Coll. Top. et Géom. Diff., Strasbourg, 1953.


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