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A GENERALIZED GLOBAL DIFFERENTIAL CALCULUS. II.
APPLICATION TO INVARIANCE UNDER A LIE GROUP
by Joseph JOHNSON

INTRODUCTION.

In the introduction to Part I of this paper (cf. [13], preceding issue of "Cahiers") the subject matter of this concluding statement was briefly explained. Here, for the reader's convenience, a second summary of these concluding pages of the paper is given.

The basic notion used here is that of a "universe group", an object $U$ of $K$ with a given factorization of $(U, ) : K \rightarrow$ sets through the category of groups (cf. §12). We then define what it means for a universe group $U$ to "act" on $C \in K$. Then if $c \in C$ and $U$ acts on $C$, we...
define what it means for \( c \) to be "invariant" (cf. Definition following (12.0.2)) and "locally invariant" (cf. Definition following (12.5)).

The essence of what is proved in the paper can now be stated, confining ourselves, for simplicity, to the case where \( U = A_\alpha, G \) a Lie group. Our first main result is (12.6), which shows the relationship between the notions "invariant" and "locally invariant". Next, we have 1) of (13.1), which says that \( c \) is locally invariant if and only if \( c \) is "infinitesimally invariant". Finally, we have 2) of (13.1) which says that the Lie bracket of the Lie algebra \( \mathfrak{g} \) of \( G \) is compatible with the Lie bracket of derivations associated to elements of \( \mathfrak{g} \).

12. ACTIONS OF A GROUP UNIVERSE ON A UNIVERSE.

In these final sections we shall work almost exclusively with \( K. "Universe" \) will mean "object of \( K \)", and all limits and colimits of universes will be taken in \( K \) except when stated otherwise. The theory will be valid for all senses of the word "admissible", even though we may refer for motivation to the differentiable case. (For the complex-analytic theory, "Lie group" will mean complex-analytic Lie group.)

A universe group consists of a universe \( U \), a functor \( F : K \to \text{groups} \) and an isomorphism \( (U, \cdot) \cong |F| \) (where \(|| \) denotes underlying set). It will usually be convenient to simply say that "\( U \) is a universe group". For instance, if \( S \) is any set, \( A_\alpha(S) \) is a universe group, since it represents the functor

\[
V \mapsto (S, V) = \text{a group},
\]

as

\[
V = \{v \in V \mid \theta v = 0\}
\]

is a group under \( + \). If \( U \) is a universe group, we have a natural transformation \( \langle U, \cdot \rangle \times \langle U, \cdot \rangle \to \langle U, \cdot \rangle \) given by the multiplication, hence \( \mu : U \to U \cup U \). Also we have \( l_U : U \to A_\alpha \) such that for each \( V \in K \),

\[
(l_U, V) : (A_\alpha, V) \to (U, V)
\]

sends the unique element of \( (A_\alpha, V) \) to the identity element of the group \( (U, V) \). We also have \( r : U \to U \) such that \( (r, V) \) is the map of \( (U, V) \) into itself that sends each \( g \in (U, V) \) to \( g^{-1} \). Conversely, if

\[
\mu : U \to U \cup U, \quad l_U : U \to A_\alpha, \quad r : U \to U
\]
are given a priori so as to satisfy (**)  

\[(U\mu_\mu)\mu = (\mu U)\mu, \quad (U\mu_\mu)\mu = U_{1\gamma} U, \quad (r, U)\mu = (A_0 U)1_{1\gamma},\]

then it is routine to show U has a unique structure of universe group that returns \(\mu, 1_{1\gamma}, r\) in the manner that was just indicated.

In what follows, G will denote a fixed (admissible) Lie group, \(e\) the identity of G. Then multiplication gives us

\[\mu : A_0 \rightarrow A_0 = A_0 \cup A_0\]

(cf. (7.9)). We also have \(e : A_0 \rightarrow A_0 \quad (e^*(f) = f(e))\) and \(r : A_0 \rightarrow A_0\), given by the inverse map of G. The required identities hold, so \(A_0\) is a universe group.

Let U be a universe group, C a universe, \(\gamma : C \rightarrow UHC\). We shall say that \(\gamma\) is an action of U on C - even though "co-action" would be more standard terminology - provided the identities (**) hold.

\[(12.0.1) \quad (UHC)\gamma = (UH\gamma)\gamma,\]
\[(12.0.2) \quad (1UHC)\gamma = C_{A_0} UHC\]

hold. Of course if G acts on the manifold X so that

\[g(hx) = (gh)x, \quad g, h \in G, \quad x \in X\text{ and } ex = x,\]

\(A_x\) is canonically acted upon by \(A_0\). If \(f \in A_x, g \in G, x \in X\), we have

\[(\gamma f)(g, x) = f(gx).\]

One always has the canonical map \(C_{\gamma} HC\) whatever U and C. It is an action of U on C that we shall refer to as the trivial action of U on C. We shall denote it by \(\alpha\) or \(\alpha_{\gamma}, c\). If \(\gamma\) is any action of U on C, \(c \in C\) will be called an invariant of \(\gamma\) if \(\gamma c = \gamma c\). The main purpose of these final sections is to prove (12.6) and (13.1). The reader should note (12.9) which shows how one can put an admissible structure on the orbit space of a group action.

\[\text{---}\]

\[\text{(*) The reader should draw the appropriate commutative diagrams.}\]

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It helps to develop some notation that is oriented toward a more geometric point of view. Let \( C \subseteq K \) be given. If \( U \subseteq K \), define \( S_U C = (C, U) \) (so that \( SC = S_{A_e} C \)). If \( x \in S_U C \), define \( c'(x) = x(c) \) for all \( c \in C \). This definition extends one of our previous uses of """. Let \( U \) be a universe group that acts on \( C \). For any \( V \subseteq K \), we can define a "multiplication"

\[
S_U C \times S_U C \rightarrow S_U C \quad \text{by} \quad gx = (g, x)_V \quad \text{if} \quad g \in S_U U, \ x \in S_U C.
\]

If \( G \) acts on the manifold \( X \), \( g \in G, \ x \in X \), then \( g'x^* = (gx)^* \) since if \( c \in A_e \),

\[
(g'x^*)^*(c) = (g'^*(x^*))^*(yc) = (g, x)^*(yc) = (yc)(g, x) = c(gx) = (gx)^*(c).
\]

If \( g, h \in S_U U \), define

\[
gh \in S_U U \quad \text{by} \quad gh = (g, h)_U.
\]

Of course if \( g, h \in G \), \( gh^* = (gh)^* \). If \( A_e \) acts on \( C \), \( g \in G, \ x \in S_U C \), we shall define

\[
 gx = g, x \in S_U C \quad \text{where} \quad gv = (A_e V g)^*.
\]

We shall use this notation to reformulate (12.0.1) and (12.0.2) (cf. (12.1)).

In what follows, \( U \) will denote a universe group that acts on a universe \( C \). In some instances it will be necessary for \( U \) to be more specifically \( A_e \) itself. For the time being it will generally be considered that the presence of \( G \) within a given paragraph will suffice to indicate when that is so. In all instances \( C \) remains a general object of \( K \), and it is mainly this generality of \( C \) that justifies the often meticulous attention to detail that follows.

The identification \( A_e W C = C \), though natural, seems to generate confusion and must be resisted for purposes of exposition. When \( x \in SW \) and the context selects an element \( h \) of \( (W, Z)_K \) in some unambiguous way, we shall let \( x = h \cdot x \). For instance, if \( W = A_e \) (\( x \in SW \)), the meaning of \( x \) is clear since \( #(A_e, Z) = 1 \).

Let \( e = 1_u \).

**Lemma** (12.1). Let \( V \subseteq K \).

1) \( (gh)x = g(hx) \) if \( g, h \in S_V U, \ x \in S_V C \).
2) \( ex = x \) if \( x \in S_V C \).

Indeed,
(gh)x = (gh,x)_{\Psi} = ((g,h)_{\mu}, x)_{\Psi} = (g,h,x)(\mu_{\Psi}C)_{\Psi} = (g,h,x)(\Psi^U)_{\Psi} = (g, (h,x)_{\Psi})_{\Psi} = (g,hx)_{\Psi} = g(hx).

Also

ex = (e_\psi,x)_{\Psi} = (A_0^{\psi V}, x)(e_{\Psi}C_{\Psi})_{\Psi} = (A_0^{\psi V}, x)(C^A_{\Psi}A_0C_{\Psi})_{\Psi} = x.

**COROLLARY (12.1.1).** \((gh)x = g(hx)\) if \(g, h \in G, x \in C\). Also

\[ ex = x. \]

If \(g \in ScU\) and \(c \in C\), define \(cg = (g, C)_{\Psi}c \in C\). We note that if \(x\) is in \(ScC\), then \(xg = xg \in ScU\), so \((xg)x \in ScC\) is defined.

**LEMMA (12.2).** \((cg)^{-1}(x) = c^{-1}(xg)\) if \(c \in C, x \in ScC, g \in ScU\).

Indeed,

\((cg)^{-1}(x) = x(cg) = x((g,C)_{\Psi}c) = (xg,x)_{\Psi}c = [xg]_{\Psi}(c) = c^{-1}(xg)x.\)

We shall often write the identity of (12.2) as

\((cg)^{-1}(x) = c^{-1}(gx), \quad gx = xg.\)

In particular, when \(g \in C\), let \(cg = cg^c\). Then we obtain

\[(12.2.1) \quad (cg)^{-1}(x) = c^{-1}(gx), \quad g \in G, c \in C, x \in ScC \]

since \(xg = gx^c = g^c = g^c x.\)

**LEMMA (12.3).** \((cg)h = c(gh)\) if \(c \in C, g, h \in G\). Also \(ce = c.\)

Let \(x = id_c\). Then

\[ (cg)h = x((cg)h) = (c(g)h)^{-1}(x) = (cg)^{-1}(hx) = c^{-1}(ghx) = (c(gh))^{-1}(x) = c(gh). \]

Also

\[ ce = x(ce) = (ce)^{-1}(x) = c^{-1}(ex) = c^{-1}(x) = x(c) = c. \]

Given \(g \in G\), define \(\rho_g c = cg\) for \(c \in C\). Then \(\rho_g = (gC,C)_{\Psi}\) is in \((C,C)\). We have

\[ \rho_g \rho_g = \rho_g, \quad g, h \in G, \quad \rho_g = Id_c. \]
Lemma (12.4) below will allow us to "move around" on SU and SC. We need some preliminary definitions. If \( g \in SU \), let \( \lambda_g : U \to U \) be defined by \( \lambda_g = (U, g_U) \mu \). A proof entirely analogous to that of (12.3) yields

\[(12.3.1) \quad g(hu) = (gh)u \quad \text{if} \quad g, h \in SU, \ u \in U.\]

We shall consider \( \lambda_g \) for \( g \in SU \) as defining

\[\lambda_g \mid C : U \lambda C \to U \lambda C,\]

and write \( \lambda_g \mid C \) also as \( \lambda_g \). Similarly we have

\[\rho_g = U \lambda \rho_g : U \lambda C \to U \lambda C.\]

For the following lemma, one needs to refer to the remark preceding (6.8).

**Lemma (12.4).** Let \( g, h \in SU, \ x \in SC \). Then:

1) \( \lambda_g \psi = \rho_g \psi : C \to U \lambda \lambda C \);
2) \( x^{-1} \rho_g = (\rho_g)_x(gx)^{-1} \);
3) \( h^{-1} \lambda_g = (\lambda_g)_h(hg)^{-1} \).

We get 2 from

\[x \rho_g = x(g, C) \psi = (xg, x) \psi = (g, x) \psi = gx\]

and 3 from

\[h \lambda_g = h(U, g_U) \mu = (h, g) \mu = hg.\]

For 1 we have

\[\lambda_g \lambda C \psi = ((U, g_U) \mu) \lambda C \psi = (U, g_U) \psi = (U, \lambda g, C) \psi = (U, \lambda \lambda g, C)(U \lambda \lambda C) \psi = (U, (g_U, C)(U \lambda \lambda C)) \psi = (U \lambda \lambda \rho_g) \psi.\]

If \( f \in C_\psi \), define \( fg \in C \) if \( g \in SU \) by \( fg = (\rho_g)_xf \). Similarly, if \( g, h \in SU, \ u \in U \lambda \rho_g, \) define \( gu = (\lambda_g)_h u \in U \rho_g \).

**Corollary (12.4.1).** Let \( c \in C, \ u \in U, \ g, h \in SU, \ x \in SC \). Then:

1) \( g \psi(c) = \psi(c)g \);
2) \( (cg)_x = (c \rho_g)g \);
3) \( (gu)_h = gu \rho_g \).
The following will turn out to be an important application of (12.4). Recall \( o = \text{CtAeW} \) is the trivial action of \( \text{Ae} \) on \( \text{C} \).

**Lemma (12.5).** Let \( c \in \text{C} \), \( x \in X \), \( g \in G \). The following are equivalent:

1) \( (\psi c)(g,x) = (\psi (cg))(g,x) \);
2) \( (\psi c)(e,gx) = (\psi c)(e,gx) \).

By using successively 1, 3 of (12.4) on \( \text{AeW} \) and 2 of (12.4) we get

\[
(\psi c)(g,x) = (g^{-1}(\psi c)g)(g,x) = g^{-1}((\psi c)(g,x)) = g^{-1}((\psi c)(e,gx)) g.
\]

Similarly we get

\[
(\psi (cg))(g,x) = (g^{-1}(\psi (cg))g)(g,x) = g^{-1}((\psi (cg))(g,x)) = g^{-1}((\psi c)(e,gx)) g.
\]

Therefore clearly 2 \( \Rightarrow \) 1, and conversely, since \( \lambda_{g^{-1}} \) and \( \rho_g \) are isomorphisms.

If \( c \in \text{C} \) and \( (\psi c)(e,x) = (\psi c)(e,x) \) for every \( x \in X \), we shall say that \( c \) is locally invariant. Obviously \( c \) invariant \( (\psi c = \psi c) \) implies \( c \) is locally invariant. In the case of an action of \( G \) on a manifold, \( c \) locally invariant means \( c \) has a constant value on \( U \cap \text{dom} \ c \) for every orbit \( U \subset X \) of the action of \( G \) on \( X \). If \( c \) is locally invariant, if \( G \) is connected and if furthermore for each orbit \( U \),

\[
U \cap \text{dom} \ c \neq \emptyset \Rightarrow U \subset \text{dom} \ c,
\]

then \( c \) is invariant. Thus, for connected \( G \), we can say that locally invariant with "invariant domain" implies invariant. We now see that this is true in general.

**Theorem (12.6).** Let \( c \in \text{C} \) be locally invariant with \( 0c \) invariant, and assume that \( G \) is connected. Then \( c \) is invariant.

From (12.5) and local invariance, \( (\psi c)(g,x) = (\psi (cg))(g,x) \) for all \( g \in G, x \in X \). Therefore

\[
(\psi c)(A_e)_{\text{C}G} = c_{\text{g}}(A_e)_{\text{C}G}, g \in G.
\]

Since \( y(0c) = \psi (0c), (g^{-1}, x)(0c) = x(0c), g \in G \). Thus \( (\psi c)(g,x) \) not a phantom means \( c \) not a phantom. Let us identify \( G \times X \) with \( S(A_eW) \) for the moment. Then given \( x \in X \), we have
Let us now use the general result (12.6.1) below with $U = A_\delta$, $w = \psi c$. We obtain $\psi c = \psi c'$ for some $c' \in C$. Then

\[
(g, x) \in \text{dom } (\psi c) \Rightarrow (g', x) \in \text{dom } (\psi c'), \quad g, g' \in G.
\]

Let us now use the general result (12.6.1) below with $U = A_\delta$, $w = \psi c$. We obtain $\psi c = \psi c'$ for some $c' \in C$. Then

\[
(g, x) \in \text{dom } (\psi c) \Rightarrow (g', x) \in \text{dom } (\psi c'), \quad g, g' \in G.
\]

so $\psi c = \psi c$. Thus we only need to show the following.

**Lemma (12.6.1).** Let $U, C \in K$ with $M = SU$ connected. Let $w \in U^*L^*C$, and assume that $w$ satisfies the following conditions:

1) $m \in M \Rightarrow w(m, ) = (m, )w$, $w(m, ) = (m, C)w$.

2) $(mx) \in \text{dom } w^* \Rightarrow Mx(x) \in C \in \text{dom } w^*, m \in M, x \in SC$.

Then $w \in \text{im } w_c$.

This lemma is vacuous when $M$ is, so we shall assume $M \neq \emptyset$.

Let $m_0 \in M$ and define $c = w(m_0, )$. We shall see that $w = \psi c$, where $\psi = \psi .c$.

Let $X = SC$, $x \in X$. For $f \in U^*L^*C$, let $f_x = fU^*L^*C_x$. Assume that (12.6.1) is valid when $C$ is local.

If $m \in M$,

\[
w_x = \psi (m, ) = w_x = \psi U^*L^*C_x, \quad \text{and} \quad w_x = \psi U^*L^*C_x.
\]

Also $\text{dom } (w_x)^* = M$ or it is empty, so 1 and 2 of (12.6.1) hold for $w_x$. The special case of (12.6.1) where $C$ is local implies that $w_x \in \text{im } w_x$, so

\[
w_x = \psi x (m, ) = \psi x C_x = (\psi x C)_x
\]

This being so for all $x \in X$, we shall have $w = \psi c$.

We can now assume that $C$ is local in completing the proof of (12.6.1). From 2, $0w = 0$ or $0w = 0\psi c$, and so we can assume $0w = 0$.

We need to show that $w = \psi c$, i.e., that $w_x = (\psi c)_x$ for every $m \in M$.

Let

\[
Z = \{m \in M \mid w_x = (\psi c)_x\}.
\]

Clearly (by 1) and the definition of $c$, $m_0 \in Z$, and $Z$ is open. To prove (12.6.1), we only need to show that $Z$ is closed.

Let $z \in \text{cl } Z$. We have $w_x = (\psi (w(z, )_x))_x$. Take $p \in 0U$ such that

\[
p^*(z) = 0 \quad \text{and} \quad w + p = \psi (w(z, ) + p
\]

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(which is possible since $\partial w = 0 \Rightarrow w_z \neq \emptyset_z$). Choose $m \in Z$ such that $p^*(m) = 0$. Then

$$\langle \Pi c \rangle_x = w_x = [\Pi (w(z, \cdot))]._x.$$  

As $o_m: C \to U_m \Pi C$ is an injection, we see that $w(z, \cdot) = c$. Therefore

$$w_x = (\Pi (w(z, \cdot))_x = (\Pi c)_x,$$

so $z \in Z$ and $Z$ is closed.

Let $\Lambda_e = (A_e)_e$ (= the localization of $A_e$ at $e$), and define $\psi: C \to \Lambda_e \Pi C$ to be $(e^{-\Pi e})\psi$. Let $\mu_e: L \to L \Pi L$ be defined by $\mu_e = (e^{-\Pi e})\mu$. This defines a "local action" of $G$ on $C$ in the sense of the following definition.

**Definition (12.7).** A morphism $\psi: C \to \Lambda_e \Pi C$ is a local action of $G$ on $C$ if

$$(\Lambda_e \Pi \psi) \psi = (\mu_e \Pi \psi) \quad \text{and} \quad (e^{-\Pi e}) \psi = C \Pi A_e \Pi C.$$  

We shall call $\psi$, as defined above, the local action associated to $\psi$. In (12.6) we have seen that when $G$ is connected, $c \in C$ is invariant iff besides $\psi c = e_{\Pi e} c$ we have also that

$$\text{dom } c^- = \{x \in X \mid x(c) \neq \emptyset_0\}$$

is an invariant set under the action of $G$. Thus the question of the invariance of $c$ decomposes into an analogous question about $\psi$ and a purely set-theoretic one about the action of $G$ on $X$. In what follows, we shall only be looking at local actions $\psi: C \to \Lambda_e \Pi C$. Of course $\Lambda_e$ is a universe group.

The Lie algebra $\mathfrak{g}$ of $G$ is commonly defined as the set of left invariant vector fields on $G$. We shall find that because we have generalized left actions of $G$ on a manifold, it will be convenient to define $\mathfrak{g}$ to be the set of right invariant vector fields on $G$ with Lie algebra "bracket" given by $[\Theta, \chi] = \Theta \chi - \chi \Theta$ for $\Theta, \chi \in \mathfrak{g}$. Thus

$$\mathfrak{g} \subseteq \operatorname{Ader} A_e, \quad [\Theta, \chi] f = \Theta (\chi f) - \chi (\Theta f) \quad \text{if } f \in A_e.$$
Since any element $\theta$ of $g$ is uniquely determined by its value $\theta_e$ at $e$, where $\theta_e f = (\theta f)(e)$, $f \in A_\theta$, it follows also that $g \in A \operatorname{der} L_\theta$. In the sequel we shall tend to consider that $g \in A \operatorname{der} L_\theta$.

We shall need to know, in terms convenient for our theory, how to formulate the idea that a vector field $\theta$ is right invariant. Let $f \in A_\theta$, $h \in G$, with $f(h) \neq 0_e$. Let $g(t)$ be any differentiable curve in $G$ such that $g(0) = e$, $g'(0) = \theta_e$. Then

$$(d/dt){}_{|t=0} f(g(t)h) = (d/dt){}_{|t=0} (\mu f)(g(t)h) = [(\Theta \Pi \Omega)(\mu f)](e,h).$$

Because $(e,h)^* = h^*(e^*_A h_\theta)$, this can be written

$$[(e^*_A h_\theta)(\Theta \Pi \Omega)(\mu f)](h).$$

Using the right invariance of $\theta$, we can calculate this same quantity as

$$(d\theta_A)(\theta_e)f = \theta(e_f) = \theta f(h).$$

Thus, for right invariant vector fields $\theta$ on $G$, we have the identity

$$(12.7.1) \quad \theta = (e^*_A h_\theta)(\Theta \Pi \Omega)\mu.$$

It is easy to see that conversely $(12.7.1)$ implies that $\theta$ is right invariant.

**Definition (12.8).** Let $U$ be a universe group. We shall say that $\theta \in A \operatorname{der} U$ is right invariant if $\theta = (e_U, U)(\Theta \Pi \Omega)\mu$.

Let $\tau: G \times X \to X$ be the multiplication map $(g, x) \mapsto gx$ of $S(A \sqcup C) = G \times X$ into $S = X$. Let $p: G \times X \to X$ be such that $p(g, x) = x$. We shall define $X/G$ to be the coequalizer in the category of sober spaces of the pair $(\tau, p)$ (cf. §8). Thus if $Y$ is the space of orbits of $X$ with the quotient topology derived from $X \to Y$, then $X/G$ is the soberization of $Y$. We shall let $\operatorname{Inv} C$ denote the set of invariant elements of $C$.

The following is a routine consequence of the definitions and $(8.5)$.

**Proposition (12.9).** For any action of $G$ on $C \in K$:

1) $\operatorname{Inv} C \in K$;
2) \( X/G \) is homeomorphic to \( S(\text{Inv } C) \).

The many writings on group actions as well as examples from the qualitative theory of ordinary differential equations show us how very perplexing the space \( X/G \) can be (cf. [12]). Therein lies the significance of 1 of (12.9).

13. ACTION OF A LOCAL GROUP UNIVERSE ON A UNIVERSE.

We continue the conventions of §12. We let \( L \) be a group universe such that \( L \) is local. We let \( \mu : L \rightarrow LUL \) be the comultiplication, and we shall assume that an action \( \gamma : C \rightarrow LUC \) of \( L \) on a universe \( C \) is given. The goal of the remainder of this paper is to demonstrate Theorem (13.1) below.

Define \( g_L \) to be the set of all right invariant elements of \( \text{Ader } L \). If \( \theta \in g = g_L \), define

\[ D_\theta = \langle e_c, C \rangle (\theta y_0) y \in \text{Ader } C. \]

If \( \theta \in g \) and \( c \in C \), define \( \theta c = D_\theta (c) \in C \).

**Theorem (13.1).** Let \( c \in C \) and assume that \( L = L_a \) acts on \( C \). Then:

1) \( c \) is invariant iff \( \theta c = 0 \) for all \( \theta \in g \);
2) If \( \theta, \chi \in g_L \), then \( D_{\theta \chi} = D_\theta D_\chi \).

If \( \theta \in g \) and \( f \in LUC \), define \( \theta f = (\theta y_0) f \), i.e., we just let \( \theta \) operate in this instance on the first summand of \( LUC \).

**Lemma (13.1.1).** Let \( \theta \in g \), \( c \in C \). Then \( \theta yc = y\theta c \).

To prove (13.1.1), we shall first need to establish the following identities:

\[(13.1.2) \quad \langle (e_c L) LUC \rangle (\theta y_0) y (\mu y) = \theta y_0. \]
\[(13.1.3) \quad \langle (e_c L) LUC \rangle (\theta y_0) y (\mu y) = \gamma (e_C, C) (\theta y_0). \]
To establish these, we notice first that replacement of $0\Pi 0$ and $0\Pi 0\Pi 0$ by $LUC$ and $LLLC$ (considered as identity maps) gives us correct equations

$$(e_L, L)\Pi C (\mu L) = LUC \quad \text{and} \quad (e_L, L)\Pi C (\Pi L) = \psi (e_c, C).$$

Indeed, the first of these is obvious. Write the second as $l = r$. Let $U \in K$, $g \in (L, U)$, $c \in (C, U)$. Then $(g, c)$ is an arbitrary element of $(LLC, U)$ and

$$(g, c) l = (g (e_L, L), c) (\Pi L) = (g e_L, g, c) (\Pi L) = (g e_L, (g, c) \gamma) = (e L, (g, c) \gamma).$$

Also

$$(g, c) r = (cg) (e_c, C) = (e_c, cg),$$

so $l = r$.

It follows that both sides of (13.1.2) are in $Ader(L)\Pi C$ and that both sides of (13.1.3) are in $Ader (\psi (e_c, C))$. Thus to establish these identities, we only need to show that both sides of the equation in question agree on $L \subset LUC$ and on $C \subset LUC$ (cf. (9.8)). On $C$, the two sides of (13.1.2) are just multiplication by 0. Equality on $L$ amounts to the identity $(e_L, L) (\theta \Pi 0) \mu = \theta$, which is so because $\theta$ is assumed to be right invariant. On $C$, both sides of (13.1.3) are multiplication by 0. On $L$ they give $e_L \psi = \psi e_c \theta$, which is true since $\psi e_c = e LUC$.

The proof of (13.1.1) is now possible. If $c \in C$,

$$\theta (\psi e) = (\theta \Pi 0) (\psi c) = ((e_L, L)\Pi C (\theta \Pi 0) (\psi c)) = ((e_L, L)\Pi C (\theta \Pi 0) (\Pi L) (\psi c)) = \psi (e_c, C) (\theta \Pi 0) (\psi c) = \psi (\theta c).$$

We can now prove 1 of (13.1). If $\theta c = 0$ for every $\theta \in \mathfrak{g}$, then $\psi c = \psi \theta c = 0$ for every right invariant $\theta \in Ader L$. Pick coordinates $z_1, \ldots, z_n$ for $G$ in the neighborhood of $e$, and write

$$L = \Lambda_0 (z_1, \ldots, z_n).$$

Let $R$ be the local ring of $L$ (so $L = R [\mathfrak{g}]$). Then $\partial / \partial z_i$ is a linear combination over $R$ of right invariant elements of $Ader L$. If $r \in R$ and $D, E \in Ader L$, we have

$$(rD)f = r(Df) \quad \text{and} \quad (D + E)f = Df + Ef$$

for every $f \in LUC$, where $D$ acts on $LUC$ as $\Pi 0$, $E$ as $\Pi 0$. Thus
By (10.2.2), this implies \( yc \in C \), so \( c \) is invariant.

14. LOW ORDER TERMS IN THE POWER SERIES EXPANSION OF \( \psi \).

Let \( x_1, \ldots, x_n \in A_\mathbb{E} \) be the functions of a coordinate system of \( G \) centered at \( e \in G \). Since we shall be working in the neighborhood of \( 0 \) in \( K^n \), we shall write \( e \) as \( 0 \) and assume \( x_1(0) = \ldots = x_n(0) = 0 \). We have \( L_a = A_0 \langle x_1, \ldots, x_n \rangle \). Let us apply (11.3) ("Taylor's Theorem") to \( yc \). We can write

\[
\psi(c) = (\psi c)(0) + (x_1c_{11} + \ldots + x_n c_{1n}) + (1/2) \sum_{i,j} x_i (c_2)_{ij} x_j + R_2(\psi c)
\]

where \( R_2(\psi c) \) is an infinitesimal of order \( > 2 \). Furthermore, since \( c_2 \) is given by partial derivatives, it is a symmetric \( h \times h \)-matrix. Its entries \( (c_2)_{ij} \in C \) and satisfy \( 0(c_2)_{ij} = 0c \). Also \( 0c_{ij} = 0c \), for \( 1 \leq j \leq h \).

Until stated otherwise, we shall steadfastly ignore all infinitesimals of order \( > 2 \). Setting \( x = 0 \), we get \( (\psi c)(0) = c \). Let us therefore use matrix notation to write

\[
\psi c = c + xc_1 + (1/2)xc_2x',
\]

here considering \( x \) as the "row vector" \([x_1, \ldots, x_n]\), with \( x' \) the "column vector" transpose of \( x \). For any set \( S \) we shall let \( \mathbb{M}_{u,v}(S) \) denote the set of \( u \times v \) matrices over \( S \).

We need to do some technical work that will lead to the proof of 2 of (13.1). Let us look at the \( i \)-th component \( c_{i1} \), of \( c_1 \in \mathbb{M}_{h,1}(C) \) in

\[
\psi c = c + xc_1 + (1/2)xc_2x',
\]

and apply \( \psi \) to it to get

\[
\psi(c_{i1}) = c_{i1} + x(c_{i1})_1 + (1/2)x(c_{i1})_2x'.
\]

We shall be needing the specific formula that one can obtain for these \( (c_{i1})_j \in C \) by using the relation \( (\mu \mu c) \psi = (L\mu \psi) \psi \). For this
note that \((C_{1i})_1 \in \mathbb{M}_{h,1}(\mathbb{C})\) and so has entries \((C_{1i})_{1j}\), for \(1 \leq j \leq h\). It is convenient to define \((C_1)_1\) in \(\mathbb{M}_{h,h}(\mathbb{C})\) by

\[
((C_1)_1)_{ij} = (C_{1i})_{1j},
\]

To derive the formula for \((C_1)_1\), we shall first need to consider the Taylor expansion of \(\mu\) at \(e\) (ignoring infinitesimals of order > 2). We shall write

\[
L_n \llcorner L_n = A_0 \langle x_1, \ldots, x_h; y_1, \ldots, y_n \rangle = A_0 \langle x; y \rangle.
\]

**Lemma (14.1).** For \(i = 1, \ldots, h\) we have

\[
\mu x_i = x_i + y_i + xb_i y^T \quad \text{where} \quad b_i \in \mathbb{M}_{h,n}(\mathbb{K}).
\]

We have

\[
(\mu x_i)(x; 0) = x_i \quad \text{and} \quad (\mu x_i)(0; y) = y_i.
\]

Thus

\[
\mu x_i = x_i + y_i + xa_i x^T + xb_i y^T + yc_i y^T
\]

for appropriate \(a_i, b_i, c_i \in \mathbb{M}_{h,n}(\mathbb{K})\), and \(a_i\) and \(c_i\) can be taken to be symmetric. But then

\[
x a_i x^T = 0 = y c_i y^T \quad \text{for all} \quad x, y \in \mathbb{K}^n,
\]

so \(a_i = 0 = c_i\).

**Lemma (14.2).** Let \(b_i\) be defined as in (14.1), \(i = 1, \ldots, h\). Then

\[
(c_1)_1 = \sum_{i=1}^{h} b_i c_{i1} + c_2.
\]

Use the relation \((\mu \llcorner \mathcal{C}) \psi = (L \llcorner \mathcal{C}) \psi\) on \(c\), writing

\[
L \llcorner \mathcal{C} = A_0 \langle x \rangle \llcorner L A_0 \langle y \rangle \llcorner \mathcal{C}.
\]

We have

\[
(L \llcorner \mathcal{C}) \langle \psi c \rangle = (L \llcorner \mathcal{C}) \langle c + x c_1 + (1/2) x c_2 x^T \rangle = \psi c + x \psi c_1 + (1/2) x \psi (c_2) x^T
\]

\[
= c + y c_1 + (1/2) y c_2 y^T + x (c_1 + (c_1) y^T) + (1/2) x c_2 x^T,
\]

as one sees by writing

\[
x \psi c_1 = x_1 \psi (c_{11}) + \ldots + x_n \psi (c_{1n})
\]

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and noting that
\[ x(c_1), y^T = x(c_1) + \sum_{i=1}^n (c_1)_i y_i. \]

Let \( b \in M_{1,n}(\mathbb{K}) \) be defined by \( b = [b_1, \ldots, b_n] \). Then also
\[
(\mu \mathbf{L}C)(y^c) = (\mu \mathbf{L}C)(c + x_0(1/2)x_2x^T) = c + (x + y + x_2y^T)c + (1/2)(x+y)c_2(x+y)^T.
\]

In view of (8.18), the expressions we have obtained for \( (L \mu y)(y^c) \) and \( (\mu \mathbf{L}C)(y^c) \) must equal each other. Equating the two expressions and performing cancellations we get
\[
x(c_1), y^T = x_2y^Tc_1 + (1/2)(yc_2x^T + xc_2y^T).
\]

As \( c_2 \) is symmetric,
\[
(1/2)(yc_2x^T + xc_2y^T) = xc_2y^T.
\]

Also
\[
x_2y^Tc_1 = \sum_{i=1}^n (x_2y^T)c_{1i} = x(\sum_{i=1}^n b_{1i}c_{1i})y^T,
\]
so (12.2) follows.

Let \( \theta \in \mathfrak{g} \) be defined by \( \theta f = (\partial f/\partial x_i)(e) \) for \( f \in L \). Abbreviate \( D_{x_i} \) by \( D_i \).

**Lemma (14.3).** Let \( c \in C \). Then:
1) \( D_1c = c_1 \); 
2) \( D_2(D_1c) = (c_1)_i \); 
3) \( [D_3, D_1](c) = \sum_{i=1}^n [(b_3)_i - (b_2)_i]c_{1i}. \)

For 1,
\[
D_1c = (D_1\mathbf{L}(0))(y^c) = (D_1\mathbf{L}(0))(c + \sum_{i=1}^n x_i(c_1)_i) = c_1.
\]

Then, because of 1, \( D_j(D_1c) = (c_1)_i \), proving 2. From 2 and (12.2),
\[
D_j(D_1c) = [(c_1)_i]_j = \sum_{i=1}^n (b_3)_i c_{1i} + (c_2)_i,
\]
so 3 follows immediately from the fact that \( c_2 \) is symmetric.

At this point we shall discontinue our practice of ignoring infinitesimals of order \( > 2 \). Write \( L_0 = \mathbf{R}L(\Omega_0) \).
**Lemma (14.4).** Let $i = 1, \ldots, h$. Then in Ader L we have

$$\theta_i = \partial/\partial x_i + xW_i \partial/\partial x$$

where $W_i \in \mathcal{M}_{n}\mathcal{M}(\mathbb{R})$ and $\partial/\partial x$ is the "column vector" with entries $\partial/\partial x_1$, $\ldots, \partial/\partial x_n$. Furthermore $W_i$ satisfies

$$(W_i)_{ij}(e) = (b_j)_{iv}, \quad 1 \leq i, v \leq h.$$ 

We can write

$$yx_j = x_j + y_j + x\delta_j y^T$$

where $b_j \in \mathcal{M}_{n}\mathcal{M}(\mathbb{R})$ and $b_j(e) = b_j$. If $\theta$ is any right invariant element of Ader L,

$$\theta x_j = (\theta W)(\theta W)(\theta W)(x_j + y_j + x\delta_j y^T) = \theta x_j + \theta x_j \delta_j x^T.$$ 

In particular,

$$\theta x_j = \delta_j + \Sigma_{v=1}^n (\delta_j)_{iv} x.$$ 

Thus

$$\theta_i = \Sigma_{v=1}^n (\theta x_j)(\theta x_j) = \partial/\partial x_i + xW_i \partial/\partial x,$$

where $(W_i)_{ij} = (\delta_j)_{iv}$. This proves (14.4).

**Lemma (14.5).** For $i = 1, \ldots, h$,

$$[\theta_i, \theta_j] = \Sigma_{q=1}^n [(b_q)_{ij} - (b_q)_{ji}] \theta_q.$$ 

From (14.4),

$$[\theta_i, \theta_j] = [\partial/\partial x_i, \partial/\partial x_j] + [W_i, \partial/\partial x_j] + [W_i, \partial/\partial x_i] + [W_i, \partial/\partial x_i, W_i].$$

Thus, since $x(e) = 0$,

$$[\theta_i, \theta_j]_e = (W_j(e))(\partial/\partial x)_e)(\theta_i)_e - (W_j(e))(\partial/\partial x)_e)(\theta_i)_e = \Sigma_{q=1}^n [(b_q)_{ij} - (b_q)_{ji}] (\theta_q)_e.$$ 

Therefore the equation of (14.5) holds, for it is an equation between right invariant vector fields that agree at $e$.

We can now prove 2 of (13.1). From 3 of (14.3) we have

$$[D_{y}, D_{i}]c = \Sigma_{q=1}^n [(b_q)_{ij} - (b_q)_{ji}] c_{ij},$$

from (14.5) that

$$[\theta_i, \theta_j] c = \Sigma_{q=1}^n [(b_q)_{ij} - (b_q)_{ji}] \theta_q c,$$

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and by 1 of (14.3), $\Theta w c = c_{1q}$, so $[D_3, D_2] = D_{(1q, q)}$. Then 2 of (13.1), $[D_2, D_4] = D_{(q, 1q)}$ is an immediate consequence of the K-bilinearity of this equation.

REFERENCES.