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Correction to “Fibrations in bicategories”


<http://www.numdam.org/item?id=CTGDC_1987__28_1_53_0>
Given homomorphisms of bicategories $J: A \to \text{Cat}$ and $S: A \to K$, the bilimit $(J, S)$ of $S$ indexed by $J$ can be constructed using biproducts, cotensor biproducts and biequalizers. However, the construction described in Section 1 (1.24), page 120, of the paper "Fibrations in bicategories" (these Cahiers, XXI-2, 1980, 111-160) is wrong. I am grateful to Max Kelly for noticing this error. He also recognized that $(J, S)$ could be constructed if $K$ admitted some further bilimits of a simple kind. In fact, no further bilimits are needed: I shall show that they can be constructed from those at hand.

Certain small categories $\text{Iso}$, $\text{End}$, $\text{Aut}$, $T$ will be required. A functor from one of these into a category amounts to an isomorphism, endomorphism, automorphism, composable pair of isomorphisms, respectively, in the category. Notice that $\text{Iso}$, $T$ are equivalent to 1 while $\text{Aut}$ is not.

The biequifier of 2-cells

$$\theta, \varphi: f \Rightarrow g : A \to B$$

in $K$ is an arrow $h: X \to A$ which, for all objects $X$, induces an equivalence of categories between $K(K, X)$ and the full subcategory of $K(K, A)$ consisting of those $a: K \to A$ for which $\theta a = \varphi a: fa \Rightarrow ga$. (If $\theta$, $\varphi$ are invertible this is the same as the biequivinerter of $\theta$, $\varphi$ as used later (4.2) in the paper.) The biequifier of an endo-2-cell

$$\gamma: f \Rightarrow f : A \to B$$

is the biequifier of $\gamma$ and the identity of $f$. If $\gamma$ is invertible, the biequifier of $\theta$, $\varphi$ is the biidentifier of $\gamma^{-1} \theta$. So to construct biequifiers of invertible pairs it suffices to construct biidentifiers of auto-2-cells.
Consider an auto-2-cell \( y: f \Rightarrow f: A \rightarrow \text{Aut}, B \) be the arrows corresponding to the automorphisms \( y, \lambda \) in \( K(A, B) \). Let \( h: H \rightarrow A, \sigma: \gamma h = f h \) be the biequalizer of \( \gamma, f \). Let \( k: K \rightarrow A, \tau: \tau k = \tau h \) be the biequalizer of \( \tau, f \). Let \( a: \text{Aut}, B \rightarrow B \) be induced by the unique functor \( 1 \rightarrow \text{Aut} \). There exist \( l: H \rightarrow K \) and \( v: kl = h \) rendering \( \sigma \) isomorphic to \( \tau l \) by definition of \( K \). Similarly, we obtain \( d: A \rightarrow K \) and \( kd = \lambda k \) rendering \( \lambda \) isomorphic to \( \tau d \). Now form the bi-pullback

\[
\begin{array}{ccc}
P & \xrightarrow{\nu} & H \\
\downarrow u & & \downarrow l \\
A & \xrightarrow{a} & K
\end{array}
\]

of \( d, l \); this is just the biequalizer of the two arrows from the biproduct of \( A, H \) into \( K \) which use \( d, l \) and the projections. I claim \( u: P \rightarrow A \) is the bi-identifier of \( y: f \Rightarrow f \). Since bilimits are defined representably, we only need to check the construction in \( \text{Cat} \). Then \( K \) can be taken to be the category of pairs \( (a, \tau) \) where \( a \) is an object of \( A \) and \( \tau: fa = fa \) in \( B \), while \( H \) can be taken to be the full subcategory of \( K \) consisting of the pairs \( (a, \sigma) \) with \( \sigma ya = \sigma \). Since \( \sigma \) is invertible, the last equation implies \( \sigma a = 1 \). Also \( l \) is the inclusion and \( d \) takes \( a \) to \( (a, 1, a) \). With this we see that the objects of \( P \) are pairs \( (\sigma, \rho) \) where

\[
\rho: a \simeq a' \text{ in } A, \ (a, \sigma) \in H \text{ and } \xi \rho \sigma = \xi \rho.
\]

This last condition implies \( \sigma = 1_{\text{reflex}} \), and, since \( y \) is natural, \( \gamma a = 1 \).

So \( P \) is equivalent to the full subcategory of \( A \) consisting of those \( a \) with \( \gamma a = 1 \).

(The above construction with \( \text{Aut} \) replaced by \( \text{End} \) yields the bi-identifier of any endo-2-cell \( y \).)

The next bilimit required is the descent object \( \text{Desc}(X) \) of a truncated bicosimplicial diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\delta_0} & X_1 \\
\downarrow \delta_1 & & \downarrow \delta_2 \\
X_1 & \xrightarrow{\delta_1} & X_2
\end{array}
\]

\( \sigma_{1,i}: \delta_1 \delta_{i-1} = \delta_1 \delta_i \) for \( i < j \), \( n_i : 1 \simeq 1 \delta_i \)
in a bicategory $K$. When $K$ is $\text{Cat}$, the category $\text{Desc}(X)$ has objects pairs $(x, \theta)$ where $x$ is an object of $X_0$ and $\theta: \delta_0 x \simeq \delta_1 x$ in $X$, such that

$$1\theta = \mu_1 \mu_0^{-1}, \quad \sigma_{12, \delta_1 \theta, \sigma_{01}} = \delta_2 \theta, \sigma_{02, \delta_0 \theta},$$

and has arrows $\chi: (x, \theta) \to (x', \theta')$ where $\chi: x \to x'$ is an arrow of $X_0$ such that $Q_1 \delta_0 \chi = \delta_1 \chi \theta$. For a general $K$, the descent object of $X$ consists of an object $D$, an arrow $h: D \to X_0$ and an invertible 2-cell $\omega: \delta_0 h \simeq \delta_1 h$ inducing an equivalence between $K(K, D)$ and $\text{Desc}(K(K, X))$. Notice that $X$ can be regarded as a homomorphism from an appropriate category $A$ into $K$ and, if we take $J: A \to \text{Cat}$ to be the functor amounting to the diagram

$$1 \leftarrow \text{Iso} \rightarrow T$$

the bilitmit $(J, X)$ is equivalent to $\text{Desc}(X)$.

The descent object can be constructed using biequalizers and bidentifiers of auto-2-cells. First, take the biequalizer

$$h: H \to X_0, \quad \theta: \delta_0 h \simeq \delta_1 h,$$

of $\delta_0, \delta_1$, then the biequifier $k: K \to H$ of the two invertible 2-cells

and then, the biequifier $m: M \to L$ of the two invertible 2-cells

Then $L, hkm, \theta km$ form $\text{Desc}(X)$.

The bilimit $(J, S)$ can be obtained as the descent object $\text{Desc}(X)$ where

$$X_0 = \Pi I_a \ (JA, SA), \quad X_1 = \Pi I_{a,b} \ (A(A_B) \times JA, SB),$$

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In (1.25) it was stated that indexed pseudo-limits in a 2-category could be constructed from cotensor products, products and equalizers. This is certainly true since pseudo-limits are particular indexed limits and all indexed limits can be so constructed (see [14], using the Bibliography of the paper). The proof outlined in (1.25) was a modification of (1.24). Using the corrected (1.24), we can squeeze out more from the method. Many naturally occurring 2-categories have iso-inserters. The iso-inserter of the diagram $f, g: A \rightarrow B$ is its limit (not pseudo-limit) indexed by the diagram

$$
1 \rightarrow \text{Iso} \quad \text{in Cat.}
$$

An iso-inserter is a biequalizer but not conversely. The strict descent object of a truncated simplicial object $X$ (this time $\mu_{x}, \sigma_{x}$, are identities) is defined as for the descent object except that we insist on an isomorphism between $K(K, D)$ and $\text{Desc}(K(K, X))$, not merely an equivalence. It can be constructed using an iso-inserter and identifiers of auto-2-cells (the latter are defined as were bidentifiers except that we ask for an isomorphism in the representation property). Then $\text{psdlim}(J, S)$ is the strict descent object for $X$ as before with biproducts and cotensor biproducts replaced by their "non-bi" versions. However, it does not seem possible to construct identifiers of auto-2-cells using a "non-bi" version of the construction of bidentifiers. The object $P$ we are led to does support an idempotent whose splitting gives the identifier; but this is already true of the iso-inverter of $\gamma^{-1}, f$. So products, cotensor products, iso-inserters, and, either identifiers of auto-2-cells or splittings of idempotents, imply all indexed pseudo-limits.

I would like to stress that I am currently using the word "weighted" in preference to "indexed" in this context.

Finally, there is a typographical error in (4.2) on page 140. The functors between 1 and Iso should have their directions reversed.