CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

ANTONIO G. RODICIO

A homological exact sequence associated with a family of normal subgroups

Cahiers de topologie et géométrie différentielle catégoriques, tome 28, nº 1 (1987), p. 79-82

<http://www.numdam.org/item?id=CTGDC_1987__28_1_79_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXVIII-1 (1987)

A HOMOLOGICAL EXACT SEQUENCE ASSOCIATED WITH A FAMILY OF NORMAL SUBGROUPS

by Antonio G. RODICIO

Résumé. R. Brown et J.-L. Loday ont obtenu, par des méthodes topologiques, une suite exacte de 8 termes qui relie l'homologie d'un groupe G avec l'homologie de deux de ses sous-groupes normaux M et N tels que MN = G. Ici on démontre que les cinq premiers termes de cette suite peuvent être obtenus pour une famille de sous-groupes $\{M_i \mid 1 \leqslant i \leqslant n\}$ telle que

$$M_k(\cap_{i\neq k} M_i) = G.$$

Introduction.

In this paper we prove the following theorems:

THEOREM 1. Let A *be a ring (with unit) and* I *a two-sided ideal of* A. *Let* $\{J_i \mid 1 \leqslant i \leqslant n\}$ *be a finite family of two-sided ideals of* A *such that*

$$J_k + (\cap_{i\neq k} J_i) = I, \quad 1 \leqslant k \leqslant n.$$

Then there exists an exact and natural sequence

$$\text{(1)} \qquad \mathrm{Tor}_2^A(A/I,\ A/I) \to \oplus_{i=1}^n \mathrm{Tor}_2^{A/J_i}(A/I,\ A/I) \to$$
$$(\cap_{i=1}^n J_i)/\Sigma_{k=1}^n(\cap_{i\neq k} J_i)J_k \to \mathrm{Tor}_1^A(A/I,\ A/I) \to \oplus_{i=1}^n \mathrm{Tor}_1^{A/J_i}(A/I,\ A/I) \to 0.$$

THEOREM 2. Let G *be a group and* $\{M_i \mid 1 \leqslant i \leqslant n\}$ *a finite family of normal subgroups of* G, *such that*

$$M_k(\cap_{i\neq k} M_i) = G, \quad 1 \leqslant k \leqslant n.$$

Then there exists an exact and natural sequence

$$H_2(G) \to \oplus_{i=1}^n H_2(G/M_i) \longrightarrow (\cap_{i=1}^n M_i)/\langle[\cap_{i\neq k} M_i, M_k] \mid 1 \leqslant k \leqslant n\rangle \to$$

(2) $$H_1(G) \to \oplus_{i=1}^n H_1(G/M_i) \longrightarrow 0$$

where $\langle[\cap_{i\neq k} M_i, M_k] \mid 1 \leqslant k \leqslant n\rangle$ *is the subgroup of* G *generated by the subgroups* $[\cap_{i\neq k} M_i, M_k]$, $1 \leqslant k \leqslant n$.

If $J_i = I$, $3 \leqslant i \leqslant n$, then Theorem 1 yields Theorem 1 of [2]. On the other hand, if $M_i = G$, $3 \leqslant i \leqslant n$, then sequence (2) yields the five first terms of the eight-term exact sequence obtained in [1] by topological methods.

Proof of the theorems.

We will systematically use the following result: If N is a submodule of an A-module M, then the sequence

$$N\otimes_A M \oplus M\otimes_A N \to M\otimes_A M \to M/N\otimes_A M/N \to 0$$

is exact.

For each k, $1 \leqslant k \leqslant n$,, we have a short exact sequence

$$\cap_i J_i \to \cap_{i\neq k} J_i \to (\cap_{i\neq k} J_i)/(\cap_i J_i) \simeq I/J_k$$

which yields a right exact sequence

$$(\cap_i J_i)\otimes_A(\cap_{i\neq k} J_i)\oplus(\cap_{i\neq k} J_i)\otimes_A(\cap_i J_i) \to (\cap_{i\neq k} J_i)\otimes_A(\cap_{i\neq k} J_i)\to(I/J_k)\otimes_A(I/J_k) \to 0.$$

Therefore we obtain the right exact sequence

$$\oplus_{k=1}^n((\cap_i J_i)\otimes_A(\cap_{i\neq k} J_i)\oplus(\cap_{i\neq k} J_i)\otimes_A(\cap_i J_i)) \to \oplus_{k=1}^n((\cap_{i\neq k} J_i)\otimes_A(\cap_{i\neq k} J_i))$$

(3) $$\to \oplus_{k=1}^n((I/J_k)\otimes_A(I/J_k)) \to 0.$$

We now consider the sequence

(4) $$\oplus_{k=1}^{n-1}(\cap_i J_i) \xrightarrow{\ \mu\ } \oplus_{k=1}^n(\cap_{i\neq k} J_i) \xrightarrow{\ \epsilon\ } I$$

where

$$\mu(a_1, a_2,\dots,a_{n-1}) = (a_1, a_2,\dots,a_{n-1}, -(a_1+a_2+\dots+a_{n-1}))$$

and

$$\epsilon(b_1, b_2,\dots, b_n) = b_1+b_2+\dots+b_n.$$

It is clear that μ is injective and Ker ϵ = Im μ. Moreover ϵ is surjective as a consequence of $I = \Sigma_{k=1}^n(\cap_{i\neq k} J_i)$. Then the sequence (4) induces the right exact sequence

$$(\oplus_{k=1}^{n-1}(\cap_i J_i))\otimes_A(\oplus_{k=1}^{n}(\cap_{i\neq k}J_i))\oplus(\oplus_{i=1}^{n}(\cap_{i\neq k}J_i))\otimes_A(\oplus_{k=1}^{n-1}(\cap_i J_i)) \to$$
$$(5) \qquad (\oplus_{r=1}^{n}(\cap_{i\neq r}J_i))\otimes_A(\oplus_{s=1}^{n}(\cap_{i\neq s}J_i)) \to I\otimes_A I \to 0.$$

Let us consider the diagram

$$\begin{array}{ccccc}
 & & (\oplus_{k=1}^{n-1}(\cap_i J_i))\otimes_A(\oplus_{k=1}^{n}(\cap_{i\neq k}J_i)) \oplus (\oplus_{k=1}^{n}(\cap_{i\neq k}J_i))\otimes_A(\oplus_{k=1}^{n-1}(\cap_i J_i)) & \xrightarrow{\gamma} & \oplus_{k=1}^{n}((\cap_i J_i)\otimes_A(\cap_{i\neq k}J_i) \oplus (\cap_{i\neq k}J_i)\otimes_A(\cap_i J_i)) \\
 & & \downarrow & & \downarrow \\
\oplus_{r\neq s}((\cap_{i\neq r}J_i)\otimes_A(\cap_{i\neq s}J_i)) & \rightarrowtail & (\oplus_{r=1}^{n}(\cap_{i\neq r}J_i))\otimes_A(\oplus_{s=1}^{n}(\cap_{i\neq s}J_i)) & \twoheadrightarrow & \oplus_{k=1}^{n}((\cap_{i\neq k}J_i)\otimes_A(\cap_{i\neq k}J_i)) \\
\vdots & & \downarrow & & \downarrow \\
\mathrm{Ker}\,\beta & \rightarrowtail & I\otimes_A I & \overset{\beta}{\twoheadrightarrow} & \oplus_{k=1}^{n}((I/J_k)\otimes_A(I/J_k))
\end{array}$$

where the first and second columns are the sequences (5) and (3) respectively, β is the canonical homomorphism and γ is the adequate epimorphism making the upper square commutative. Then it follows that the induced homomorphism

$$\oplus_{r\neq s}((\cap_{i\neq r}J_i)\otimes_A(\cap_{i\neq s}J_i)) \to \mathrm{Ker}\,\beta$$

is surjective. From that we obtain the right exactness of the top row in the diagram

$$\begin{array}{ccccc}
\oplus_{r\neq s}((\cap_{i\neq r}J_i)\otimes_A(\cap_{i\neq s}J_i)) & \to & I\otimes_A I & \overset{\beta}{\twoheadrightarrow} & \oplus_{i=1}^{n}((I/J_i)\otimes_A(I/J_i)) \\
\downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow \\
\cap_{i=1}^{n}J_i & \rightarrowtail & I & \twoheadrightarrow & \oplus_{i=1}^{n}(I/J_i)
\end{array}$$

in which the homomorphisms are the usual ones. Now Im ϕ is

$$\Sigma_{r\neq s}(\cap_{i\neq r}J_i)(\cap_{i\neq s}J_i) = \Sigma_{k=1}^{n}(\cap_{i\neq k}J_i)J_k.$$

On the other hand, there exist exact sequences

$$0 \to \mathrm{Tor}_2^A(A/I, A/I) \to I\otimes_A I \to I \to \mathrm{Tor}_1^A(A/I, A/I) \to 0$$

and

$$0 \to \mathrm{Tor}_2^{A/J_i}(A/I, A/I) \to (I/J_i)\otimes_A(I/J_i) \to I/J_i \to \mathrm{Tor}_1^{A/J_i}(A/I, A/I) \to 0.$$

Therefore, application of the ker-coker Lemma to the diagram yields the sequence (1).

In order to show the exactness of the sequence (2) let $A = ZG$ be the integral group ring of G, $I = IG$ the augmentation ideal of G and $J_i = I_{M_i}$ the kernel of the ring homomorphism $ZG \twoheadrightarrow Z(G/M_i)$. It is clear that

$$I_{M_k} + (\cap_{i \neq k} I_{M_i}) = IG, \quad 1 \leqslant k \leqslant n.$$

Moreover we have an isomorphism of abelian groups

$$(\cap_{i=1}^{n} I_{M_i}) / \Sigma_{i=1}^{n} (\cap_{i \neq k} I_{M_i}) I_{M_k} \simeq (\cap_{i=1}^{n} M_i) / \langle [\cap_{i \neq k} M_i, M_k] \mid 1 \leqslant k \leqslant n \rangle$$

which is obtained as in the case $n = 2$ (see [2]).

Therefore sequence (1) yields sequence (2).

References.

1. R. BROWN & J.-L. LODAY, Excision homotopique en basse dimension, *C.R.A.S. Paris* 298, Ser. I (1984), 353-356.

2. A.G. RODICIO, On some five-term exact sequences in homology, *Comm. in Algebra*, 14 (1986), 1357-1364.

Universidad de Santiago
Facultad de Matematicas
SANTIAGO, SPAIN