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WEDGE PRODUCT OF FORMS IN SYNTHETIC DIFFERENTIAL GEOMETRY

by M. Carmen MINGUEZ

RÉSUMÉ. En Géométrie Différentielle Synthétique, nous définissons le "wedge" produit de formes lorsque les p-formes sont des fonctions définies sur les p-tangentes $D^p \to M$. Ce produit vérifie les propriétés algébriques usuelles de même que la formule de Leibniz. Nous comparons les p-formes et les p-cochaînes cubiques, à l'aide d'un homomorphisme défini par intégration, ainsi que les "wedge" et "cup" produits sur des cubes infinitésimaux. Un homomorphisme similaire entre le complexe de de Rham et le complexe des cochaînes singulières est défini; cet homomorphisme devient un morphisme entre R-algèbres quand il descend à la cohomologie.

1. WEDGE PRODUCT.

We suppose the Axiom K-L (Axiom 1 in [1]) and we refer the reader to [5] for the calculations.

There are several ways of introducing the notion of form (differential form) in the synthetic context (see [1]). The closest with the classical one is:

(1.1) A *quasi-classical p-form* $\omega^\sim$ on M is a law which to any p-tuple $(t_1,\ldots,t_p)$ of tangents to M (at the same point) associates an element $\omega^\sim(t_1,\ldots,t_p) \in$ R such that

(i) $\omega^\sim(t_1,\ldots,\lambda t_i,\ldots,t_p) = \lambda\omega^\sim(t_1,\ldots,t_p)$, $\lambda \in$ R, $i = 1,\ldots,p$,

(ii) $\omega^\sim(t_{\pi(1)},\ldots,t_{\pi(p)}) = \mathrm{sig}(\pi)\omega^\sim(t_1,\ldots,t_p)$, $\pi \in S_p$.

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It is obvious to give the wedge product of a quasi-classical p-form $\omega^{\sim}$ and a quasi-classical q-form $\theta^{\sim}$ by direct translation of the classical notion, that is $(n = p+q)$

(1.2) $$\omega^{\sim}\wedge\theta^{\sim}(t_1,\dots,t_n) = (1/p!q!)\Sigma_{\pi\in S_n}\operatorname{sig}(\pi)\omega^{\sim}(t_{\pi(1)},\dots,t_{\pi(p)})\theta^{\sim}(t_{\pi(p+1)},\dots,t_{\pi(n)}).$$

It seems that it is not possible to define a quasi-classical $(p+1)$-form $d\omega^{\sim}$ which corresponds to concept of exterior differential except when we can use coordinates on M. For this we consider a wider notion of form given in [2] (also [1]).

DEFINITION 1. A *p-form* on M is a map $\omega\colon M^{D^p} \to R$ which is
(i) homogeneous:
$$\omega(\lambda._i\tau) = \lambda\omega(\tau),$$
where
$$\lambda \in R,\ i = 1,\dots,p \text{ and } \lambda._i\tau(d_1,\dots,d_p) = \tau(d_1,\dots,\lambda d_i,\dots,d_p);$$

(ii) alternating:
$$\omega(\tau.D^{\pi}) = \operatorname{sig}(\pi)\omega(\tau),$$
where
$$\pi \in S_p \text{ and } D^{\pi}(d_1,\dots,d_p) = (d_{\pi(1)},\dots,d_{\pi(p)}).$$

We will write $A^p(M)$ for the R-module of p-forms on M. The differential exterior operator $d\colon A^p(M) \to A^{p+1}(M)$ is built making true the "Infinitesimal Stokes Theorem" (see [1]). But, for this kind of forms the wedge product is not obvious, we attempt to give a definition.

By restricting $\tau\colon D^p \to M$ to the p "axes" through $\underline{0} \in D^p$ we obtain a p-tuple of tangent vectors at $\tau(\underline{0})$. Let $k\colon M^{D^p} \to M^D\times_M\dots\times_M M^D$ be this map. Each quasi-classical p-form defines a p-form by composition $\omega = \omega^{\sim}\circ k$. (This correspondence becomes a bijection for a large class of objects M, see [2], [4].)

Now we give a definition of $\omega\wedge\theta$ compatible with the above comparison, that is satisfying
$$(\omega^{\sim}\circ k)\wedge(\theta^{\sim}\circ k) = (\omega^{\sim}\wedge\theta^{\sim})\circ k.$$

DEFINITION 2. We will call *wedge product* of $\omega \in A^p(M)$ and $\theta \in A^q(M)$ to the $(p+q)$-form given by

(1.3) $$\omega\wedge\theta(\tau) = (1/p!q!)\Sigma_{\pi\in S_n}\operatorname{sig}(\pi)\omega(\tau\circ D^{\pi}\circ\alpha^{+})\theta(\tau\circ D^{\pi}\circ\alpha^{+})$$

where $n = p+q$, τ is a n-tangent on M, $\alpha^{+}: D^{p} \to D^{p}\times D^{q}$ is $\alpha^{+}(\underline{d}) = (\underline{d},\underline{0})$ and $\alpha^{\vdash}: D^{q} \to D^{p}\times D^{q}$ is $\alpha^{\vdash}(\underline{d}) = (\underline{0},\underline{d})$.

PROPOSITION 3. *The map* $\wedge: A^{p}(M)\times A^{q}(M) \to A^{p+q}(M)$ *is bilinear, associative, anticommutative* $(\theta\wedge\omega = (-1)^{pq}\omega\wedge\theta)$ *and functorial* $(f^{*}(\omega\wedge\theta) = f^{*}\omega \wedge f^{*}\theta$ *for every* $f: M \to N)$.

If we put $M^{(i)}: M^{D^{p+1}} \to (M^{D^{p}})^{D}$, $M^{(i)}(\tau) = \tau_i$ being

$$\tau_i(h)(\underline{d}) = \tau(d_1,\dots,d_{i-1},h,d_i,\dots,d_p),$$

and $\gamma: R^{D} \to R$ is the second projection ($R^{D} \simeq R\times R$), then we obtain an explicit expression of $d\omega$. Writting $D_i\omega = \gamma\circ\omega^{D}\circ M^{(i)}$ and by applying Axiom K-L we have that

$$d\omega = \Sigma_{i=1}^{p+1} (-1)^{i+1} D_i\omega. \tag{1.4}$$

PROPOSITION 4 *(Leibniz's Formula). For each p-form* ω *and q-form* θ *on* M, *we have*

$$d(\omega\wedge\theta) = d\omega\wedge\theta + (-1)^{p}\omega\wedge d\theta.$$

PROOF (Sketch). For any $\tau: D^{n+1} \to M$ and $j = 1,\dots,n+1$, as γ is a derivation we get to

$$D_j(\omega\wedge\theta)(\tau) = \Sigma_{\pi\in S_n} (\mathrm{sig}(\pi)/p!q!)\{\gamma[\omega(\tau_j(-)\circ D^{\pi}\circ\alpha^{+})]\theta(\tau_j(\underline{0})\circ D^{\pi}\circ\alpha^{\vdash}) + \omega(\tau_j(\underline{0})\circ D^{\pi}\circ\alpha^{+})\gamma[\theta(\tau_j(-)\circ D^{\pi}\circ\alpha^{\vdash})]\}.$$

On the other hand

$$d\omega \wedge \theta(\tau) = \Sigma_{i=1}^{p+1}((-1)^{i+1}/(p+1)!q!)\Sigma_{\sigma\in S_{n+1}} \mathrm{sig}(\sigma)D_i\omega(\tau\circ D^{\sigma}\circ\alpha^{+})\theta(\tau\circ D^{\sigma}\circ\alpha^{\vdash})$$

but, if we take into account that

$$(\tau\circ D^{\sigma}\circ\alpha^{+})_i(-) = \tau_j(-)\circ D^{\sigma_{ij}}\circ\alpha^{+}, \quad \tau\circ D^{\sigma}\circ\alpha^{\vdash} = \tau_j(\underline{0})\circ D^{\sigma_{ij}}\circ\alpha^{\vdash}$$

where $j = \sigma(i)$ and $\sigma_{ij} \in S_n$ is canonically built from σ (we note that α^{+} and $\alpha^{\vdash}$ are different in each side), we obtain that

$$d\omega \wedge \theta(\tau) = \Sigma_{j=1}^{n+1}(-1)^{j+1}\Sigma_{\pi\in S_n} (\mathrm{sig}(\pi)/p!q!)\gamma[\omega(\tau_j(-)\circ D^{\pi}\circ\alpha^{+})]\theta(\tau_j(\underline{0})\circ D^{\pi}\circ\alpha^{\vdash}).$$

Using the anticommutativity of the wedge product we get a similar expression for $\omega\wedge d\theta(\tau)$. #

To make a comparison between the wedge product of forms and the cup product of cubical cochains, we take the following digression in [7], Ch. IV:

An *infinitesimal n-cube* (a n-tangent with a point $\underline{h} \in D^n$) can be regarded as a finite n-cube, in fact we have the map ϕ: $M^{D^n} \times D^n \to M^I$ given by

$$\phi(\tau, \underline{h})(x_1, \ldots, x_n) = \tau(h_1 x_1, \ldots, h_n x_n).$$

If we suppose that M has the extension property

(E) "The canonical map $M^{D^{2n}} \to M^{D^n}$ is an epic",

we can prove that for evey n-form ω and $(\tau, \underline{h}) \in M^{D^n} \times D^n$ it is satisfied

$$\int_{\phi(\tau,\underline{h})} \omega = h_1 \ldots h_n \omega(\tau),$$

that is

$$\rho(\omega) \circ \phi(\tau, \underline{h}) = \prod_{i=1}{}^{n} h_i \omega(\tau) \tag{1.6}$$

where ρ: $A^n(M) \to C^n(M)$ is the integration homomorphism given on the generators γ: $I^n \to M$ of $C^n(M)$ by

$$\rho(\omega)(\gamma) = \int_\gamma \omega = \int_{I^n} \gamma^*(\omega).$$

On the complex of cubical cochains $C^\bullet(M)$ we have the following cup product ([3], Ch. VI, VIII): Let $c \in C^p(M)$, $c' \in C^q(M)$ and γ: $I^n \to M$ $(n = p+q)$, then it is defined by

$$(c \cup c')(\gamma) = \Sigma_{K\subset[n], \mathrm{card}K=p} \operatorname{sig}(KH) c(A_H\gamma) c'(B_K\gamma) \tag{1.7}$$

where H is the complementary set of K in $[n] = \{1,2,\ldots,n\}$ and if we put ϕ_K: $K \to [p]$ for the unique bijective, order-preserving map, $A_H\gamma$: $I^p \to M$ and $B_K\gamma$: $I^q \to M$ are given by:

$$A_H\gamma(x_1, \ldots, x_n) = \gamma(y_1, \ldots, y_n) \quad \text{with } y_i = \begin{cases} 0 & \text{if } i \in H \\ x_{\phi_K(i)} & \text{if } i \in K \end{cases}$$

$$B_K\gamma(x_1, \ldots, x_q) = \gamma(y_1, \ldots, y_n) \quad \text{with } y_i = \begin{cases} 1 & \text{if } i \in K \\ x_{\phi_H(i)} & \text{if } i \in H. \end{cases}$$

PROPOSITION 5. *Let* M *have the property* (E) (*). *Then for each* ω *in*

(*) See Note at the end of the paper.

$A^p(M)$ *and* θ *in* $A^q(M)$ *we have*

$$(1.8) \qquad (\rho(\omega)\cup\rho(\theta))\circ\phi = \rho(\omega\wedge\theta)\circ\phi.$$

PROOF. First we note that, with the above notations, taking into account that ω and θ are alternating, we have:

$$\omega\wedge\theta(\tau) = \Sigma_{K\subset[n],\mathrm{card}K=p}\ \mathrm{sig}(KH)\omega(A_H\tau)\theta(A_K\tau).$$

Then, by (1.6), it is enough to show that for each $\tau\colon D^n \to M$ and $\underline{h} \in D^n$ we have

$$(1.9) \quad (\rho(\omega)\cup\rho(\theta))(\phi(\tau,\underline{h})) = \Pi_{i=1}{}^n\, \Sigma_{K\subset[n],\mathrm{card}K=p}\ \mathrm{sig}(KH)\omega(A_H\tau)\theta(A_K\tau).$$

We obtain easily that

$$A_H\phi(\tau,\underline{h}) = \phi(A_H\tau,\underline{h}_K) \quad \text{with } \underline{h}_K = \{h_i \mid i \in K\},$$

and also that

$$B_K\phi(\tau,\underline{h})(x_1,\ldots,x_q) = (y_1,\ldots y_n)$$

with

$$y_i = \begin{cases} h_i & \text{if} \quad i \in K \\ h_i x_{\phi_H(i)} & \text{if} \quad i \in H, \end{cases}$$

but, applying the Axiom K-L to the map $\rho(\theta)(B_K\phi(\tau,-))\colon D^n \to R$ we will have

$$\rho(\theta)(B_K\phi(\tau,\underline{h})) = \rho(\theta)(A_K\phi(\tau,\underline{h})) + \Sigma_{i\in K}\ h_i a_i$$

for some $a_i \in R$. Hence

$$\rho(\omega)(A_H\phi(\tau,\underline{h}))\rho(\theta)(B_K\phi(\tau,\underline{h}))) = \Pi_{i=1}{}^n\, h_i\omega(A_H\tau)\theta(A_K\tau)$$

which, substituted in (1.7), gives us (1.9). #

We could go further taking into account that

$$(\rho(\omega)\cup\rho(\theta))\circ\phi\colon M^{D^n}\times D^n \to R$$

verifies the conditions in [1], Prop. 14.4.I. Then there exists a unique n-form Ω such that

$$(\rho(\omega)\cup\rho(\theta))(\phi(\tau,\underline{h})) = \prod_{i=1}^{n} h_i\Omega(\tau).$$

Thus $\omega\wedge\theta$ is the unique n-form verifying (1.8).

2. THE DE RHAM HOMOMORPHISM.

Moerdijk and Reyes construct in [6] the de Rham cohomology of M, $H_R(M)$, and the complex of singular chains $(S_\cdot(M),\delta)$. Now, we consider the dual complex which we will note $(S^\cdot(M),\delta)$:

$$S^n(M) = \mathrm{Hom}_R(S_n(M),R), \qquad \delta = -.\delta: S^n(M) \to S^{n+1}(M)$$

(we will identify an element of $S^n(M)$ with a map $f: M^{\Delta_n} \to R$, where

$$\Delta_n = \{(x_0,\ldots,x_n) \in R^{n+1} \mid 0 \leqslant x_i \leqslant 1 \quad \text{and} \quad \Sigma x_i\}$$

is the standard n-simplex). As usually, we can build the singular cohomology, $H_S{}^\cdot(M)$, of M.

By integration of a n-form ω on a n-simplex $\sigma: \Delta_n \to M$ through the map

$$\pi: I^n \to \Delta_n, \quad \pi(t_1,\ldots,t_n) = (1-t_1, t_1(1-t_2),\ldots,t_1\ldots t_p)$$

we define

$$\text{(2.1)} \qquad \rho_R: A^n(M) \to S^n(M), \quad \rho_R(\omega)(\sigma) = \int_\sigma \omega =: \rho(\omega)(\sigma\pi)$$

which is a morphism of complexes (because of Stokes Theorem) and derives in a homomorphism called "de Rham homomorphism":

$$\text{(2.1)} \qquad \rho_R: H_R{}^n(M) \to H_S{}^n(M).$$

The cup product on $S^\cdot(M)$ is defined by

$$f\cup g\,(\sigma) = f(\sigma\alpha^+)g(\sigma\alpha^-)$$

(here $\alpha^+: \Delta_p \to \Delta_n$ is $\alpha^+(\underline{x}) = (\underline{x},\underline{0})$ and $\alpha^-: \Delta_q \to \Delta_n$ is $\alpha^-(\underline{x}) = (\underline{0},\underline{x})$), and induces a product on the de Rham cohomology (because of Leibniz's formula) which we will write:

$$\text{ext}: H_S{}^p(M)\times H_S{}^q(M) \to H_S{}^{p+q}(M).$$

THEOREM 1. *The de Rham homomorphism* ρ_R: $H_R{}^{\bullet}(M) \to H_S{}^{\bullet}(M)$ *is multiplicative, that is, it commutes with* ext *and* cup.

PROOF (Sketch). It is sufficient to get a homotopy ϕ: $\mathrm{cup}\circ\otimes\rho_R \to \rho_R\circ\mathrm{ext}$, that is a family of R-linear maps

$$\phi_n = \phi_n(M): (\otimes A)^n(M) \to S^{n-1}(M)$$

natural in M and verifying

$$\delta_{n-2}\phi_{n-1} + \phi_n \tilde{d}_{n-1} = \rho_R\circ\mathrm{ext} - \mathrm{cup}\circ\otimes\rho_R \ .$$

We build ϕ_n by induction beginning with $\phi_1 = 0$. Now, if $\phi_{n-1}(X)$ exists, let

$$\mu_{n-1}(X): (\otimes A)^{n-1}(X) \to S^{n-1}(X)$$

be given by

$$\mu_{n-1}(X) = -\delta_{n-2}\phi_{n-1}(X) + \rho_R\circ\mathrm{ext} - \mathrm{cup}\circ\otimes\rho_R \ ,$$

and, for every $\Omega \in (\otimes A)^n(M)$ and σ: $\Delta_{n-1} \to M$, we define

$$\phi_n(M)(\Omega)(\sigma) = [(\mu_{n-1}(\Delta_{n-1})\circ \tilde{K}(\Delta_{n-1})\circ\otimes\sigma^*)(\Omega)](\mathrm{id}_{\Delta_{n-1}})$$

where

$$\tilde{K}_n(\Delta_{n-1}): (\otimes A)^n(\Delta_{n-1}) \to (\otimes A)^{n-1}(\Delta_{n-1})$$

is a homotopy constructed after a contraction of Δ_{n-1} is fixed and

$$\otimes\sigma^*: (\otimes A)^n(M) \to (\otimes A)^n(\Delta_{n-1})$$

is induced from σ.

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NOTE ADDED IN PROOF.

The referee and the author have subsequently eliminated the need for assuming Property (E) in the formula (1.6) (so in Proposition 5): firstly we reduce the question to the case where ω is a n-form on D^n and τ is the generic tangent id_{D^n}. In this case (1.6) follows considering the restriction of ω to $D(n+1)^n$ and proving that any function g: $(D(n+1))^n \to R$ which is homogeneous and alternating (in a certain sense) is a restriction of the n-form $k\, dx_1 \ldots dx_n$ on R^n for some unique $k \in R^n$.

Note that the technique of the generic n-tangent forces us to abandon property (E), since the object $M = D^n$ does not satisfy such property.

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