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THE RELATION BETWEEN HOMOTOPY LIMITS AND BAUER'S SHAPE SINGULAR COMPLEX

by Hanns THIEMANN

RÉSUMÉ. On montre qu'il existe une équivalence d'homotopie faible naturelle entre le complexe singulier de forme $\bar{S}(Y)$ et holim$\underline{Y}$, où $\underline{Y}$ est une ANR approximation d'un espace compact métrisable Y (Théorème 1). La preuve utilise les propriétés universelles du complexe singulier de forme et de la limite homotopique.

0. INTRODUCTION.

Let *Com* be the category of compact metrizable spaces, then for any $Y \in$ *Com* and ANR approximation $\underline{Y}$ of Y (cf. Definition 1.1) we have the homotopy limit holim$\underline{Y}$ [3, 4] and the geometric realization of the shape singular complex $|\bar{S}(Y)|$ [1]. The purpose of this paper is to provide proofs of the following theorems:

THEOREM 1. *There exists a weak homotopy equivalence*

$$q_Y : |\bar{S}(Y)| \longrightarrow \mathrm{holim}\,\underline{Y}$$

which is natural in the following sense: Let $[f] \in Com_h(Y,Z)$ *be the homotopy class of a continuous mapping* $f \in Com(Y,Z)$, *then there exists an induced*

$$\mathrm{holim}\, f \in Top_h(\mathrm{holim}\,\underline{Y},\ \mathrm{holim}\,\underline{Z})$$

(cf. §2.a) *such that*

$$\mathrm{holim}\, f \cdot [q_Y] = [q_Z] \cdot |\bar{S}(f)|$$

holds.

THEOREM 2. *A compact metric space* Y *is of the same shape as a CW space iff* q_Y *is a homotopy equivalence.*

The proofs are based entirely on universal properties of the functors holim and $\bar{S}(\cdot)$ rather than on any explicit constructions. However in order to verify one direction of Theorem 2 we need a strengthened version of (6) (the necessary link between shape theory and the homotopy limit) the proof of which is deferrred to an Appendix.

In §1 we collect some results on strong shape theory; §2 is devoted to the homotopy limit and the shape singular complex, while §3 contains the proofs of Theorems 1 and 2.

REMARK 1. The author of [7] introduces the complex $\bar{S}_c(Y)$ which turns out to be homotopy equivalent to $\bar{S}(Y)$. As a consequence from Theorem 1, Problem 2' in [7] has an affirmative answer.

REMARK 2. Apart from Theorem 2 there exist various other results concerning the stability of compacta (see e.g. [5]).

1. STRONG SHAPE THEORY.

Strong shape (for arbitrary, resp. compact metric spaces) was introduced independently and by different methods by F.W. Bauer [1], and D.A. Edwards and H.M. Hastings [6]. In the meantime it has turned out that for compact metric spaces, on the homotopy level, both approaches to strong shape theory are equivalent, i.e., we have a natural isomorphism

(1) $$\overline{Top_h}(X,Y) \approx ssh(X,Y), \quad X, Y \in Com$$

([8], Remark 3.1, 3.3) where $\overline{Top_h}$ and ssh denote the strong shape categories of [1] and [6].

We note here that $\overline{Top_h}(X,Y)$ consists of homotopy classes of strong shape morphisms $\bar{f} \in \overline{Top}(X,Y)$ (which are defined to be 2-functors $\bar{f}\colon P_Y \to P_X$, P_Y and P_X being 2-categories of ANR spaces over X resp. Y (cf. [1,2]).

We need:

1.1. DEFINITION. Let Y be a compact metric space, then an ANR *approximation of* Y is a tower

$$\underline{Y} = \{Y_0 \supset Y_1 \supset \ldots\}$$

of ANR subspaces of some space $Q \supset Y$ satisfying

$$\bigcap_{i=0}^{\infty} Y_i = Y.$$

Observe that for every compactum an ANR approximation exists (e.g. [1], p. 31, Example 1).

1.2. PROPOSITION. *Suppose we have* $Y \in Com$ *and an ANR approximation* $\underline{Y}$ *of* Y, *then we obtain a bijection (CW = category of CW spaces):*

(2) $\overline{Top_h}(X,Y) \approx \textit{Ho-pro-Top}(c(X),\underline{Y})$, $X \in Com$ *or* $X \in CW$,

which is natural with respect to continuous mappings in the first variable (c(X) denoting the trivial pro-object associated with X).

PROOF. a) Let X be a compact metrizable space, $\underline{X}$ an arbitrary ANR approximation; then we have the following natural bijections:

(3) $ssh(X,Y) \approx \textit{Ho-pro-Top}(\underline{X},\underline{Y})$ ([5], 6.5.5)

(4) $\textit{Ho-pro-Top}(\underline{X},\underline{Y}) \approx \textit{Ho-pro-Top}(c(X),\underline{Y})$ ([4], 2.16).

As a result the bijection (2) follows from (1), (3) and (4).

b) If X is a CW space then (3) serves as a definition for $ssh(X,Y)$ (setting $\underline{X} = c(X)$). The proof of (2) consists in an immediate translation of the proof of (1) given in [8].

2. REMARKS ON holim AND THE SHAPE SINGULAR COMPLEX.

a) The homotopy limit.

The homotopy limit for an inverse system $\underline{Y}$ has been introduced by A.K. Bousfield and D.M. Kan in [3] as a substitute for the (non-existing) inverse limit in the homotopy category. More specifically we need the following natural bijection:

(5) $\textit{Ho-pro-Top}(c(X),\underline{Y}) \approx Top_h(X,\mathrm{holim}\,\underline{Y})$ ([6], 4.2.12)

i.e., the homotopy limit

$$\text{holim}: \textit{Ho-pro-Top} \longrightarrow \textit{Top}_h$$

is a right adjoint to the functor

$$c: \textit{Top}_h \longrightarrow \textit{Ho-pro-Top}.$$

Let Y be a compact metrizable space, $\underline{Y}$ any ANR approximation of Y (Definition 1.1), then it turns out that the homotopy type of holim$\underline{Y}$ does not depend on the choice of $\underline{Y}$. Let $f \in \textit{Com}(Y,Z)$ be continuous, then we find a morphism

$$\underline{f} \in \textit{Ho-pro-Top}(\underline{Y},\underline{Z})$$

(only depending on the homotopy class of f) which induces a homotopy class

$$\text{holim} f \in \textit{Top}_h(\text{holim}\, Y,\ \text{holim}\, Z).$$

The bijection

$$(6) \qquad \varphi: \textit{Top}_h(X, \text{holim}\,\underline{Y}) \longrightarrow \overline{\textit{Top}_h}(X,Y), \quad X \in \textit{Com} \text{ or } X \in \textit{CW},\ Y \in \textit{Com},$$

which is natural with respect to continuous mappings is an immediate consequence of (2) and (5).

b) The shape singular complex.

The strong shape category $\overline{\textit{Top}}$ operates with individual strong shape morphisms (rather than with their homotopy classes). This turns out to be essential for establishing a shape singular complex $\bar{S}(Y)$, $Y \in \textit{Top}$; in complete analogy to the classical case an n-simplex $\bar{\sigma}^n \in \bar{S}(Y)_n$ is a shape morphism $\bar{\sigma}^n \in \overline{\textit{Top}}(\Delta^n,Y)$. Equipped with the ordinary boundary and degeneracy operators

$$\bar{S}: \textit{Top} \to S_E \quad (= \text{category of Kan complexes})$$

becomes a functor and there is a natural transformation

$$\bar{\omega}_Y: |\bar{S}(Y)| \to Y \quad ([1],\ 3.5).$$

Let P be a CW space, then $\bar{\omega}_Y$ induces a natural bijection

$$(7) \qquad \bar{\omega}_{Y*}: \overline{\textit{Top}_h}(P,|\bar{S}(Y)|) \longrightarrow \textit{Top}_h(P,Y).$$

While the injectivity of $\bar{\omega}_{Y*}$ follows by standard arguments of ordinary homotopy theory, the surjectivity is the subject of [1], Theorem 3.6.

3. PROOFS.

Proof of Theorem 1.

Suppose X is either a CW space or compact metric, then (6) ensures the existence of a mapping

$$\eta_{X,Y} = \eta = (\varphi_{X,Y})^{-1}(\bar{\omega}_Y)_*,$$

$$\text{(8)} \qquad \eta\colon \mathit{Top}_h(X,|\bar{S}(Y)|) \longrightarrow \mathit{Top}_h(X,\operatorname{holim}\underline{Y}),\ Y \in \mathit{Com}.$$

We observe that:

1. η is natural with respect to continuous maps

and 2. according to (7) η is a bijection whenever $X \in CW$.

So we are allowed to choose

$$(q_Y\colon |\bar{S}(Y)| \longrightarrow \operatorname{holim}\underline{Y}) \in \eta_{|\bar{S}(Y)|,Y}(1_{|\bar{S}(Y)|})$$

rendering the following diagram commutative

$$\text{(9)} \qquad \begin{array}{ccc} \mathit{Top}_h(X,|\bar{S}(Y)|) & \xrightarrow{\ \approx\ } & \overline{\mathit{Top}_h}(X,|\bar{S}(Y)|) \\ \Big\downarrow (q_Y)_* & & \Big\downarrow (\bar{\omega}_Y)_* \\ \mathit{Top}_h(X,\operatorname{holim}\underline{Y}) & \xrightarrow{\ \varphi_{X,Y}\ } & \overline{\mathit{Top}_h}(X,Y) \end{array}$$

a) q_Y *is a weak homotopy equivalence.* Since because of (9) q_Y induces the bijection $\eta_{P,Y}$ ($P \in CW$) the result follows from [9], Chapter IV, 7.17.

b) q_Y *is natural.* Let $f \in \mathit{Com}(Y,Z)$ be continuous, then we show:

$$\operatorname{holim} f \cdot [q_Y] = [q_Z][|\bar{S}(f)|].$$

We have

$$\varphi_{|\bar{S}(Y)|,Y}([q_Y]) = \bar{\omega}_Y \text{ and } \varphi_{|\bar{S}(Z)|,Z}([q_Z]) = \bar{\omega}_Z.$$

The naturality of φ implies

$$\varphi_{|\bar{S}(Y)|,\operatorname{holim}\underline{Z}}(\operatorname{holim} f \cdot [q_Y]) = [f]\cdot\varphi_{|\bar{S}(Y)|,\operatorname{holim}\underline{Y}}([q_Y]) = [f]\cdot\bar{\omega}_Y,$$

$$\varphi_{|\bar{S}(Y)|,\operatorname{holim}\underline{Z}}([q_Z][|\bar{S}(f)|]) = \varphi_{|\bar{S}(Z)|,\operatorname{holim}\underline{Z}}([q_Z])[|\bar{S}(f)|] = \bar{\omega}_Z \cdot [|\bar{S}(f)|].$$

Because $\bar{\omega}_\cdot$ is natural we have

$$\varphi_{|\bar{S}(Y)|,\mathrm{holim}\underline{Z}}(\mathrm{holim}\, f\circ [q_Y]) = \varphi_{|\bar{S}(Y)|,\mathrm{holim}\underline{Z}}([q_Z][|\bar{S}(f)|]).$$

The assertion follows now since $\varphi_{|\bar{S}(\ )|,\mathrm{holim}\underline{Z}}$ is injective.

Proof of Theorem 2.

a) Suppose that q_Y is a homotopy equivalence, then $(\bar{\omega}_Y)_*$ in (9) turns out to be a bijection for $X \in Com$ or $X \in CW$, which implies that $\bar{\omega}_Y$ is a shape equivalence.

b) Assume that Y is of the same shape as a CW space P, then $\bar{\omega}_Y : |\bar{S}(Y)| \to Y$ is a shape equivalence. On the other hand (6') in the Appendix yields a commutative diagram (9) but now also for $X \in Top$. So $(q_Y)_*$ is a bijection (for all admissible X, in particular for $X = \mathrm{holim}\,\underline{Y}$) and the assertion follows from the Yoneda Lemma.

4. APPENDIX.

The following Proposition 4.1 asserts that (6) in §2 is also valid for any $X \in Top$. The proof does not depend on [4,6,8]. Proposition 4.1 is needed in the proof of Theorem 2.

PROPOSITION 4.1. *Let $Y \in Com$ be a space, $\underline{Y}$ any ANR approximation of Y, then for all spaces $X \in Top$ one has a natural bijection*

(6') $$\overline{Top_h}(X, \mathrm{holim}\,\underline{Y}) \approx Top_h(X, Y).$$

The proof uses the following notations:

1. To each tower $\underline{Y}$ there exists a tower

$$\underline{\tilde{Y}} = \mathrm{Ex}^{\infty}\underline{Y} = \{\tilde{Y}_n, p_n\}$$

with Hurewicz-fibrations p_n as bonding maps. Moreover there are SDR-mappings a_n (i.e., one has a mapping $r_n : \tilde{Y}_n \to Y_n$ with $r_n a_n = 1$ and a homotopy $a_n r_n \approx 1$ rel a_n) fitting into the commutative diagram (10) hereafter (cf. [6], §4.3; [4], 2.5).

2. The objects of the category $\overline{Top}^{N}$ are defined to be towers

$$\underline{X} = \{X_n, \xi_n\}\ ,\ \xi_n \in Top(X_{n+1}, X_n).$$

A morphism

$$\underline{\bar{f}} = \{(f_n, \varphi_n)\} \in \overline{Top}^{\mathrm{N}}(\underline{\mathrm{X}},\underline{\mathrm{Y}}),\ \underline{\mathrm{Y}} = \{\mathrm{Y}_n, \upsilon_n\}$$

is defined to be a sequence of mappings $f_n: \mathrm{X}_n \to \mathrm{Y}_n$ and homotopies φ_n: $\upsilon_n f_{n+1} \approx f_n \xi_n$ (cf. [2]). The related homotopy category $\overline{Top}^{\mathrm{N}}_h$ is established by using mappings

$$\underline{\bar{\mathrm{H}}} \in \overline{Top}^{\mathrm{N}}(\underline{\mathrm{X}}\times \mathrm{I},\underline{\mathrm{Y}}) \quad (\underline{\mathrm{X}}\times\mathrm{I} = \{\mathrm{X}_n\times\mathrm{I}, \xi_n\times 1_{\mathrm{I}}\})$$

as homotopies.

$$(10)\qquad \begin{array}{ccccc} \mathrm{Y}_\mu & \xleftarrow{\ i_\mu\ } & \mathrm{Y}_1 & \xleftarrow{\ i_1\ } & \mathrm{Y}_2 \\ \downarrow a_\mu & & \downarrow a_1 & & \downarrow a_2 \\ \tilde{\mathrm{Y}}_\mu & \xleftarrow{\ p_\mu\ } & \tilde{\mathrm{Y}}_1 & \xleftarrow{\ p_1\ } & \tilde{\mathrm{Y}}_2 \end{array}$$

Proof of Proposition 4.1.

Step 1. We have a natural bijection

$$(11)\qquad Top_h(\mathrm{X},\mathrm{Y}) \approx \overline{Top}^{\mathrm{N}}_h(c(\mathrm{X}),\underline{\mathrm{Y}}).$$

PROOF. It is well-known [1,2] that every $\bar{f} \in \overline{Top}(\mathrm{X},\mathrm{Y})$, $\mathrm{Y} \in Com$, is induced up to homotopy by a functor $\mathrm{T}: \mathrm{P}_{\dot{\mathrm{Y}}} \to \mathrm{P}_{\mathrm{X}}$ ($\mathrm{P}_{\dot{\mathrm{Y}}}$ being in the terminology of [1] a st-category). This fact reveals itself simply as a reformulation of (11).

Step 2. We have a natural bijection

$$(12)\qquad \overline{Top}^{\mathrm{N}}_h(c(\mathrm{X}),\underline{\mathrm{Y}}) \approx \overline{Top}^{\mathrm{N}}_h(c(\mathrm{X}),\underline{\tilde{\mathrm{Y}}}).$$

PROOF. We associate with $\{a_n: \mathrm{Y}_n \to \tilde{\mathrm{Y}}_n\} \in pro\text{-}Top(\underline{\mathrm{Y}},\underline{\tilde{\mathrm{Y}}})$ (cf. (10)) a morphism

$$\bar{a} = \{(a_n, \underline{1}: p_n a_{n+1} = a_n i_n)\}$$

($\underline{1}$ = stationary homotopy) in $\overline{Top}^{\mathrm{N}}(\underline{\mathrm{Y}},\underline{\tilde{\mathrm{Y}}})$ and conclude:

LEMMA 4.2. $[\bar{a}] \in \overline{Top}^{\mathrm{N}}_h(\underline{\mathrm{Y}},\underline{\tilde{\mathrm{Y}}})$ *is an isomorphism.*

PROOF. Since a_n is an SDP-mapping, we find a homotopy ω_n: $a_n r_n \approx 1$ rel a_n (for suitable r_n, satisfying $r_n a_n = 1$). It turns out that $\bar{r} := \{(r_n, r_n p_n \omega_{n+1})\}$ becomes a homotopy inverse of $\bar{a}$: The relation $\bar{r}\bar{a} = \underline{1}$ is immediate. The verification of $\bar{a}\bar{r} = \underline{1}$ in $\overline{Top}^{\mathrm{N}}$ consists in the detection of mappings H_n: $\underline{\tilde{\mathrm{Y}}}_{n+1}\times\mathrm{I}\times\mathrm{I} \to \underline{\tilde{\mathrm{Y}}}_n$

satisfying

$$H_n(y,s,0) = p_n\omega_{n+1}(y,s) = \omega_n(p_n\omega_{n+1}(y,s),1),$$
$$H_n(y,s,1) = \omega_n(p_n(y),s) = \omega_n(p_n\omega_{n+1}(y,1),s),$$
$$H_n(y,0,t) = a_n r_n p_n\omega_{n+1}(y,t) = \omega_n(p_n\omega_{n+1}(y,t),0),$$
$$H_n(y,1,t) = p_n(y) = \omega_n(p_n\omega_{n+1}(y,1),1)$$

on the boundary. This mapping $B_n: \underline{\tilde{Y}}_{n+1}\times \mathrm{bd}(I\times I) \to \underline{\tilde{Y}}_n$ is homotopic to $B'_n: \underline{\tilde{Y}}_{n+1}\times \mathrm{bd}(I\times I) \to \underline{\tilde{Y}}_n$ defined by

$$B'_n(y,s,t) = a_n r_n p_n \omega_{n+1}(y,\max(s,t)),$$

which obviously admits an extension over $\underline{\tilde{Y}}_{n+1}\times I\times I$, furnishing us with the required H_n. The homotopy $\underline{\bar{H}}$: $\bar{a}\ \bar{r} \approx \underline{1}$ is nothing else than $\underline{\bar{H}} = \{(\omega_n, H_n)\}$.

The assertion (12) follows because the required bijection is induced by $\bar{a}$.

Step 3. We have a natural bijection

$$\overline{Top}^{N}_h(c(X),\underline{\tilde{Y}}) \approx Top_h(X, \mathrm{holim}\,\underline{Y}). \tag{13}$$

PROOF. According to [6], 4.2.10 we have $\mathrm{holim}\,\underline{Y} = \lim \underline{\tilde{Y}}$ (lim = inverse limit). Suppose that

$$\bar{f} = \{(f_n,\varphi_n)\} \in \overline{Top}^{N}(c(X),\underline{\tilde{Y}})$$

then we define an assignment

$$\mu(\bar{f}) = g = \lim(g_n : X \to \tilde{Y}_n) \in Top(X, \mathrm{holim}\,\underline{Y})$$

as follows: We set $g_0 = f_0$ Since p_0 is a Hurewicz fibration we can lift the homotopy φ_0: $p_0 f \approx f_0 = g_0$ thus obtaining a Φ_0: $X\times I \to \underline{\tilde{Y}}$ and set $g = \Phi_0(\cdot,1)$. The construction of g is now performed by induction on n. The assignment

$$\mu: \overline{Top}^{N}(c(X),\underline{\tilde{Y}}) \longrightarrow Top(X, \mathrm{holim}\,\underline{Y})$$

respects (by virtually the same argument) homotopies so that we come to an assignment

$$\tilde{\mu} : \overline{Top}^{N}_h(c(X),\underline{\tilde{Y}}) \longrightarrow Top_h(X, \mathrm{holim}\,\underline{Y}).$$

There exists an immediate transformation

$$\lambda: Top(X, \mathrm{holim}\,\underline{Y}) \longrightarrow \overline{Top}^{N}(c(X),\underline{\tilde{Y}})$$

inducing a

$$\tilde{\lambda} : Top_h(X, \mathrm{holim}\,\underline{Y}) \longrightarrow \overline{Top}^{N}_h(c(X),\underline{\tilde{Y}}).$$

The proof of $\tilde{\lambda}\ \tilde{\mu} = 1$, $\tilde{\mu}\ \tilde{\lambda} = 1$ is now purely technical.

As a result Proposition 4.1 follows from (11)-(13).

REFERENCES.

1. F.W. BAUER, A shape theory with singular homology, *Pacific J. Math.* 64 (1976), 25-65.
2. F.W. BAUER, Duality in manifolds, *Ann. di Mate. pura ed appl.* 136 (1984), 241-302.
3. A.K. BOUSFIELD & D.M. KAN, Homotopy limits, completions and localizations, *Lecture Notes in Math.* 304, Springer (1972).
4. F.W. CATHEY, Strong shape theory, *Lecture Notes in Math.* 870, Springer (1981), 215-238.
5. D.A. EDWARDS & R. GEOGHEGAN, Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction, *Ann. of Math.* 101 (1975), 521-535. Correction 104 (1976), 389.
6. D.A. EDWARDS & H.M. HASTINGS, Čech and Steenrod homotopy theories with applications to geometric topology, *Lecture Notes in Math.* 542, Springer (1976).
7. A. KOYAMA, Coherent singular complexes in strong shape theory, *Tsukuba J. Math.* 5-2 (1984), 261-293.
8. YU.T. LISICA, Strong shape theory and Steenrod-Sitnikov homology (Russian), *Sibirski Mat. Z.* 24 (1983), 81-99.
9. G.W. WHITEHEAD, *Elements of Homotopy Theory*, Berlin 1978.

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