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J. ROSICKÝ

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MULTIPLICATIVE LATTICES AND C*-ALGEBRAS

by J. ROSICKÝ

RÉSUMÉ. Soit R(A) le treillis des idéaux à droite fermés d'une C*-algèbre A. D'après la dualité de Gelfand, toute C*-algèbre commutative A est uniquement déterminée par R(A). On a cherché à étendre ce résultat au cas non-commutatif en munissant R(A) d'une structure supplémentaire [2,7,11,3 et 4]). On poursuit ici ces recherches en munissant R(A) d'une multiplication $I \circ J = \overline{I \cdot J}$. Les treillis multiplicatifs ainsi obtenus sont appelés "quantum frames", et leurs propriétés sont étudiées.

Our aim is to introduce a new structure on the lattice R(A) of all closed right ideals of a C*-algebra A. Recall that a C*-algebra is a complex Banach algebra A with an involution * satisfying

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

By the Gelfand duality, commutative C*-algebra correspond to locally compact Hausdorff spaces ($\hat{A}$ will denote the topological space determined by A). Since R(A) is isomorphic to the lattice $O(\hat{A})$ of all open sets of $\hat{A}$, a commutative C*-algebra A is completely determined by R(A). In the non-commutative case, the situation is much more complicated and the search for satisfactory invariants is still continuing. The lattice R(A) may be considered as a non-commutative substitute of a topological space but it does not determine A. It motivates efforts to consider R(A) as a richer structure than a mere lattice. C.J. Mulvey [11] equipped R(A) with the natural multiplication $\overline{I \cdot J}$ of closed right ideals. The resulting multiplicative lattices were called quantales and their theory was further developped in [3] and [4]. The second paper proves that any unital postliminary C*-algebra can be reconstructed from its quantale. We carry this investigation forward by considering another multiplication

$\circ$ on R(A): $I \circ J = \overline{I^* . J}$.

Recall that a *frame* is a complete lattice satisfying the distributivity law

$$x \wedge \bigvee y_i = \bigvee x \wedge y_i$$

(see [8]). Open set lattices of topological spaces are frames. For instance, R(A) is a frame for any commutative C*-algebra A. Quantales were introduced as a non-commutative generalization of frames. A *quantale* is a complete lattice Q equipped with an associative and idempotent multiplication $\cdot$ such that

$$x.1 = x, \ x.\bigvee_{i \in I} x_i = \bigvee_{i \in I} x.x_i \text{ and } (\bigvee_{i \in I} x_i).x = \bigvee_{i \in I} x_i.x$$

for any $x, x_i \in Q$, $i \in I$ (1 is the top element of Q, 0 is the bottom one). Since the concept of a quantale was varying, we point out that our quantales are idempotent. Frames coincide with commutative quantales ($\cdot$ equals to $\wedge$). For any C*-algebra A, R(A) is a quantale. Besides many nice features, quantales have some deficiencies. For instance, one cannot require separation properties because any regular quantale is automatically commutative [13]. We introduce *quantum frames* as quantales together with a new multiplication $\circ$ subjected to some axioms. Any frame is a quantum frame with $\circ$ taken as $\wedge$. Frames coincide with quantum frames having $\circ$ idempotent. Any R(A) is a quantum frame. Separation axioms can be imposed upon quantum frames and, analogously as in the commutative case, R(A) is completely regular for any A and normal for A unital. Moreover, R(A) is compact iff A is unital. Like discrete frames are connected with Boolean algebras, discrete quantum frames lead to ortholattices, i.e., to the usual quantum logic.

Even the quantum frame R(A) does not determine A. The reason is that R(A) is an invariant of the Jordan algebra A_h of hermitian elements of A (non-isomorphic C*-algebras may have isomorphic Jordan parts). The author does not know whether the quantum frame R(A) determines the Jordan algebra A_h. Following C.A. Akemann [2] and R. Giles, H. Kummer [7], a unital C*-algebra A can be reconstructed from an embedding of R(A) into a bigger lattice P which is given by irreducible representations of A. In the commutative case, it is the embedding of $O(\hat{A})$ into the lattice $2^{\hat{A}}$ of all subsets of $\hat{A}$. In a general case, P is a discrete quantum frame but it is not determined by the quantum frame R(A).

1. QUANTUM FRAMES.

Recall that an element x of a quantale Q is called *2-sided* if $1\cdot x = x$. The set $\tilde{Q}$ of all 2-sided elements of Q is a complete sublattice of Q closed with respect to the multiplication. Hence for any $x \in Q$ there is the smallest 2-sided element $\hat{x} \geq x$. It holds

(1) $$x.y = x \wedge \hat{y}.$$

Moreover, there is a one-to-one correspondence between quantale multiplications on a complete lattice Q and complete sublattices $\tilde{Q} \subset Q$ satisfying

(2) $$(\bigvee_{i\in I} x_i) \wedge a = \bigvee_{i\in I} x_i \wedge a \qquad \text{for } x_i \in Q,\ a \in \tilde{Q},$$

(3) $$x \wedge \bigvee_{i\in I} a_i = \bigvee_{i\in I} x \wedge a_i \qquad \text{for } x \in Q,\ a_i \in \tilde{Q},$$

(4) $$\widehat{x \wedge a} = \hat{x} \wedge a \qquad \text{for } x \in Q,\ a \in \tilde{Q}$$

(see [3]). Further, $\tilde{Q}$ is a frame.

1.1. DEFINITION. A *quantum frame* is a quantale Q together with a new multiplication $\circ$ satisfying:

(QF1) $x \circ y \in \tilde{Q}$ for $x,y \in Q$.

(QF2) $a \circ b = a.b$ for $a,b \in \tilde{Q}$,

(QF3) $x \circ y = y \circ x$ for $x,y \in Q$,

(QF4) $x \circ \bigvee_{i\in I} y_i = \bigvee_{i\in I} x \circ y_i$ for $x,y_i \in Q$, $i \in I$,

(QF5) $x \leq x \circ x$ for $x \in Q$.

(QF6) $(x \circ a) \circ y = x \circ (a \circ y)$ for $x,y \in Q$, $a \in \tilde{Q}$,

(QF7) $(x \circ y).a = (x.a) \circ (y.a)$ for $x,y \in Q$, $a \in \tilde{Q}$.

An immediate consequence of the first two axioms is

(5) $$\tilde{Q} = \{x \circ y \mid x,y \in Q\}.$$

Hence the multiplication $\circ$ determines 2-sided elements and therefore the quantale multiplication, as well. We could formulate Definition 1.1 without mentioning $\cdot$ at all and $\cdot$ was used for the convenience only. Another easy consequence is that

(6) $$x \wedge y \leq x \circ y \leq \hat{x} \wedge \hat{y}$$

for any $x,y \in Q$

$$(x \wedge y \leq (x \wedge y) \circ (x \wedge y) \leq x \circ y \leq \hat{x} \circ \hat{y} = \hat{x} \wedge \hat{y}).$$

Further

(7) $$\hat{x} = 1 \circ x = x \circ x$$
because $x \leq x \circ x \in \tilde{Q}$, i.e.,

$$\hat{x} \leq x \circ x \leq \hat{x} \wedge \hat{x} = \hat{x}.$$

Consequently, if $\circ$ is idempotent then it coincides with $\wedge$. Hence idempotent quantum frames coincide with frames.

1.2. LEMMA. *The following conditions are equivalent for any quantum frame* Q :

(i) Q *is associative.*

(ii) Q *satisfies the equation* $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$.

(iii) $x \circ y = \hat{x} \wedge \hat{y}$ *for any* $x, y \in Q$.

PROOF. Evidently (iii) ⇒ (i), (ii).

(ii) ⇒ (iii):

$$\hat{x} \wedge \hat{y} = (1 \circ x).(1 \circ y) = 1 \circ (x \circ y) = x \circ y.$$

(i) ⇒ (iii):

$$\hat{x} \wedge \hat{y} = (1 \circ x).(1 \circ y) = x \circ y.$$

1.3. LEMMA. *In the definition of a quantum frame, the sixth axiom can be weakened to:*

(QF6') $$(x \circ 1) \circ y \leq x \circ (1 \circ y) \qquad \text{for } x, y \in Q.$$

PROOF. Using (QF6') and (QF3) we get

$$x \circ (1 \circ y) = (y \circ 1) \circ x \leq y \circ (1 \circ x) = (x \circ 1) \circ y.$$

Therefore

$$(x \circ 1) \circ y = x \circ (1 \circ y).$$

Now

$$(x \circ a) \circ y = ((x \circ (a \circ 1)) \circ 1) \circ y = ((x \circ 1) \circ a) \circ (1 \circ y) =$$
$$= x \circ (1 \circ ((a \circ 1) \circ y)) = x \circ (a \circ y).$$

Let A be a C*-algebra and denote by R(A) its lattice of closed right ideals. Following [3], R(A) is a quantale with I·J being the closed right ideal generated by ab, $a \in I$, $b \in J$. Let $I \circ J$ be the closed right ideal generated by $a^* b$, $a \in I$, $b \in J$. In fact, $I \circ J = \overline{I^* J}$ is the usual product of a closed left ideal I^* with a closed right ideal J.

1.4. PROPOSITION. R(A) *is a quantum frame for any* C*-*algebra* A.

PROOF. (QF1) is evident because $\overline{I^* J}$ is 2-sided. (QF2) follows

by the self-adjointness of any 2-sided ideal.
(QF3) Since the involution is continuous in the norm topology, we have

$$I \circ J = \overline{I^*J} = \overline{I^*J}^* = \overline{(I^*J)^*} = \overline{J^*I} = J \circ I.$$

(QF4) only needs that $x\cdot -$ is continuous in the norm topology.
(QF5) Taking a left approximate unit u_t for I (see [5]), we obtain that

$$x \in \overline{\{u_t x\}} \subset I \circ I \text{ for any } x \in I.$$

(QF6) For K 2-sided, one has

$$(I \circ K) \circ J = (K \circ I) \circ J = \overline{(\overline{KI})^* J} = \overline{I^*KJ} = I \circ (K \circ J).$$

(QF7) Let K be 2-sided and u_t its left approximate unit. Then

$$(I.K) \circ (J.K) = \overline{KI^*JK} \subset \overline{I^*JK} = (I \circ J).K$$

and

$$K \in \overline{\{u_t x\}} \subset \overline{KI^*JK} \text{ for any } x \in \overline{I^*JK}.$$

Let A be a W*-algebra of operators in a Hilbert space H and $r(A)$ its lattice of weakly closed right ideals. Replacing the norm closure $^-$ by the weak closure cl we get multiplications $\cdot$ and $\circ$ on $r(A)$ (i.e.,

$$I \cdot J = \mathrm{cl}(IJ) \text{ and } I \circ J = \mathrm{cl}(I^*J)).$$

1.5. PROPOSITION. *$r(A)$ is a quantum frame for any W*-algebra A.*

Proof is analogous to that of 1.4 (the involution and translations $x\cdot -$ are weakly continuous, see [6]).

Let A be a C*-algebra and A_h its set of hermitian elements. For $a,b \in A_h$, let $a \circ b = (ab+ba)/2$ be their Jordan product.

1.6. LEMMA. *Let $I,J \in R(A)$. Then $I \circ J$ is the closed ideal generated by $a \circ b$, $a \in I_h$, $b \in J_h$.*

PROOF. Denote by K the closed ideal generated by $a \circ b$, $a \in I_h$, $b \in J_h$. Since

$$a \circ b = ab + (ab)^* \in I \circ J$$

for $a \in I_h$, $b \in J_h$, we have $K \subset I \circ J$. Let $a \in I^+$, $b \in J^+$ be positive elements. Then $a^{1/2} \in I^+$ and $b^{1/2} \in J^+$ (see [5], 12.4). Hence K contains elements

$$a^{1/2}b+ba^{1/2},\quad a^{1/2}(a^{1/2}b+ba^{1/2}) = ab+a^{1/2}ba^{1/2},$$
$$(a^{1/2}b+ba^{1/2})a^{1/2} = a^{1/2}ba^{1/2}+ba,$$
$$(a^{1/2}ba^{1/2}+ba)-(ab+ba) = a^{1/2}ba^{1/2}-ab,$$
$$(ab+a^{1/2}ba^{1/2})-(a^{1/2}ba^{1/2}-ab) = 2ab.$$

We proved that $ab \in K$ for any $a \in I^+$, $b \in J^+$. The assertion therefore follows by

(8) $I \circ J$ is the closed ideal generated by ab, $a \in I^+$, $b \in J^+$.

To verify (8), let u_s, v_t be left approximate units for I,J resp. and $x \in I$, $y \in J$. Then

$$x^*y \in \overline{\{u_s x\}}^* \, \overline{\{v_t y\}} \subset \overline{\{x^* u_s v_t y\}}\ ,$$

which yields (8).

1.7. PROPOSITION. *Let* A *be a* C**-algebra. The quantum frame* R(A) *is completely determined by the Jordan algebra* A_h.

PROOF. The lattice R(A) is an invariant of the ordered real vector space A_h (see [5] 12.4.1). The result follows by 1.6.

The ordered set of real numbers will be denoted by $\mathbb{R}$. An isotone mapping e: $\mathbb{R} \to L$ into an ordered set L is called *bounded* if $e(a) = 0$ and $e(b) = 1$ for some real numbers $a < b$.

1.8. PROPOSITION. *Let* A *be a unital* C**-algebra. Hermitian elements of* A *correspond to bounded* $\vee$*-preserving mappings* $\mathbb{R} \to R(A)$.

PROOF. Let $\alpha_A: R(A) \to P(\tilde{A})$ be the embedding mentioned in the Introduction. By the spectral Theorem, hermitian elements of $\tilde{A}$ correspond to $\vee$-preserving bounded mappings e: $\mathbb{R} \to P(\tilde{A})$. The fundamental result of Akemann, Giles and Kummer (see [2] and [7]) asserts that e comes from a hermitian element of A iff it factorizes through R(A).

The author is able to prove that the quantum frame R(A) also determines $\|a\|$, ra and a^2 for $a \in A_h$ and $r \in \mathbb{R}$. He does not know whether the sums $a+b$, $a, b \in A_h$ are determined too, i.e. whether the whole Jordan structure of A_h can be reconstructed.

2. FUNCTORIALITY.

A *homomorphism* of quantum frames is a mapping $Q_1 \to Q_2$ preserving $\vee$, 1, $\cdot$ and $\circ$. Any homomorphism $f: A \to B$ of C^*-algebras determines a mapping $R(f): R(A) \to R(B)$ such that $R(f)(J)$ is the closed right ideal generated by $f(J)$ (see [3]). But $R(f)$ does not need to be a homomorphism of quantum frames because 2-sided ideals are not generally preserved. To restore the functoriality we will introduce weak homomorphisms of quantum frames.

2.1. DEFINITION. Let Q_1 and Q_2 be quantum frames. A mapping $h: Q_1 \to Q_2$ will be called a *weak homomorphism* if it preserves $\vee, 1$ and satisfies

(9) $$h(x) \circ h(y) = \widehat{h(x \circ y)} \text{ for any } x, y \in Q_1.$$

Any $\vee, 1$ and $\circ$ preserving mapping is a weak homomorphism. Weak homomorphisms of frames coincide with homomorphisms.

2.2. LEMMA. *Let $f: A \to B$ be a unital homomorphism (i.e., unit preserving) of unital C^*-algebras. Then $R(f): R(A) \to R(B)$ is a weak homomorphism. For f surjective, $R(f)$ is a homomorphism.*

PROOF. It is easy to see that $R(f)$ preserves $\vee$ and 1. (9) follows from the fact that $R(f)(I \circ J)$ is the closed right ideal generated by $f(a)^* f(b)$, $a \in I$, $b \in J$ and $R(f)(I) \circ R(f)(J)$ is similarly generated by

$$(f(a)x)^* f(b) = x^* f(a)^* f(b) \text{ for } x \in B,\ a \in I \text{ and } b \in J,$$

i.e., it is a closed ideal generated by $f(a^*)f(b)$, $a \in I$, $b \in J$.

Let f be surjective. Then $R(f)$ preserves $\cdot$ (see [4]) and since for any $x \in B$ there is a $y \in A$ with $x = f(y)$, $R(f)$ also preserves $\circ$.

It is easy to see that for a weak homomorphism $h: Q_1 \to Q_2$ we have

(10) $$\widehat{h(x)} = \widehat{h(\hat{x})} \text{ for any } x \in Q_1$$

$$(\widehat{h(\hat{x})} = h(x \circ x)^\wedge = (h(x) \circ h(x)) = \widehat{h(x)}).$$

Hence

$$h(x \cdot y) = h(x \wedge \hat{y}) \leq h(x) \wedge h(\hat{y}) \leq h(x) \wedge \widehat{h(y)} = h(x) \cdot h(y).$$

We proved that

$$h(x \cdot y) \leq h(x) \cdot h(y) \tag{11}$$

holds for any weak homomorphism h. Another consequence of (10) is that the composition of weak homomorphisms $h: Q_1 \to Q_2$, $g: Q_2 \to Q_3$ is a weak homomorphism:

$$(gh(x \circ y))^\wedge = g(h(x \circ y)^\wedge)^\wedge = g(h(x) \circ h(y))^\wedge = gh(x) \circ gh(y).$$

Let $\boldsymbol{Q}$ denote the category of quantum frames and weak homomorphisms and $\boldsymbol{C}$ be the category of unital C*-algebras and unital homomorphisms. We have obtained the functor R: $\boldsymbol{C} \to \boldsymbol{Q}$ and the next result gives some of its properties. They follow by [1] and [7]; assertion (b) is III.3 from [1]. Recall that a functor F is faithful if for any morphisms $f, g: A \to B$ from the domain category the implication $F(f) = F(g) \Rightarrow f = g$ holds.

2.3. PROPOSITION. *The functor* R: $\boldsymbol{C} \to \boldsymbol{Q}$ *is faithful and*

(a) R(f) *is one-to-one iff* f *is one-to-one*

(b) R(f) *is surjective iff* f *is surjective.*

PROOF. Let $f: A \to B$ be a unital homomorphism and $a \in A_h$. Consider the corresponding bounded $\vee$-preserving mapping e_a: $\mathbb{R} \to R(A)$ from 1.8. Let $x \in \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(y) = 0$ for $y \geq x$ and $g(y) > 0$ otherwise. Following [7] 3.2, it holds

$$e_a(x) = \overline{g(a)A} \tag{12}$$

where $g(a)$ is defined by the functional calculus. Hence

$$R(f)\, e_a(x) = \overline{f(\overline{g(a)A})B} = \overline{f(g(a)A)B} = \overline{g(f(a))B} = e_{f(a)}(x).$$

We proved

$$R(f)\, e_a = e_{f(a)} . \tag{13}$$

Consequently, R is faithful.

If R(f) is one-to-one then $\ker(f) = 0$ and f must be one-to-one. The converse follows by (13). It is evident that R(f) is surjective for any surjective f. Consider $f: A \to B$ with R(f) surjective. Let

$$A \longrightarrow f(A) \xrightarrow{g} B$$

be the factorization through the inclusion g: $f(A) \to B$. Then R(g) is an isomorphism and (13) yields that g is an isomorphism. Hence f is surjective.

3. REGULARITY.

We say that elements x,y of a quantum frame Q are *disjoint* if $x \circ y = 0$. The notation will be $x \perp y$. By (QF4), for any x in Q there is a greatest element $x^{\perp}$ disjoint to x. The following properties are evident:

(14) $$(\bigvee_{i \in I} x_i)^{\perp} = \bigwedge_{i \in I} x_i^{\perp},$$

(15) $$x \leq x^{\perp\perp},$$

(16) $$a \in \tilde{Q} \Rightarrow a^{\perp} \in \tilde{Q}$$

(concerning (16),

$$a \circ x = 0 \Rightarrow a \circ \hat{x} = a \circ (1 \circ x) = (a \circ 1) \circ x = a \circ x = 0).$$

Having a weak homomorphism h, we get

(17) $$x \perp y \Rightarrow h(x) \perp h(y).$$

The concept of disjointness makes possible to consider topological properties of quantum frames. We will write $x \lhd y$ for $x,y \in Q$ if $x \leq y$ and $y \vee x^{\perp} = 1$. Recall that a set S is directed if for any $s,t \in S$ there is $u \in S$, $s,t \leq u$.

3.1. DEFINITION. Let Q be a quantum frame. An element $y \in Q$ is called *regular* if there is a directed set of elements

$$x_i \lhd y,\ i \in I \text{ such that } y = \bigvee_{i \in I} x_i.$$

Q is called *regular* if any of its elements is a directed join of regular elements.

3.2. REMARKS. (a) For frames, we get the usual concept of regularity (see [8]) extending that of topological spaces.

(b) There is a weaker concept of regularity for quantum frames (any element y is a join of elements $x \lhd y$) and a stronger one (any element is regular).

(c) The disjointness based on the quantale multiplication (i.e., $x \cdot y = 0$) is too strong. Using it, any regular quantale would be a frame (see [13]).

We will write $x \lhd\!\lhd y$ for $x,y \in Q$ if there are elements z_t for any rational number $0 \leq t \leq 1$ such that

$$x \leq z_{\mu},\ z_1 \leq y \text{ and } z_s \lhd z_t \text{ for any } s < t.$$

Using $\lhd\!\lhd$ instead of $\lhd$ we will get the concept of a *completely regular* quantum frame. Any completely regular quantum frame is regular and we get the usual complete regularity for frames (see [8]).

3.3. PROPOSITION. *The quantum frame* R(A) *is completely regular for any* C*-*algebra* A.

PROOF. Any C*-algebra A is a closed ideal in a unital C*-algebra A' derived from A by adjoining a unit. It is easy to see that the mapping

$$u\colon R(A') \to R(A) \text{ given by } u(J) = J \cap A$$

is a homomorphism of quantum frames. Since a homomorphic image of a completely regular quantum frame is completely regular, it suffices to prove 3.3 for unital C*-algebras.

Let A be a unital C*-algebra. If $x,y \in A^+$ and $x \le y$ then

$$\overline{xA} \subset \overline{yA}$$

by [5] 12.4.1. Hence the set

$$\{\overline{xA} \mid x \in J^+\}$$

is directed for any $J \in R(A)$. Since

$$J = \bigvee_{x \in J^+} \overline{xA}$$

(see [5] 2.9.3), it remains to prove that any element

$$\overline{xA} \in R(A)\ ,\ x \in A^+$$

is completely regular. Let B be a sub-C*-algebra of A generated by x and 1. Since B is commutative, R(B) is completely regular. Following 2.2, $R(h)\colon R(B) \to R(A)$ is a weak homomorphism where $h\colon B \to A$ denotes the inclusion. It is an immediate consequence of (17) that weak homomorphisms preserve complete regularity of elements. Since $\overline{xA}$ belongs to the image of $R(h)$, it is completely regular.

An element x of a complete lattice L is called *compact* if for any directed set S with $x \le \bigvee S$, there is an $s \in S$ such that $x \le s$. L is called *compact* if 1 is compact.

3.4. PROPOSITION. *A* C*-*algebra* A *is unital iff* R(A) *is compact.*

PROOF. Clearly, R(A) is compact for A unital. Let R(A) be compact. Then A is a compact 2-sided element of R(A') (see the proof of 3.3). Since R(A') is regular, A is a directed join of regular elements of R(A'). Using compactness, we get that A is a regular element of R(A'). Repeating the argument we obtain that $A \lhd A$ in R(A'), i.e., $A \vee A^{\perp} = A'$. Assume that $A^{\perp} \subset A$. Then $A^{\perp} = 0$, i.e., $A = A'$, which is impossible. Hence $A^{\perp} \not\subset A$ and by [12] §7.4.VI, $0 = A \cap A^{\perp}$ is regular ideal of A (do not confuse it with regular elements in our sense). Hence $A \approx A \backslash 0$ is unital.

A quantum frame Q will be called *normal* if for any x, $y \in Q$, $x \vee y = 1$ there is $z \lhd x$ with $z \vee y = 1$. Again, it extends the usual concept of normal frames (see [8]).

3.5. LEMMA. *Any compact regular quantum frame is normal.*

PROOF. Let Q be compact and regular, $x, y \in Q$ and $x \vee y = 1$. Then x is a directed join of regular elements x_i and any x_i is a directed join of elements $x_{ij} \lhd x_i$. Hence 1 is the directed join of elements $x_i \vee y$, i.e., $1 = x_i \vee y$ for some i. We can repeat the argument and we find $z \in Q$ ($z = x_{ij}$ for some i and j) such that

$$z \lhd x \text{ and } z \vee y = 1.$$

3.6. COROLLARY. R(A) *is normal for any unital* C*-*algebra* A.

Proof follows by 3.3, 3.4 and 3.5.

4. DISCRETENESS.

4.1. DEFINITION. A quantum frame Q will be called *discrete* if it is dually atomic (i.e., any element is a meet of dual atoms) and $x = x^{\perp\perp}$ for any $x \in Q$.

Discrete frames, complete atomic Boolean algebras and open set lattices of discrete topological spaces mean the same concept. We will show that discrete quantum frames are closely connected with ortholattices. Recall that an *ortholattice* is a lattice L equipped with a unary operation ' satisfying

(OL1) $$x \leq y \Rightarrow y' \leq x',$$

(OL2) $$x'' = x\,,$$

(OL3) $$x \vee x' = 1,\ x \wedge x' = 0$$

for any $x,y \in L$ (see [9]). The third condition can be weakened to

(OL3') $$x \wedge x' = 0 .$$

Since ' is an anti-isomorphism of L, an ortholattice is atomic iff it is dually atomic. Complete ortholattices are ortholattices which are complete as lattices. For $x,y \in L$, it is said that x *commutes* with y if

$$x = (x \wedge y) \vee (x \wedge y').$$

The notation is xCy. The *centre* C(L) of L is defined as the set

$$\{ z \in L \mid xCz \text{ for any } x \in L\}$$

(see [10]). Central elements have properties

(18) $$a \in C(L) \Rightarrow a' \in C(L),$$

(19) $$a \wedge \bigvee x_i = \bigvee a \wedge x_i \text{ for any } a \in C(L),\ x_i \in L,\ i \in I.$$

The first is evident and the second is in [10], p. 603.

4.2. LEMMA. *In a complete atomic ortholattice* L, (3) *holds for any* $x \in L$ *and* $a_i \in C(L)$, $i \in I$.

PROOF. Assume that (3) is not true for $x \in L$ and $a_i \in C(L)$. Since

$$\bigvee x \wedge a_i \leq x \wedge \bigvee a_i,$$

there is an atom $p \in L$ such that $p \leq x \wedge \bigvee a_i$ but $p \wedge \bigvee x \wedge a_i = 0$. Hence

$$p \wedge a_i = p \wedge x \wedge a_i = 0 \text{ for any } i \in I.$$

By (19),

$$a_i = (a_i \wedge p) \vee (a_i \wedge p') = a_i \wedge p'$$

and therefore $a_i \leq p'$ for any i. Consequently

$$p \leq x \wedge \bigvee a_i \leq p'$$

giving $p = 0$, i.e., a contradiction.

4.3. LEMMA. *Let* Q *be a quantum frame such that* $x = x^{\perp\perp}$ *for any* $x \in Q$. *Then*

(a) $(Q,\perp)$ *is an ortholattice.*

(b) $\tilde{Q}$ *is a complete Boolean subalgebra of* Q *and* $\tilde{Q} \subset C(Q)$.

(c) For any $x,y \in Q$ *and* $a \in \tilde{Q}$,

(20) $$x \circ y \leq a \Leftrightarrow x \leq a \vee y^{\perp}$$

holds.

PROOF. (a) (OL1) is evident, (OL2) is assumed and (OL3') follows by (6).

(b) follows by (16) and (3).

(c) If $x \circ y \le a$ then

$$x = (a^{\perp} \wedge y) \le (x \circ a^{\perp}) \wedge (x \circ y) \le a^{\perp} \wedge a = 0 .$$

Hence

$$x \le (a^{\perp} \wedge y)^{\perp} = a \vee y^{\perp}.$$

Conversely, $x \le a \vee y^{\perp}$ implies

$$x \circ y \le (a \vee y^{\perp}) \circ y = a \circ y \le a.$$

Consequently, if a quantum frame Q satisfies $x = x^{\perp\perp}$ for any $x \in$ Q, then the multiplication $\circ$ is given by the formula

(21) $$x \circ y = \bigwedge\{ a \in \tilde{Q} \mid x \le a \vee y^{\perp}\}.$$

4.4. PROPOSITION. *Let* L *be a complete ortholattice and* B *a complete Boolean subalgebra of* L *such that* $B \subset C(L)$ *and* (3) *holds for any* $x \in L$ *and* $a_i \in B$. *Then* L *is a quantum frame with* $x^{\perp} = x'$ *for any* $x \in L$, *with* $B = \tilde{L}$ *and with the multiplication given by* (21), i.e..

$$x \circ y = \bigwedge\{a \in B \mid x \le a \vee y'\}.$$

PROOF. We will start with the verification that $\tilde{L} = B$ makes L a quantale. Condition (2) follows by (19), (3) is assumed and concerning (4) let $x \in L$ and $a \in B$. Evidently

$$(x \wedge a)\hat{} \le \hat{x} \wedge a \text{ and } (x \wedge a)\hat{} \vee a' \ge (x \wedge a) \vee (x \wedge a') = x$$

by (3). Hence

$$\hat{x} \le (x \wedge a)\hat{} \vee a' \text{ , i.e., } \hat{x} \wedge a \le (x \wedge a)\hat{} .$$

For proving that L is a quantum frame, we will need to know that (20) holds for any $x,y \in L$ and $a \in B$. It follows by the fact that

$$(x \circ y) \vee y' = y' \vee \bigwedge\{a \in B \mid x \le a \vee y'\} =$$
$$= \bigwedge\{y' \vee a \mid a \in B,\ x \le a \vee y'\} \ge x.$$

Now, (QF1) is evident.

(QF2) is an immediate consequence of (20).

(QF3) also follows by (20) because $x \le a \vee y'$ implies

$$a \vee x' \le a \vee (a' \wedge y) \ge y \text{ for any } x,y \in L \text{ and } a \in B.$$

(QF4) By (19),

$$a \vee (\bigvee y_i)' = a \vee \bigwedge y_i' = \bigwedge a \vee y_i'$$

for any $y_i \in$ L and $a \in$ B. Hence

$$x \circ \bigvee y_i \le a \Leftrightarrow x \le a \vee (\bigvee y_i)' \Leftrightarrow x \le a \vee y_i' \text{ for any } i$$
$$\Leftrightarrow x \circ y_i \le a \text{ for any } i \Leftrightarrow \bigvee x \circ y_i \le a.$$

We used (20) and, since $x \circ \bigvee y_i$, $\bigvee x \circ y_i \in$ B and $a \in$ B was arbitrary, we proved (QF4).

(QF5) If $x \le a \vee x'$ for $x \in$ L and $a \in$ B then

$$x \le (a \vee x) \wedge (a \vee x') = a.$$

(QF6) Let $x \circ (1 \circ y) \le a$ for $x, y \in$ L and $a \in$ B. By (20),

$$x \le a \vee (1 \circ y)' \text{ , i.e., } x \circ 1 = \hat{x} \le a \vee (1 \circ y)' \le a \vee y' \text{ .}$$

Therefore $(x \circ 1) \circ y \le a$ and (QF6') is verified. It remains to apply 1.3.

(QF7) For $x, y \in$ L and $a \in$ B, we have

$$(x \circ y) \cdot a = (x \circ y) \wedge a = [((x \wedge a) \vee (x \wedge a')) \circ y] \wedge a =$$
$$[((x \wedge a) \circ y) \vee ((x \wedge a') \circ y)] \wedge a = [((x \wedge a) \circ y) \wedge a] \vee [((x \wedge a') \circ y) \wedge a] =$$
$$= (x \wedge a) \circ y = (x \cdot a) \circ y.$$

(QF7) follows.

The next result immediately follows by 4.2, 4.3 and 4.4. Its formulation uses the fact that C(L) is a complete Boolean subalgebra of an atomic complete ortholattice L (see [10] 3.3).

4.5. COROLLARY. *Discrete quantum frames uniquely correspond to couples* (L,B) *where* L *is a complete atomic ortholattice and* B *a complete Boolean subalgebra of* C(L).

Emphasize that disjointness and orthogonality means the same for discrete quantum frames. There is another case where 4.3 and 4.4 fit together. It is the case of orthomodular lattices, i.e., ortholattices satisfying the identity

$$x \vee (x' \wedge y) = y \text{ for any } x \le y.$$

4.6. COROLLARY. *Let* L *be a complete orthomodular lattice. Then quantum frame structures on* L *uniquely correspond to complete Boolean subalgebras* B *of* C(L).

PROOF. Any complete orthomodular lattice satisfies (3) for any

$x \in L$ and $a_j \in C(L)$ (see [9] §3, 2 and 4). Moreover, C(L) is a complete Boolean subalgebra of L ([9] §3, Ex. 16).

A quantum frame Q is called *simple* if $\tilde{Q} = \{0,1\}$. Following 4.5, discrete simple quantum frames coincide with complete atomic ortholattices (via $x \circ y = 0$ for $x \leq y'$ and $x \circ y = 1$ otherwise). Another possibility for a discrete quantum frame Q is $\tilde{Q} = C(Q)$. Since $C(\Pi Q_i) = \Pi C(Q_i)$, it follows by [10] 3.4 that these discrete quantum frames coincide with products of discrete simple quantum frames.

Let A be a W*-algebra and P(A) its lattice of projections. By [6] §3.4, the lattices P(A) and r(A) are isomorphic and 2-sided ideals correspond to central projection. P(A) is orthomodular and its centre consists of central projections. Hence P(A) and r(A) are isomorphic quantum frames (by 4.6).

Let A be a C*-algebra and $\pi: A \to \tilde{A}$ its reduced atomic representation. It means that π is induced by a maximal set $\pi_i: A \to B(H_i)$, $i \in \hat{A}$ of inequivalent non-null irreducible representations

$$(\tilde{A} = \prod_{i \in \hat{A}} B(H_i)).$$

The prescription $\alpha(J) = \mathrm{cl}\,\pi(J)$ provides, leaning on the isomorphism $r(\tilde{A}) \approx P(\tilde{A})$, a mapping $\alpha: R(A) \to P(\tilde{A})$, which is the embedding mentioned in the Introduction. It was proved in [4] that α is a quantale homomorphism and, quite analogously and easily, it can be shown that α preserves $\circ$. Hence $\alpha: R(A) \to P(\tilde{A})$ is an embedding of quantum frames.

Pursuing the analogy with frames, a quantum frame may be called *spatial* if it is a quantum subframe of a discrete quantum frame.

4.7. PROPOSITION. *The quantum frame* R(A) *is spatial for any* C*-*algebra* A.

PROOF. R(A) is isomorphic to a quantum subframe of $P(\tilde{A})$ and $P(\tilde{A})$ is discrete (even

$$P(\tilde{A}) = \prod_{i \in \hat{A}} L(H_i)$$

where $L(H_i)$ is the lattice of closed subspaces of H_i).