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**MOPHISMS BETWEEN SPACES OF LEAVES  
VIEWED AS FRACTIONS**

*by Jean PRADINES*

**RÉSUMÉ.** Après avoir transféré au cadre différentiable la notion algébrique d'équivalence de groupoïdes, nous montrons que les morphismes de la catégorie de fractions correspondante sont représentés par une unique fraction irréductible (calcul de fractions simplifié) que nous identifions aux morphismes de Connes-Skandalis-Haefliger entre espaces de feuilles. Dans cette catégorie de fractions, le groupe fondamental de l'espace d'orbites au sens de Haefliger-van Est s'interprète comme réflecteur sur la sous-catégorie pleine des sous-groupes discrets.

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**0. INTRODUCTION.**

The basic references for the present text are the papers by W. T. van Est [19] and A. Haefliger [8], in which various approaches to the transverse structure of foliations are described and certain concepts of transverse morphisms are introduced. The second approach is more general in that it considers topological groupoids which may be unequivalent to pseudogroups.

A very careful scrutiny of these papers would show that (when restricted to the common case of pseudogroups) the notions of morphisms considered by these authors are *not* equivalent in general, though they are in the special case of submersive morphisms and equivalences. More recently the "generalized morphisms" of A. Haefliger, attributed to G. Skandalis, have been used extensively, under the name of "K-oriented morphisms" by Skandalis, Hilsum [9] and the school of A. Connes [3].

Here we start with van Est's *geometrical* approach of pseudo-groups viewed as "generalized atlases", but we extend this (very illuminating) geometrical language to the "non étale" case, considering general groupoids as "non étale atlases". In this framework, a "non étale change of base" is an induction (or pullback) along a surmersion, which is a special case of equivalence (which turns out to generate the most general concept).

It is then natural, from an *algebraic* point of view, to define morphisms by formally inverting these surmersive equivalences, which is always possible in an abstract non-sense way [7]. However the conditions for the classical calculus of fractions [7] are *not* fulfilled, but it turns out that we are able to unfold a "simplified" calculus of fractions in the sense that our fractions admit unique irreducible representatives, as in the elementary case of integers. Now we have the remarkable fact that the irreducible fractions can be identified (in a non-obvious way) with the Skandalis-Haefliger morphisms.

The consideration of possibly non-reduced representatives gives a significant increase in flexibility. For instance the composition of morphisms becomes a routine diagram chasing (note that in the locally trivial topological case considered in [19] this composition is defined but in very special cases).

The irreducible fractions may also be viewed as special instances of J. Bénabou's distributors or profunctors (a more symmetric notion). However the intersection of the two theories reduces to a rather trivial part of each one, and we let it to the informed reader [2,10].

As an illustration we give a very simple characterization of the fundamental group of a foliation (in the sense of van Est and Haefliger) by means of a reflection of our category of fractions into the full subcategory of discrete groups.

The present paper gives, essentially, ideas and results without detailed proofs. Our general policy throughout will be first to describe algebraic set-theoretic constructions by means of suitable diagrams in which we stress the injections and surjections, and secondly to replace injections by (regular) embeddings and surjections by surmersions (i.e. surjective submer-

sions). Then the proofs work by diagram chasing, using the formal properties of embeddings and surmersions listed in [13] under the name of "diptych" and the formal properties of commutative squares stated in the basic proposition A2 of [16].

In the following, a *pseudogroup* of transformations (always assumed to be complete or completed) will always be identified with the *groupoid* of its germs, provided with the (étale) sheaf topology.

## 1. THE LANGUAGE OF (GENERALIZED) ATLASES.

Let us first consider a (smooth) manifold  $Q$  and a (classical) atlas of  $Q$ . i.e., a collection of charts  $p_i: V_i \rightarrow U_i$  (open sets in some  $\mathbb{R}^n$ ), or equivalently of cocharts  $q_i = p_i^{-1}$ . It is equivalent to consider the étale surjective map  $q: U \rightarrow Q$  where  $U$  is the (trivial) manifold coproduct (or disjoint sum) of the  $U_i$ 's. The fibered product  $R = U \times_Q U$ , with its projections  $\alpha = pr_2$ ,  $\beta = pr_1$  may be viewed either as the graph of the equivalence relation in  $U$  defined by  $q$  or as the pseudogroup of changes of charts, which is a (very special kind of) groupoid with base  $U$ . Conversely the data of  $R$  with its manifold and groupoid structures determine  $Q$  and  $q$  up to isomorphisms.

In that context, a refinement of the given atlas is viewed as an étale surjective map  $u: U' \rightarrow U$  and then the corresponding graph  $R'$  is obtained by pulling back along  $u$ . Two atlases are equivalent if they admit a common refinement.

This situation admits a twofold generalization.

First following van Est a pseudogroup may be viewed as a generalized (étale) atlas of its space of orbits (which is no longer a manifold in general). This applies to any regular foliation, using a totally transverse manifold  $T$  and the corresponding holonomy pseudogroup, whose space of orbits is the space of leaves. Various choices of  $T$  lead to equivalent atlases in a generalized sense explained below.

Second replacing  $q$  by a (possibly non-étale) surmersion  $q: B \rightarrow Q$ , we can view the graph  $R = B \times_Q B$  (with its manifold and groupoid structures) as a "non-étale atlas" of  $Q$  with base  $B$ . A non-étale refinement is then a surmersion  $B' \rightarrow B$  and the new "atlas"  $R'$  is again obtained by pulling back. If moreover  $q$  is "retroconnected" (i.e., the fibres are connected), the manifold  $Q$  is the space of leaves of the simple foliation of  $B$  defined by  $q$ .

A further generalization is required for a non-simple (regular) foliation, the previous construction being valid only locally. The local pieces can be glued together into the holonomy

groupoid introduced by Ehresmann in [5] and renamed as the (smooth) graph of the foliation by Winkelkemper [20] and A. Connes [3]. Though this groupoid has special properties which we emphasized in [15], we do not use them in the sequel. So we are led to the following common generalization. This generalization makes use of the general notion of smooth (or differentiable) groupoid introduced by Ehresmann [4] which we recall first.

**2. SMOOTH GROUPOIDS AND ORBITAL ATLASES.**

In the sequel  $D$  will denote the category of (morphisms between) smooth manifolds. We consider the following subcategories:

- $D^*$  = diffeomorphisms;
- $D_e$  = étale maps (or local diffeomorphisms);
- $D_i$  = (regular) embeddings, denoted  $\succ$ ;
- $D_s$  = surmersions, denoted  $\dashv$ ;
- $D_{ei} = D_e \cap D_i$ ;  $D_{es} = D_e \cap D_s$ .

The subclass  $D_i D_s$  (which is not a subcategory!) is denoted by  $D_r$  (= regular morphisms).

Let

$$(P) \quad \begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ u \downarrow & & \downarrow v \\ A & \xrightarrow{f} & B \end{array}$$

be a commutative square of  $D$ , and denote by  $R$  the (set theoretic) fibered product  $A \times_B B'$ . Then  $P$  is called:

- i-faithful* if  $(u, f'): A' \rightarrow A \times_B B'$  lies in  $D_i$ ;
- universal* (resp. *s-full*) if  $R$  is a submanifold of  $A \times B'$  and moreover the canonical map  $A' \rightarrow R$  lies in  $D^*$  (resp.  $D_s$ ).

Note that universal *implies*  $D$ -cartesian (i.e., pullback square in  $D$ ) but *the converse is false*. Note also that the transversality of  $f$  and  $v$  *implies* that the pullback is universal but *the converse is false*: we shall say that  $f$  and  $v$  are *weakly transversal* when they can be completed into a universal square.

These notions are stable by the tangent functor  $T$ . The basic properties of these squares are stated (with a different terminology) in Proposition A2 of [16], which we complete by the following: *If  $f$  is a surmersion and  $P$  and  $QP$  are universal, then  $Q$  is universal.*

Now we remind that a (small) *groupoid* is a (small) category with all arrows invertible. Usually a groupoid will be loosely denoted by its sets of arrows  $G$ . The *base*  $B = G_0$  is the set of *objects*, identified by the *unit map*  $\omega: B \rightarrow G$  with the set of *units*  $\omega(B) \subset G$ . The *source* and *target* maps are denoted by  $\alpha, \beta: G \rightarrow B$ . The map  $\tau = (\beta, \alpha): G \rightarrow B \times B$  will be called the *transitor* (anchor map in [11]). The image of  $\tau$  is the graph of an equivalence relation in  $B$  whose classes are the *orbits* of  $G$  in  $B$ . The inverse images of the orbits are the *transitive components* of  $G$ . The map

$$\delta: \Delta G \rightarrow G, (x, y) \mapsto xy^{-1}$$

(where  $\Delta G \subset G \times G$  is the set of pairs of arrows with the same source) may be called the *divisor*.

The morphisms  $f: G' \rightarrow G$  between groupoids are just the *functors* and are the arrows of a category  $\mathcal{G}$ . The restriction  $f_0: B' \rightarrow B$  of  $f$  to the bases of  $G', G$  may be called the *objector* of  $f$ : when  $f_0$  is the identity of  $B$ ,  $f$  is said to be *uniferous*. The subcategory of uniferous functors will be denoted by  $G_0$ , and  $G_B$  when the base  $B$  is fixed.

We say that the *groupoid*  $G$  is *smooth* (or *differentiable*) [4, 11] when  $G$  and  $B$  are provided with manifold structures such that  $\omega \in D$ ,  $\alpha \in D_s$  (which implies that  $\Delta G$  is a submanifold of  $G \times G$ ), and  $\delta \in D$ . This implies easily  $\omega \in D_i$ ,  $\beta \in D_s$ ,  $\delta \in D_s$ .

A *functor*  $f: G' \rightarrow G$  is *smooth* if the underlying map is smooth: if moreover it lies in  $D_i$  (resp.  $D_s$ ) we say that  $f$  is an *i-functor* (resp. *s-functor*): note that this implies  $f_0$  is also in  $D_i$  (resp.  $D_s$ ). The category of smooth functors between smooth groupoids is denoted by  $GD$ .

A smooth functor is *split* when it admits a section in  $GD$ .

To any smooth functor  $f: G' \rightarrow G$  there are associated two commutative squares:

$$\begin{array}{ccc}
 G' & \xrightarrow{f'} & G \\
 \alpha_{G'} \downarrow & & \downarrow \alpha_G \\
 B' & \xrightarrow{f_0} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 G' & \xrightarrow{f'} & G \\
 \tau_{G'} \downarrow & & \downarrow \tau_G \\
 B' \setminus B' & \xrightarrow{f_0 \vee f_0} & B \setminus B
 \end{array}$$

$a(f)$                        $t(f)$

the first one in  $D$ , the second in  $GD$ .

A smooth functor is called *i-faithful* (*s-full*, an *inductor*) when the square  $t(f)$  is *i-faithful* (*s-full*, universal). These notions are stable by the tangent functor  $T$ . From Proposition A2

of [16] we get:

**PROPOSITION 2.1.** *Let  $h=gf$  be the composite of two smooth functors.*

- (i) *If  $f$  and  $g$  are  $i$ -faithful (resp. inductors), so is  $h$ ;*
- (ii) *If  $h$  is  $i$ -faithful, so is  $f$ .*
- (iii) *Assume  $f$  is an  $s$ -functor and an inductor (briefly an  $s$ -inductor): then if  $h$  is  $i$ -faithful (an inductor), so is  $g$ ;*
- (iv) *Assume  $g$  is an inductor: then  $f$  is  $s$ -full (an inductor) iff  $h$  is.*

Now the considerations of §1 lead us to set:

**DEFINITION 2.1.** An *orbital atlas* on a set  $Q$  is a pair  $(G, q)$  where  $G$  is a smooth groupoid with base  $B$  and  $q: B \rightarrow Q$  a surjection whose fibres are the orbits of  $G$  in  $B$ .

$Q$  will be provided with the finest topology making  $q$  continuous. Then  $q$  is *open*.

A basic example is the holonomy groupoid viewed as an orbital atlas of the space of leaves. Note that transitive smooth groupoids (particularly Lie groups) define various unequivalent orbital atlases for a singleton.

### 3. SURMERSIVE EQUIVALENCES AND EXTENSORS.

If  $u: B' \rightarrow B$  is in  $D_s$ , the fibered product  $G' = u^*(G)$  of the arrows  $\tau_G$  and  $u \times u$  has a canonical structure of groupoid called the pullback of  $G$  along  $u$ , for which  $f: G' \rightarrow G$  is an  $s$ -inductor. Any smooth functor  $g: H \rightarrow G$  with its objector  $g_0 = u$  admits a unique factorization  $g = f h$ .

**DEFINITION 3.1.** An  $s$ -inductor will be called also an  $s$ -equivalence; an  $s$ -full  $s$ -functor is called an  $s$ -extensor.

The following statements are proved in [17]:

**THEOREM 3.1.** (i) *An  $s$ -equivalence induces an equivalence between the categories  $(G \downarrow GD_B)$  and  $(G' \downarrow GD_{B'})$  of groupoids under  $G$  and  $G'$  [12].*

(ii) *Let  $f: H \rightarrow G$  be a smooth functor and  $N = f^{-1}(B)$  its set-theoretic kernel: then the following statements are equivalent:*

- a)  *$f$  is an  $s$ -extensor;*

b)  $N$  is a regular smooth groupoid embedded in  $H$  and the square

$$\begin{array}{ccc}
 N & \xrightarrow{\quad} & B \\
 \downarrow & & \downarrow \\
 H & \xrightarrow{\quad f \quad} & G
 \end{array}$$

is a pushout in  $GD$ :

c)  $N$  is a regular smooth groupoid embedded in  $H$ .  $f$  is an  $s$ -functor, and the relation  $f(x) = f(y)$  is equivalent to  $x \in Ny$  ( $N$  two-sided coset).

Keeping the above notations, if  $(G, q)$  is an orbital atlas of  $Q$ , then  $(G', q')$ , where  $q' = u \circ q$  (with  $u$  an  $s$ -equivalence) is again an orbital atlas of  $Q$  called a *refinement* of  $(G, q)$ . Two atlases of  $Q$  are said to be *equivalent* if they admit a common refinement.

It is convenient to think an equivalent class of orbital atlases on  $Q$  as defining a (generalized) structure on the set  $Q$ , called *orbital structure*. But one should notice carefully that the morphisms we shall introduce will be defined only at the atlas level and not between such structures.

Two smooth groupoids  $G_i$  ( $i = 1, 2$ ) are called (smoothly) *equivalent* if there exists a pair of  $s$ -equivalences  $f_i: G \rightarrow G_i$ ; this is indeed an equivalence relation.

**PROPOSITION 3.1.** *Let  $h = gf$  be the composite of two smooth functors. Then:*

- (i) *if  $f$  and  $g$  are  $s$ -extensors, so is  $h$ ;*
- (ii) *assume  $g$  is an  $s$ -equivalence and  $f_0 \in D_s$ : then if  $h$  is an  $s$ -extensor or an  $s$ -equivalence, so is  $f$ ;*
- (iii) *assume  $f$  is an  $s$ -extensor: then if  $h$  is an  $s$ -extensor or an  $s$ -equivalence, so is  $g$ .*

#### 4. SOME IMPORTANT SPECIAL SMOOTH GROUPOIDS.

Let  $G$  be a smooth groupoid with base  $B$ . We consider various special cases.

- (i)  $G$  is *discrete*; we can identify  $G$  with the full subcategory of discrete smooth groupoids in  $GD$ ;
- (ii)  $B$  is a singleton:  $G$  is (identified with) a *Lie group*.

(iii)  $\alpha_G = \beta_G$ :  $G$  is called a smooth *plurigroup*: the full subcategory of smooth (pluri)groupoids will be denoted by  $gD$  ( $\bar{g}D$ ).

(iv)  $\omega_G \in D^\dagger$ :  $G$  is *null*: we may identify  $D$  with the full subcategory of null smooth groupoids in  $GD$ ;

(v)  $\tau_G \in D^*$ :  $G$  is *coarse* (this refers here to the algebraic structures, not the topology).

(vi)  $\tau_G \in D_j$ :  $G$  is *principal* (or *Godement*): by Godement's Theorem,  $G$  is identified with the graph of a regular equivalence relation in  $B$ .

(vii)  $\tau_G \in D_s$ :  $G$  is *s-transitive* or a *Lie groupoid* [11]; the fibres of  $\alpha_G$  are principal bundles with base  $B$  and  $G$  is identified with their gauge groupoid [4.11].

(viii)  $\tau_G \in D_{es}$ :  $G$  is *es-transitive* or a *Galois groupoid* (gauge groupoid of a Galois or normal covering).

(ix)  $\tau_G \in D_r$ :  $G$  is *regular*.

(x)  $\tau_G$  is a weak embedding:  $G$  is a *Barre groupoid* (its space of orbits is a Barre  $Q$ -manifold) [1].

(xi)  $\tau_G$  is a faithful immersion:  $G$  is a *graphoid* [15].

**PROPOSITION 4.1.**  *$G$  is principal (a Lie groupoid, a Galois groupoid, a graphoid) iff it is equivalent to a null groupoid (Lie group, discrete group, pseudogroup).*

The holonomy groupoid of a regular foliation is equivalent to any of its transverse holonomy pseudogroups.

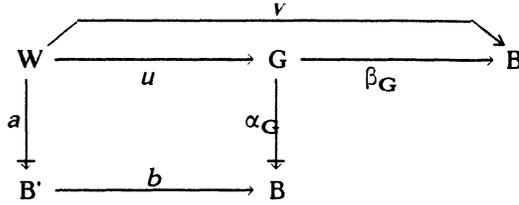
**DEFINITION 4.1.** A smooth functor is called *principal* if its source groupoid is principal.

**PROPOSITION 4.2.** *Assume the smooth functor  $f: H \rightarrow G$  is  $i$ -faithful (resp.  $s$ -full, an  $s$ -extensor). Then if  $G$  is principal (resp. Lie, resp. regular), so is  $H$ .*

## 5. SMOOTH EQUIVALENCES.

Following our general policy, we give a smooth version of the algebraic notions of essential (or generic) surjectivity and equivalences between groupoids (more general than the surjective equivalences).

Let be given a smooth groupoid  $G$  with base  $B$  and a map  $b: B' \rightarrow B$  (in  $D$ ). Let  $W$  be the fibered product (in  $D$ ) of  $\alpha_G$  and  $b$  and consider the following diagram in  $D$ .



**DEFINITION 5.1.** We say  $b$  is transversal to  $G$  when  $v$  lies in  $D_s$ , and that a functor  $f: G' \rightarrow G$  is essentially surmersive when  $f_0$  is transversal to  $G$ .

**PROPOSITION-DEFINITION 5.2.** If  $b$  is transversal to  $G$ , then the fibered product of  $b \cdot b$  and  $\tau_G$  does exist in  $GD$  and the pullback we get is universal in  $D$ . We say that  $G'$  is the (smooth) groupoid induced by  $G$  along  $b$  (or the pullback of  $G$  along  $b$ ).

A smooth functor  $f: H \rightarrow G$  with  $f_0 = b$  is called a (smooth) equivalence if it is essentially surmersive and if the canonical factorization  $H \rightarrow G' \rightarrow G$  is an isomorphism.

**PROPOSITION 5.3.** (i) The equivalences and the essentially surmersive functors make up subcategories of  $GD$ .

(ii) If  $g$  is an equivalence and  $gf$  is essentially surmersive (resp. an equivalence), then  $f$  is essentially surmersive (resp. an equivalence).

(iii) If  $f_0$  lies in  $D_s$  (resp. if  $f$  is an  $s$ -extensor) and  $gf$  is essentially surmersive (resp. an equivalence), then  $g$  is essentially surmersive (resp. is an equivalence and  $f$  is an  $s$ -equivalence).

**6. HOLOMORPHISMS.**

If  $\square G$  denotes the smooth groupoid of commutative squares of  $G$  with the horizontal composition law, the two canonical projections  $\pi_1, \pi_2$  on  $G$  are  $s$ -equivalences while the two canonical injections  $\iota_1, \iota_2$  are  $i$ -equivalences.

A (smooth) natural transformation between two smooth functors  $f_1, f_2: G \rightarrow H$  may be described either as a smooth functor  $\square G \rightarrow H$  or a smooth functor  $G \rightarrow \square H$ . As a consequence:

**PROPOSITION 6.1.** The following properties of a smooth functor

are preserved by a smooth functorial isomorphism: *i*-faithful, *s*-full, inductor, essentially surmersive, equivalence, *s*-extensor.

By the horizontal composition of natural transformations, the isomorphism between smooth functors is compatible with the composition of functors.

This gives rise to a new category (with the same objects as  $GD$ ) denoted by  $[G]D$ , the arrows of which will be called *holomorphisms*, and a canonical full functor  $f \mapsto [f]$  from  $GD$  to  $[G]D$ .

The holomorphisms between Lie groups are just the conjugacy classes of homomorphisms. So the notion of holomorphism extends the notion of outer automorphism (this suggests the alternative terminology of exomorphism).

## 7. ACTORS, EXACTORS, SUBACTORS.

After the diagram  $t(f)$ , which measures the faithfulness of  $f$ , we turn now to the diagram  $a(f)$ , which measures its "activity". (In the purely algebraic context several variants of the notions below have been used by many authors such as Ehresmann, Grothendieck, Higgins, R. Brown, van Est et alii, under various names, notably (discrete) (op)fibrations, coverings, and others, which we cannot carry over to the smooth case.)

**DEFINITION 7.1.** A smooth functor  $f$  is called an *actor* (*inactor*, *exactor*) when the square  $a(f)$  (§2) is universal (*i*-faithful, *s*-full). More precisely we speak of  $G$ -actor, etc... when the target  $G$  of  $f$  is fixed.

There is an equivalence of categories between the category of (morphisms between)  $G$ -actors and the category of (equivariant morphisms between) smooth action laws of  $G$  on manifolds over the base  $B$  of  $G$  (hence the terminology) [11].

**REMARKS.** (i) The image of an actor is a possibly non-smooth subgroupoid of  $G$ .

(ii) Any *s*-extensor is an *s*-exactor: any inactor is *i*-faithful.

(iii) An exactor is essentially surmersive iff it is an *s*-exactor.

**PROPOSITION 7.1.** A smooth functor which is an equivalence

and an actor is an isomorphism (of smooth groupoids). If it is an exactor and an inductor, it is an  $s$ -equivalence.

**PROPOSITION 7.2.** *If  $f: G' \rightarrow G$  is an  $s$ -exactor,  $H$  a smooth groupoid, and  $h: G \rightarrow H$  a (set-theoretic) map such that  $hf: G' \rightarrow H$  is a smooth functor, then  $h: G \rightarrow H$  is a smooth functor.*

**PROPOSITION 7.3.** *Let  $h = gf$  be the composite of two smooth functors.*

- (i) *If  $f, g$  are (ex)(in)actors, so is  $h$ .*
- (ii) *Assume  $g$  is an actor. Then if  $h$  is an (ex)actor, so is  $f$ ;*
- (iii) *Assume  $f$  is an  $s$ -exactor. Then if  $h$  is an (ex)actor, so is  $g$ .*
- (iv) *If  $h$  is an inactor, so is  $f$ .*
- (v) *Assume  $f$  is an  $s$ -actor. Then if  $h$  is an (ex)actor, so is  $g$ .*

**PROPOSITION 7.4.** *Let  $f: G' \rightarrow G$  be an (ex)actor, and  $u: H \rightarrow G$  a smooth functor. Assume  $f_0$  and  $g_0$  to be weakly transversal*

- (i) *Then the fibered product exists in  $GD$ , the pullback square is universal in  $D$ , and  $g: H' \rightarrow H$  is an (ex)actor. The induced map  $k: \text{Ker } g \rightarrow \text{Ker } f$  is an actor.*
- (ii) *If moreover  $f$  is an  $s$ -extensor (an  $s$ -equivalence), so is  $g$ .*
- (iii) *If  $u$  is an inactor ( $i$ -faithful, essentially surmersive, an inductor, an equivalence), so is  $u': H' \rightarrow G'$ . If moreover  $f$  is an  $s$ -exactor, then if  $u'$  is an (in)(ex)actor (essentially surmersive, an equivalence), so is  $u$ .*

As a consequence any exactor  $f$  has a kernel in  $GD$ :  $f$  will be an actor iff this kernel is null.

The more general case when this kernel is principal is of importance too:

**PROPOSITION-DEFINITION 7.5.** *Let  $f: H \rightarrow G$  be an exactor. The following are equivalent:*

- (i)  *$\text{Ker } f$  is principal (§4.vi):*
  - (ii)  *$f$  is  $i$ -faithful:*
  - (iii)  *$f = ae$  where  $e$  is an  $s$ -equivalence and  $a$  an actor.*
- The decomposition (iii) is essentially unique.*

Then  $f$  is called a *subactor*.

**REMARK.** It will be proved elsewhere that any  $i$ -faithful functor is the composite of an equivalence and an actor.

The two following propositions generalize and extend to the smooth case a lemma of van Est [19].

**PROPOSITION 7.6.** *Assume  $ae' = ea'$  where  $a, a'$  are actors,  $e'$  an equivalence, and  $e$  an  $s$ -equivalence. Then the square is a pull-back.*

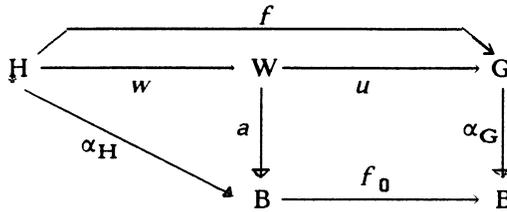
Now let  $u: G' \rightarrow G$  be an  $s$ -equivalence. Then pulling back along  $u$  determines a functor  $u^*: (\text{Act} \downarrow G) \rightarrow (\text{Act} \downarrow G')$  from the category of  $G$ -actors to the category of  $G'$ -actors. Conversely we define the direct image of a  $G'$ -actor  $a'$  by taking for  $u_*(a')$  the first factor of the decomposition (iii).

**THEOREM 7.1.**  *$(u^*, u_*)$  defines an adjoint equivalence [12] between  $(\text{Act} \downarrow G)$  and  $(\text{Act} \downarrow G')$ .*

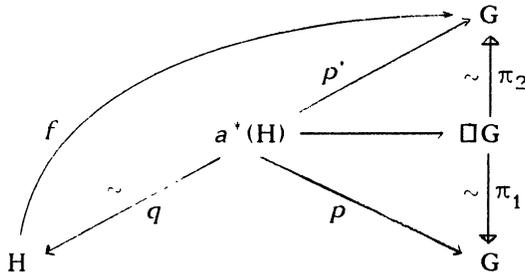
**8. HOLOGRAPH OF A FUNCTOR.**

The following smooth construction is known in the algebraic context of profunctors [2.10]. It turns out to be crucial for defining the (non-trivial) functor from the functors to the fractions.

We start again with the square  $a(f)$  (§2) and we display the pullback factorization in  $D$ :



Noting  $a$  is in  $D_s$  we can construct the commutative diagram in  $GD$  :



**PROPOSITION-DEFINITION 8.1.** For any smooth functor  $f$ ,  $p$  is an exactor and  $q$  a split  $s$ -equivalence. We call  $(p, q)$  the *holograph* of  $f$  and  $p = p(f)$  the *expansion* of  $f$ :  $p$  is isomorphic to  $p' = fq$ .

The holograph of the identity is  $(\pi_1, \pi_2)$ .

**PROPOSITION 8.2.** A smooth functor  $f$  is essentially surmersive ( $i$ -faithful, an equivalence) iff its expansion  $p(f)$  is an  $s$ -exactor (a subactor, an  $s$ -equivalence).

**EXAMPLE.** The holograph of the unit map  $\omega_G: B \rightarrow G$  is  $(\delta_G, q)$  where  $q$  is the canonical projection  $\Delta G \rightarrow B$ .

## 9. TRANSVERSAL SUBGROUPOIDS.

Let  $K$  be a smooth groupoid with base  $E$ , and  $M, N$  two uniferous embedded subgroupoids,  $i, j$  the canonical injections,  $S$  the (generally non-smooth) subgroupoid  $M \cap N$ .

Let  $L$  be the fibered product of  $\alpha_M$  and  $\alpha_N$ , which is a submanifold of  $\Delta K$ .

**DEFINITION 9.1.**  $M$  and  $N$  are called *transversal* in  $K$  (denoted by  $M \Psi N$ ) if the restriction of  $\delta_K$  to the submanifold  $L$  is a surmersion on  $K$ . They are called *transverse* ( $M \perp N$ ) if it is a diffeomorphism.

Then it can be proved that  $S$  is a smooth subgroupoid embedded in  $M$  and  $N$ : in particular, if  $M$  or  $N$  is principal, so is  $S$ .

**REMARK.** The data  $M, N$  with  $M \perp N$  determine on  $K$  a structure of smooth double groupoid [6]:  $M$  and  $N$  are the respective bases of the horizontal and vertical laws and the source map  $K \rightarrow M$  of the horizontal law is an  $s$ -actor when  $K$  and  $M$  are considered with the vertical law. The converse is true. We do not develop these facts that are not needed here.

**PROPOSITION-DEFINITION 9.2.** Let  $p: K \rightarrow G$  be an exactor and assume  $N = \text{Ker } p$ . Let  $M$  be another uniferous subgroupoid embedded in  $K$ . Then one has  $M \Psi N$  (resp.  $M \perp N$ ) iff  $u = pi$  is an exactor (resp. an actor: when such an  $M$  exists, we say  $p$  is *inessential*). (Note that for surjective homomorphisms of groups the notions of inessential and split coincide.)

As a consequence, if  $M$  is also the kernel of an exactor  $q: K \rightarrow H$ , then  $u = pi$  is an (ex)actor iff  $v = qj$  is. If such is the case we say the exactors  $p$  and  $q$  are *cotransvers(al)*.

**10. FRACTIONS AND MEROMORPHISMS: THE SIMPLIFIED CALCULUS OF FRACTIONS.**

We consider here the category whose objects are pairs  $(p, q)$  of exactors  $p: K \rightarrow G$ ,  $q: K \rightarrow H$  with the same source, and arrows  $k: (p', q') \rightarrow (p, q)$  are smooth functors  $k: K' \rightarrow K$  making the diagram commutative.

The *isomorphy class* of the pair  $(p, q)$  will be denoted by  $p/q$  and called a *fraction* with source  $H$  and target  $G$ .

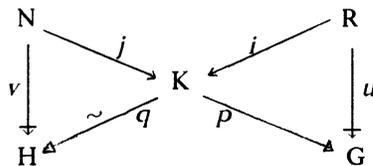
Two pairs  $(p_i, q_i)$  ( $i = 1, 2$ ) are *equivalent* if there exist two  $s$ -equivalences  $k_i: (p, q) \rightarrow (p_i, q_i)$ . The *equivalence class* of  $(p, q)$  is denoted by  $pq^{-1}: H \dashrightarrow G$ .

**PROPOSITION-DEFINITION 10.1.** *The following properties are preserved by equivalence:*

- (i)  $q$  is an  $s$ -equivalence;
- (ii)  $p$  and  $q$  are cotransversal.

When they are both satisfied,  $pq^{-1}$  is called a *meriedric morphism* or briefly *meromorphism* from  $H$  to  $G$ . If moreover  $p$  is an  $s$ -equivalence too,  $pq^{-1}$  is called a *meriedric equivalence* (from  $H$  to  $G$ ).

Setting  $N = \text{Ker } p$ ,  $R = \text{Ker } q$  (the latter principal), we have the commutative "*butterfly diagram*":



in which  $v$  is an  $s$ -exactor and  $u$  a *principal* exactor.

From the previous section we know that  $S = N \cap R$  is a smooth embedded principal subgroupoid of  $K$ .

**PROPOSITION-DEFINITION 10.2.** *The following are equivalent:*

- (i)  $S$  is null;
- (ii)  $N$  and  $R$  are transverse in  $K$  :

(iii)  $p$  and  $q$  are cotransverse:

(iv)  $(p, q)$  is a terminal object in its equivalence class.

Then  $p/q$  is called a *reduced* or *irreducible fraction*.

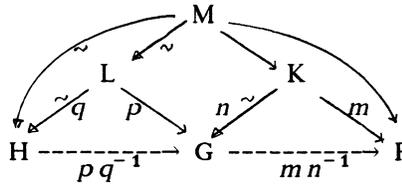
**REMARK.** If  $H$  is null ( $H=E$ ) and  $p/q$  irreducible, then  $p$  is a principal actor; the orbit space of the corresponding action is the underlying space of the null groupoid  $E$ :  $pq^{-1}$  is a non-abelian cohomology class on  $E$ .

Using the theory of smooth quotients [17] to divide by  $S$ , we get:

**PROPOSITION 10.3.** *Every meromorphism is represented by a unique irreducible fraction with which it will be identified. In turn this irreducible representative may be identified (up to equivariant isomorphism) with a Skandalis-Haefliger morphism [8,9].*

The two commuting actions are defined by the  $s$ -actor  $v$  and the principal actor  $u$ ; the base of  $H$  is the orbit manifold of the principal action of  $G$  on the base of  $K$ .

Now the use of non-irreducible representatives allows a very simple definition of the composite of two meromorphisms by means of the diagram:



where the square is a pullback. By diagram chasing and a repeated use of the general properties stated in the previous sections, it can be proved that the equivalence class of the composite depends only upon the classes  $pq^{-1}$  and  $mn^{-1}$  and is again a meromorphism.

The category of meromorphisms will be denoted by  $\tilde{GD}$ .

Now we define the (non-obvious) functor from  $GD$  to  $\tilde{GD}$  by means of the holograph.

**PROPOSITION 10.4.** *Let  $f: H \rightarrow G$  be a smooth functor,  $(p, q)$  its holograph.*

(i)  $p/q$  is an irreducible fraction which we identify with the meromorphism  $\tilde{f} = pq^{-1}$ .

(ii) Two functors  $f, g$  define the same meromorphism  $\tilde{f} = \tilde{g}$

iff they define the same holomorphism  $[f]=[g]$  (§6). (Hence we can identify  $[f]$ ,  $\tilde{f}$ ,  $p/q$ , and  $pq^{-1}$ .)

(iii)  $f \longmapsto \tilde{f}$  defines a uniferous functor  $\gamma:GD \rightarrow \tilde{G}D$  for which  $\tilde{f}\tilde{q} = \tilde{p}$ , and  $\gamma$  admits a factorization through the canonical full functor  $[G]D \rightarrow [G]D$  and an injective (hence faithful) canonical functor  $[G]D \rightarrow \tilde{G}D$ , by which we identify the category of holomorphisms with a uniferous subcategory of the category of meromorphisms.

(iv) A meromorphism is a holomorphism iff it admits a representative  $p/q$  with  $q$  split. Then  $v$  is split too.

(In particular any meromorphism with source a group or a pluri-group with discrete base is a holomorphism.)

**THEOREM-DEFINITION 10.5.** (i) The functor  $\gamma:GD \rightarrow \tilde{G}D$  is the universal solution of the problem of fractions [7] of  $GD$  for the subcategory  $\Sigma$  made up with the  $s$ -equivalences:

(ii)  $\gamma(f)$  is an isomorphism iff  $f$  is a smooth equivalence: then  $\gamma(f)$  is called a *holoedric equivalence*:

(iii)  $pq^{-1}$  is an isomorphism in  $\tilde{G}D$  iff  $p$  is an  $s$ -equivalence: then  $pq^{-1}$  is called a *meriedric equivalence*.

(iv) The  $s$ -equivalences, the smooth equivalences, the holoedric equivalences and the meriedric equivalences generate the same notion of equivalence between smooth groupoids.

The equivalence class of a smooth groupoid is therefore its isomorphy class in  $\tilde{G}D$ . Equivalent orbital atlases are isomorphic in  $\tilde{G}D$ .

**REMARK.** (i) The classical conditions for the calculus of right (nor left) fractions [7] are not fulfilled; we can say that we have got a *simplified calculus* of right fractions in that sense that our fractions are equivalent to an irreducible (or simple) one.

(ii) If we identify any manifold with a null groupoid,  $D$  is identified with a full subcategory of  $\tilde{G}D$ .

(iii) The category  $[G]D$  of conjugacy classes of homomorphisms between Lie groups is identified with a full subcategory of  $\tilde{G}D$ . This is valid too for plurigroups with discrete bases.

(iv) In the case of meriedric equivalences, the butterfly diagram becomes symmetric and reversible; this special case had been presented in [14] and will be developed elsewhere; the principal  $s$ -actors  $u$  and  $v$  are called *conjugate*.

(v) Given two orbital structures  $Q, Q'$  and choosing orbital atlases  $G, G'$  for these structures, the set  $\tilde{G}D(G, G')$  depends on the choices but up to bijection. But this does not allow to take the orbital structures for objects of a category. However this is

possible when there is a *canonical* choice of a meriedric equivalence between two equivalent orbital atlases: this is the case for graphoids [16] and more generally convectors in the sense of [15].

## 11. APPLICATION TO THE FUNDAMENTAL GROUP.

In the present framework we can restate the Theorem 2 of [18] in a more striking form.

**THEOREM 11.1.** *The full subcategory of discrete plurigrroups is reflective [12] in  $\tilde{G}D$ .*

In particular to any connected orbital structure (i.e., if the associated topological space is connected), there is associated a well defined (up to isomorphism) discrete group which, in the case of the orbital structure of the space of leaves of a foliation, coincides with the fundamental group in the sense of van Est-Haefliger [8,19] (and in the case of a connected smooth manifold with the Poincaré group).

This group is invariant under a wider equivalence in which uniferous retroconnected (i.e., the fibres are connected) extensions are admitted too. This will be studied and developed elsewhere.

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