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FUNCTIONALLY HAUSDORFF SPACES

by Harriet LAZOWICK LORD

RÉSUMÉ. Un espace topologique X est dit fonctionellement Hausdorff si, quels que soient les éléments distincts x et y de X, il existe une fonction continue f: X→ ℝ telle que $f(x) \neq f(y)$. La catégorie FH des espaces fonctionellement Hausdorff et fonctions continues a la propriété que ses morphismes réguliers ne sont pas fermés par composition. De plus, FH est un exemple d'une sous-catégorie extrémale-épiréflective de TOP pour laquelle il n'y a pas de structure de factorisation (E,M) sur TOP telle que X ∈ FH ssi $\Delta_X \in M$.

INTRODUCTION.

A number of versions of the Diagonal Theorem have been proved by many authors. (See, for example, [2, 4, 5, 8].) The Theorem states that if a category C satisfies certain conditions and if A is an extremal-epireflective subcategory of C, then $X \in A$ iff Δ_X is A-regular. As a Corollary, we showed that if the A-regular morphisms are closed under composition, then there exists a strong factorization structure (E', M') on C such that

$$X \in A \text{ iff } \Delta_X \in M'.$$

The category of functionally Hausdorff spaces and continuous functions (denoted FH) is one of the few known examples of a full subcategory of TOP (the category of topological spaces and continuous functions) in which the composition of regular morphisms is not necessarily regular. FH is an example of the fact that if the condition on A-regular morphisms is omitted from the statement of the Corollary to the Diagonal Theorem, the conclusion of the Corollary may be false.

In Section 1 we define the category FH and characterize FH-epimorphisms and FH-regular morphisms. In Section 2 we obtain a factorization structure (for single morphisms) on TOP in which the left factor is the family of FH-epimorphisms. We show in Section 3 that the conclusion to the Corollary of the Diagonal Theorem is false for FH in TOP.

1. FH-EPIMORPHISMS AND FH-REGULAR MORPHISMS.

In this section we define functionally Hausdorff spaces, FH-epimorphisms, and FH-regular morphisms. We then characterize both of these classes of morphisms, and show that FH-epimorphisms are closed under composition, but FH-regular morphisms are not.

The category of functionally Hausdorff spaces has been studied by Dikranjan and Giuli [1] and by Schröder [11]. Examples of Hausdorff spaces that are not functionally Hausdorff can be found in [12], where functionally Hausdorff spaces are called Urysohn spaces, and Urysohn spaces are called completely Hausdorff.

DEFINITION 1.1. A topological space X is called *functionally Hausdorff* if for each pair of distinct points x and y in X there exists a continuous function $f: X \to \mathbb{R}$ such that $f(x) \neq f(y)$. We denote the category of functionally Hausdorff spaces and continuous functions by FH. FH is an extremal-epireflective subcategory of TOP, the category of topological spaces and continuous maps.

DEFINITION 1.2 [6, 8, 9]. A morphism $f: X \to Y$ in TOP will be called an FH-*epimorphism*, or FH-epi, if for all spaces A in FH,

$$r \circ f = s \circ f \text{ implies } r = s, \text{ where } r, s: Y \to A.$$

PROPOSITION 1.3. *$e: X \to Y$ is an* FH-*epimorphism iff for all morphisms $f: Y \to \mathbb{R}$, $f \circ e$ a constant implies that f is a constant.*

PROOF. Assume that e is an FH-morphism, and $f \circ e$ is a constant. Let $c = f(e(x))$, for $x \in X$. If f is not constant, then there exists $y \in Y$, $y \notin e(X)$, such that $f(y) \neq c$. Define $g: Y \to \mathbb{R}$ by

$$g(y) = c \text{ for all } y \in Y.$$

Then

$$f \circ e = g \circ e \quad \text{but} \quad f \neq g.$$

This contradicts the fact that e is an FH-epimorphism.

Assume that $f \circ e$ constant implies that f is constant for all morphisms $f: Y \to \mathbb{R}$. If e is not an FH-epimorphism, there exist α, β such that $\alpha \circ e = \beta \circ e$ but $\alpha \neq b$ where $\alpha, \beta : Y \to A$, A an object in FH. Thus there exists $y \in Y$, $y \notin e(X)$ such that $\alpha(y) \neq \beta(y)$. There exists, therefore, g: $A \to \mathbb{R}$ such that

$$g(\alpha(y)) \neq g(\beta(y)).$$

Let $h = g \circ \alpha - g \circ \beta$. Then $h \circ e$ is constant. (In fact, $h \circ e(x) = 0$ for $x \in X$.) However, $h(y) \neq 0$, and h is therefore not constant. Thus, e must be an FH-epimorphism.

COROLLARY 1.4. *In the category* FH, $e : X \to Y$ *is an epimorphism iff for all morphisms* $f : Y \to \mathbb{R}$, $f \circ e$ *a constant implies that* f *is a constant.*

PROPOSITION 1.5. FH-*epimorphisms are closed under composition.*

PROOF. Let $e_1 : X \to Y$, $e_2 : Y \to Z$ be FH-epimorphisms, and let f: $Z \to \mathbb{R}$ be a morphism such that $f \circ (e_2 \circ e_1)$ is constant. $f \circ (e_2 \circ e_1)$ constant implies $(f \circ e_2) \circ e_1$ constant. Thus $f \circ e_2$ is constant since e_1 is an FH-epimorphism. This implies that f is constant since e_2 is also an FH-epimorphism. Therefore, we have that if $f \circ (e_2 \circ e_1)$ is constant, then f must be constant, and so $e_2 \circ e_1$ is an FH-epimorphism.

DEFINITION 1.6 [1, 8, 10]. Let $i : S \to X$ be a morphism in TOP such that there exist r, $s : X \to A$, A in FH, with $(S, i) = \mathrm{Equ}(r, s)$. (Recall that in TOP, the equalizer of r and s is given by

$$\mathrm{Equ}(r,s) = \{x \in X \mid r(x) = s(x)\}.)$$

i will be called an FH-*regular morphism.*

PROPOSITION 1.7 [1]. $i : S \to X$ *is a regular morphism in* FH *iff* i *is a closed embedding, and, for each* $x \notin i(S)$, *there exists* f: $X \to \mathbb{R}$ *such that* $f(x) \neq 0$ *and* $f(i(s)) = 0$ *for all* $s \in S$.

PROOF. Assume that i is a closed embedding such that for each $x \notin i(S)$ there exists $f : X \to \mathbb{R}$ such that

$$f(x) \neq 0, \; f(i(s)) = 0 \text{ for all } s \in S.$$

We will show that

$$(S, i) = \mathrm{Equ}(q \circ u_1, q \circ u_2),$$

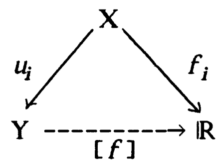
where u_1, u_2 are the injections from X into $Y = X \dot{\amalg} X$, the disjoint union of X with X, and q is the quotient map from Y onto Z, where $Z = Y/\sim$ and $\sim$ is the relation defined on Y by the following:

$$y \sim y \text{ for all } y \text{ in } Y,$$

and $y_1 \sim y_2$ if $y_1 = u_1(i(s))$ and $y_2 = u_2(i(s))$, for some s in S.

Both Y and Z are functionally Hausdorff.

To show that Y is functionally Hausdorff, let $y_1, y_2 \in Y$, $y_1 \neq y_2$. Consider the following diagram:



If $y_1 = u_1(x_1)$, $y_2 = u_1(x_2)$, let f be a function $f: X \to \mathbb{R}$ with $f(x_1) \neq f(x_2)$. Let $f_1 = f_2 = f$, and let $[f]$ be defined so that the above diagram commutes. Then

$$[f](y_1) \neq [f](y_2).$$

The proof is similar for $y_1 = u_2(x_1)$, $y_2 = u_2(x_2)$. If $y_1 = u_1(x_1)$ and $y_2 = u_2(x_2)$, let

$$f_1(x) = 0 \text{ for all } x \in X, \text{ and } f_2(x) = 1 \text{ for all } x \in X.$$

Then

$$[f](y_1) = 0 \text{ and } [f](y_2) = 1.$$

Thus Y is functionally Hausdorff.

We now show that Z is functionally Hausdorff. Let z_1, $z_2 \in Z$, $z_1 \neq z_2$. $z_1 = q(y_1)$, $z_2 = q(y_2)$. If

$$y_1 = u_1(x_1) \text{ and } y_2 = u_1(x_2),$$

let $f_i = f$, where $f: X \to \mathbb{R}$, $f(x_1) \neq f(x_2)$. (Define f similarly for the case $y_1 = u_2(x_1)$ and $y_2 = u_2(x_2)$.) If $y_1 = u_1(x_1)$ and $y_2 = u_2(x_2)$, define f_i as follows: if $x_1 \neq x_2$, define f as above; if $x_1 = x_2$, then $x_1 \notin i(S)$. Thus, there exists $f: X \to \mathbb{R}$ such that

$$f(i(s)) = 0 \text{ for all } s \in S, \ f(x_1) \neq 0.$$

Let $f_1 = f$, $f_2 = g$, where $g(x) = 0$ for all $x \in X$. Then

$$[f] \circ u_1(x) = f(x), \ [f] \circ u_2(x) = 0,$$

and so

$$[f](y_1) \neq 0 \text{ and } [f](y_2) = 0.$$

Define $f': Z \to \mathbb{R}$ by:

$$f'(z) = [f](y), \text{ where } z = q(y).$$

$f'(z_1) \neq 0$ and $f'(z_2) = 0$. That f' is well-defined and continuous follows from the definition of $[f]$. That $(S,i) = \mathrm{Equ}(q \circ u_1,\ q \circ u_2)$ is immediate.

Assume that $i: S \to X$ is an FH-regular morphism. Then

$$(S,\ i) = \mathrm{Equ}(f,g), \text{ where } f,\ g: X \to Y,$$

Y functionally Hausdorff.

$$i(S) = \{\, x \in X \mid f(x) = g(x) \,\}$$

is closed since Y is Hausdorff. If $x \notin i(S)$, then $f(x) \neq g(x)$. Thus there exists $h: Y \to \mathbb{R}$ such that $h(f(x)) = 0$ and $h(g(x)) \neq 0$. Then

$$(h \circ g - h \circ f)(i(s)) = 0 \text{ for all } s \in S \text{ and } (h \circ g - h \circ f)(x) \neq 0.$$

COROLLARY 1.8. *$i: S \to X$ is an FH-regular morphism iff i is a closed embedding. and. for each $x \notin i(S)$. there exists $f: X \to \mathbb{R}$ such that $f(x) \neq 0$ and $f(i(s)) = 0$ for all $s \in S$.*

PROOF. The corollary is an immediate consequence of Proposition 1.7 and the fact that i is an FH-regular morphism iff there exists a regular morphism $e: B \to A$ such that A is functionally Hausdorff. e is a regular morphism in FH, and (S,i) is a pullback of (B,e). (See Lemma 3.3 in [8].)

The following example shows that FH-regular morphisms are not necessarily closed under composition. Let X denote the set of real numbers and T the Smirnov topology on X [12]. T is defined as follows: Let T' denote the Euclidean topology on the reals, and let

$$A = \{1/n \mid n = 1,2,3,\dots\}.$$

$V \in T$ iff $V = U - B$, where $U \in T'$, $B \subset A$. Thus every open set in T that contains A intersects every open set that contains 0. Therefore, there is no $f: X \to \mathbb{R}$ with $f(a) = 0$ for all $a \in A$ and $f(0) = c \neq 0$, since this would imply

$$f^{-1}(0 - |c|/3,\ 0 + |c|/3) \cap f^{-1}(c - |c|/3, c + |c|/3) = \varnothing,$$

where

$$A \subset f^{-1}(0 - |c|/3,\ 0 + |c|/3) \text{ and } 0 \in f^{-1}(c - |c|/3,\ c + |c|/3).$$

Thus i_A: $A \to \mathbb{R}$ is not an FH-regular morphism.

Let $B = A \cup \{0\}$. Since B is closed in T', the Euclidean topology, and $T' \subset T$, for each $x \notin B$ there exists $f: X \to \mathbb{R}$ so that $f(b) = 0$ for all $b \in B$ and $f(x) \neq 0$. This follows from the fact that (X,T') is completely regular. Define $g: B \to \mathbb{R}$ by $g(a) = 0$ for all a in A, $g(0) = 1$. g is clearly continuous. Thus $i'_A: A \to B$ and $i_B: B \to X$ are FH-regular, but their composition, $i_A: A \to X$, is not.

2. AN (FH-epi, M)-FACTORIZATION.

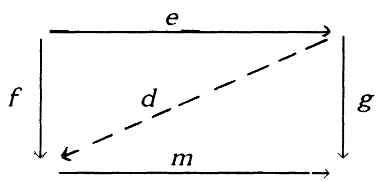
In this section we describe a class of morphisms M' that has the property that (FH-epi,M') is a factorization structure on TOP. We begin by recalling the definition of a factorization structure.

DEFINITION 2.1 [7,8]. Let E and M be classes of morphisms in the category C. (E, M) is a *factorization structure* (for single morphisms) if:

1. E and M are closed under composition;
2. $E \cap M$ contains all isomorphisms;
3. Each morphism f in C is factorizable; i.e.,

$$f = m \circ e, \text{ where } e \in E \text{ and } m \in M;$$

4. C has the (E,M) unique diagonalization property, i.e., if $m \circ f = g \circ e$, with $m \in M$, $e \in E$, there exists a unique morphism d that makes the following diagram commute:



If M is contained in the class of embeddings in C, (E, M) is called a *strong factorization structure.*

Let

$$M' = \left\{ m \;\middle|\; \begin{array}{l} m: X \to Y \text{ is an embedding, and if } m(X) \subsetneqq S \subset Y, \\ \text{then there exists } f: S \to \mathbb{R} \text{ such that } f(m(x)) = 0 \\ \text{for all } x \in X \text{ and } f(s) \neq 0 \text{ for some } s \in S. \end{array} \right\}$$

LEMMA 2.2. *M' is closed under composition.*

PROOF. Let $m_1: X \to Y$, $m_2: Y \to Z \in M'$. Then $m_2 \circ m_1: X \to Z$. Assume $m_2 \circ m_1(X)$ is contained in, but different from, S.

If $S \subset m_2(Y)$, then

$$m_1(X) \subset m_2^{-1}(S) \subset Y,$$

so there exists $g: m_2^{-1}(S) \to \mathbb{R}$ such that

$$g(m_1(x)) = 0 \text{ for all } x \in X, \; g(m_2^{-1}(s)) \neq 0 \text{ for some } s \in S.$$

Let $m: Y \to m_2(Y)$ be defined by $m(y) = m_2(y)$ for all $y \in Y$, and let $f = g \circ m^{-1}|S$.

If S is not contained in $m_2(Y)$, there exists $f: S \to \mathbb{R}$ such that $f(m_2(y)) = 0$ for all $y \in Y$, $f(s) \neq 0$ for some $s \in S$. Thus $f(m_2 \circ m_1(x)) = 0$ for all $x \in X$ and $f(s) \neq 0$ for some $s \in S$.

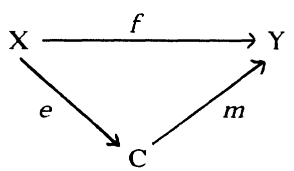
LEMMA 2.3. *M' is closed under intersections.*

PROOF. Let $X_i \subset Y$ be such that the inclusions $i_{X_i}: X_i \to Y$ are in M'. For each i, if $X_i \subset S \subset Y$, then there exists $f_i: S \to \mathbb{R}$ such that $f_i(x) = 0$ for all $x \in X_i$, $f_i(s) \neq 0$ for some $s \in S$. If $\cap X_i \subset S \subset Y$ and $\cap X_i \neq S$, then there exists X_{i_0} such that $X_{i_0} \neq S$. Define T by $T = X_{i_0} \cup S \neq X_{i_0}$. Since $X_{i_0} \subset T \subset Y$, there exists $f: T \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in X_{i_0}$, $f(t) \neq 0$ for some $t \in T$, and so $t \in S$ since $t \notin X_{i_0}$. Thus $f(x) = 0$ for all $x \in \cap X_i$, $f(s) \neq 0$ for some $s \in S$. $f|S$ is the desired function.

THEOREM 2.4. (FH-epi, M') *is a strong factorization structure on* TOP.

PROOF. In Proposition 1.5, we showed that the FH-epimorphisms are closed under composition. In Lemma 2.2, we showed that M' is closed under composition. That FH-epi and M' both contain all isomorphisms is obvious.

To show that each morphism f has an (FH-epi, M')-factorization, consider the following diagram



where

$$(C, m) = \cap\{(S, i_S) \mid S \subset Y,\ f(X) \subset S, \text{ and } i_S \in M'\}.$$

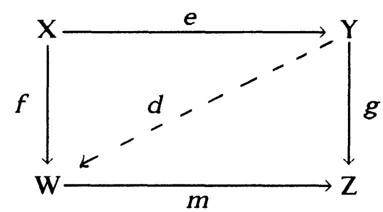
$m \in M'$ since M' is closed under intersections (Lemma 2.3). It remains to prove $e \in$ FH-epi, where $e(x) = f(x)$ for all $x \in X$. Let

$$D = \{c \mid c \in C \text{ and } \alpha(f(x)) = 0\ \forall x \in X \text{ implies } \alpha(c) = 0 \text{ for all } \alpha: C \to \mathbb{R}\}.$$

$f(X) \subset D \subset C$. If $D \neq C$, then $i_D: D \to Y$ is not in M' since $i_D \in M'$ contradicts the fact that C is the smallest subspace of Y that contains $f(X)$ and whose inclusion into Y belongs to M'. Thus $D \neq C$ implies $i_D \notin M'$. We will show that $i_D \in M'$, and therefore $D = C$. That $D = C$ implies $e \in$ FH-epi is immediate.

To show that $i_D \in M'$, we must show that if $D \subset S$, $D \neq S$, there exists $g: S \to \mathbb{R}$ such that $g(d) = 0$ for all $d \in D$ and $g(s) \neq 0$ for some $s \in S$. If $S \subset C$, we know that $D \neq C$ implies that there exists $\alpha: C \to \mathbb{R}$ such that $\alpha(f(x)) = 0$ for all $x \in X$, $\alpha(c) \neq 0$ for some $c \in C$. Since $\alpha(f(x)) = 0$ implies $\alpha(d) = 0$ for all $d \in D$, we can let $g = \alpha|S$. If S is not contained in C, let $T = S \cup C$. Since $i_C \in M'$ and $C \neq T$, there exists $j: T \to \mathbb{R}$ such that $j(c) = 0$ for all $c \in C$, $j(t) \neq 0$ for some $t \in T$. Thus $j(s) \neq 0$ for some $s \in S$, $j(f(x)) = 0$ for all $x \in X$ and so $j(d) = 0$ for all $d \in D$. Let $g = j|S$. We have shown $i_D \in M'$, and so $D = C$ and $e \in$ FH-epi.

In order to complete the proof that (FH-epi, M') is a factorization structure on TOP, we must show that if $g \circ e = m \circ f$, there exists a unique d such that the following diagram commutes:



If $m(W) = Z$, then m is a homeomorphism and $d = m^{-1} \circ g$. If $m(W) \neq Z$, we will show that $g(Y) \subset m(W)$. Then $d = r^{-1} \circ g$, where r: $W \to m(W)$ is defined by $r(w) = m(w)$ for all $w \in W$.

Suppose that $m(W) \subset S \subset Z$ and $m(W) \neq S$. Since $m \in M'$, there exists $k: S \to \mathbb{R}$ such that $k(m(w)) = 0$ for all $w \in W$ and $k(s) \neq 0$ for some $s \in S$. Thus

$$\alpha(m(f(x))) = 0 \text{ for all } x \in X,$$

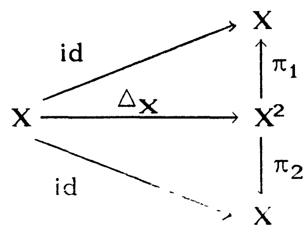
and so $\alpha(g(e(x))) = 0$ for all $x \in X$. Therefore $\alpha \circ g \circ e$ is constant, and thus $\alpha \circ g$ is constant since e is an FH-epi. If there is some $y_0 \in Y$ with $g(y_0) \not\in m(W)$, then if $S = m(W) \cup \{g(y_0)\}$, we have $\alpha(g(y_0)) \neq 0$ since $m \in M'$ and $\alpha \circ g(y_0) = 0$ since e is an FH-epi. Therefore $g(Y) \subset m(W)$.

COROLLARY 2.5. *In the category* FH, $m: X \to Y$ *is an extremal monomorphism iff whenever* $m(X) \subset S \subset Y$, $m(X) \neq S$, *there exists* $f: X \to \mathbb{R}$ *such that* $f(m(x)) = 0$ *for all* $x \in X$ *and* $f(s) \neq 0$ *for some* $s \in S$.

PROOF. Let M_{FH} denote those morphisms in FH that belong to the class M'. (epi, M_{FH}) is a factorization structure for FH. (This follows from the fact that extremal-epireflective subcategories of TOP are mono-hereditary.) Since the right factor is uniquely determined by the left factor (see Corollary 33.7 in [7], for example) M_{FH} must be the class of all extremal monomorphisms in FH.

3. THE DIAGONAL MORPHISM.

DEFINITION 3.1 [7]. The *diagonal* $\Delta_X: X \to X^2$ is the unique morphism defined by the categorical product $(X \times X, \pi_1, \pi_2)$; i.e., Δ_X is the unique morphism such that the following diagram commutes:



In TOP, the diagonal $\Delta_X: X \to X^2$ is the function $\Delta_X(x) = (x, x)$ for all $x \in X$.

We can conclude from the Diagonal Theorem [4, 5, 8] that X is functionally Hausdorff iff Δ_X is FH-regular. The corollary to that theorem [8] states that if A is an extremal-epireflective subcategory of TOP with the property that the A-regular morphisms are closed under composition, then there exists a factorization structure (E, M) on TOP such that X is an object in A iff $\Delta_X \in M$.

We have already shown that the FH-regular morphisms are not closed under composition. The following theorem shows that without this condition on the A-regular morphisms, the corollary is false.

THEOREM 3.2. *There is no factorization structure* (E, M) *on* TOP *that has the property that* X *is functionally Hausdorff iff* $\Delta_X \in M$.

PROOF. Let X be a Hausdorff space that is not functionally Hausdorff which contains exactly 2 points a, b such that there exists no continuous $f: X \to \mathbb{R}$, with $f(a) \neq f(b)$. Let $\Delta_X: X \to X^2$ be the diagonal map (i.e., $\Delta_X(x) = (x, x)$ for all $x \in X$). If $\Delta_X(X) \subset S \subset X^2$ and $\Delta_X(X) \neq S$, then if there exists $(x, y) \in S$ for which there exists $f: X \to \mathbb{R}$, with $f(x) \neq f(y)$, define

$$f': S \to \mathbb{R} \text{ by } f'(w, z) = f(w) - f(z).$$

Then $f'((x, x)) = 0$ for all $x \in X$, $f'((x, y)) \neq 0$. If there exists no such pair (x, y), then

$$S = \Delta_X(X) \cup \{(a, b)\}, \ \Delta_X(X) \cup \{(b, a)\}, \text{ or } \Delta_X(X) \cup \{(a, b)\} \cup \{(b, a)\}.$$

Define $f: S \to \mathbb{R}$ as follows:

$$f((x,x)) = 0 \text{ for all } x \in X, \; f((a,b)) = f((b,a)) = 1.$$

f is continuous since X is Hausdorff. Thus $\Delta_X \in M'$.

If (E,M) is a factorization structure on TOP such that $E \subset$ FH-epi, then $M' \subset M$. If there were a factorization structure (E, M) on TOP with the property that X is functionally Hausdorff iff $\Delta_X \in M$, then $E \subset$ FH-epi. (See Lemma 2.8 in [8].) The space X cited above is an example of a Hausdorff space that is not functionally Hausdorff and whose diagonal Δ_X is in M whenever (E,M) is a factorization structure on TOP with $E \subset$ FH-epi.

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