CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

BRIAN J. DAY

Onto Gelfand maps : appendix

Cahiers de topologie et géométrie différentielle catégoriques, tome 30, nº 4 (1989), p. 345-347

<http://www.numdam.org/item?id=CTGDC_1989__30_4_345_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

VOL. XXX-4 (1989)

ONTO GELFAND MAPS: APPENDIX

by Brian J. DAY

RÉSUMÉ. Si X est un espace compact, toute fonctionelle multiplicative préservant l'unité de C(X;K) vers K est un projecteur, lorsque K est le corps réel ou complexe. Ce résultat est étendu ici à d'autres algèbres topologiques.

INTRODUCTION.

It is well known that if X is a compact Hausdorff space, then each multiplicative identity-preserving functional from the space C(X;K) to K is a projection map if K is either the real or complex field. In this article this result is extended to include certain other real topological algebras.

The proof is based on a combination of the fact (see [2,3]) that metric compact Hausdorff spaces form a codense full subcategory of the category of all compact Hausdorff spaces and continuous maps and the final result proved in [1] on multiplicative functionals (non-commutative case - same proof). For notation and further implications concerning Gelfand dualities, we refer the reader back to [1].

1. THE FIRST REDUCTION.

If A is any metric topological ring and the canonical evaluation map

$$\varepsilon(M): M \longrightarrow \mathrm{Alg}([M,A],A)$$

is a homeomorphism whenever M is a compact Hausdorff metric space, then $\varepsilon(X)$ is a homeomorphism whenever X is a compact Hausdorff space. In order to show this, we first consider the codensity expression:

$$X \approx \int_M(\mathrm{Lim}(X,M), M)$$

where Lim(X,M) denotes all the continuous maps from X to M, and the brackets denote set-indexed powers. Then, since

$$\mathrm{Lim}(X,M)\times [M,A] \longrightarrow [X,A]$$

is a jointly surjective family of continuous maps, we deduce that the canonical map

$$\mathrm{Alg}([X,A],A) \longrightarrow \int_M (\mathrm{Lim}(X,M), \mathrm{Alg}([M,A],A))$$
$$\approx \int_M (\mathrm{Lim}(X,M), M) \approx X.$$

is an injection which is left inverse to $\varepsilon(X)$, hence is a homeomorphism as required.

2. THE SECOND REDUCTION.

Now suppose that A denotes a real topological algebra (with identity) structure, with no small subspaces, on the space of all continuous functions from a fixed compact Hausdorff space into the real numbers. Since each metric space is sequential it now follows from the Tietze extension Theorem that, for all M, $\varepsilon(M)$ is a homeomorphism if $\varepsilon(N)$ is a homeomorphism where N is the one-point compactification of the discrete positive integers. Then, from the last result in [1], we obtain:

THEOREM. *Let* A *be as above but with no central idempotents, and let* X *be any compact Hausdorff space. Then* $\varepsilon(X)$ *is a homeomorphism.*

For further implications in the case where the fixed compact Hausdorff space for A is finite, see [1]. Examples of such an algebra A include Weil algebras and certain convolution group algebras.

Note that, once a duality has been established for a finite-dimensional real algebra A, there results a relative Stone-Weierstrass Theorem, based on this A, which states that, if B is a left-right point-separating sub-A-algebra of a given exponent [X,A], where X is a compact Hausdorff space, then the induced set map

$$\mathrm{Alg}_0([X,A],A) \longrightarrow \mathrm{Alg}_0(B,A)$$

is a *bijection* (see [1] Section 1). This, in turn, allows "upgrading" to Gelfand dualities, with respect to A, over Lim-categories such as bornological limitspaces, simplicial limitspaces, "smooth" limitspaces, and so on (see [1] Theorem 2.2) via reduction to the compact case.

REFERENCES.

1. DAY B.J., Onto Gelfand transformations, *Cahiers Top. et Géom. Diff. Cat.* XXVIII-3 (1987), 227-234.
2. GABRIEL P. & ULMER F., Lokal präsentierbare Kategorien, *Lecture Notes in Math.* 221, Springer (1971).
3. ISBELL J., Adequate subcategories, *Illinois J. Math.* 4-4 (1960), 541-552.

Department of Mathematics
Macquarie University
NORTH RYDE N.S.W. 2109
AUSTRALIA