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COMBINATORIAL-GEOMETRIC ASPECTS OF POLYCATEGORY THEORY: PASTING SCHEMES AND HIGHER BRUHAT ORDERS (LIST OF RESULTS)

by M.M. KAPRANOV and V.A. VOEVODSKY

RÉSUMÉ: Nous étudions la combinatoire et la topologie des 'schémas de composition' n-catégoriques introduits par M. Johnson. Ces schemas ont certaines relations avec polytopes convexes, leurs triangulations, les arrangements d'hyperplans et les matroïdes orientés. En particulièr, nous presentons une description n-catégorique des 'ordres de Bruhat supérieurs' introduits par Y.I. Manin and V.V. Schechtman pour l'étude des généralisations 'supérieures' des équations de Yang-Baxter.

This talk is devoted to detailed study of free n-categories and their relations with more "classical" geometric objects. Among these objects we list convex polytopes, their triangulations, configurations of hyperplanes, oriented matroids, and so-called "higher Bruhat orders", introduced by Y.I. Manin and V.V. Schechtman.

The base for our study is the notion of a pasting diagram for n-categories introduced by M. Johnson [J]. Though we feel that much of Johnson's theory can be substantially simplified, even in its present state it yields a lot of combinatorial objects, some of which are known, and the others new and unexpected.

§1. PASTING SCHEMES.

The notion of a polycategory we use is the "globular" one ([S1], [J], [MS], compare also [Gr]), in contrast with the cubical version [BH]. An n-category is a category enriched in the cartesian closed category (n-1)-cat of (n-1)-categories. Sometimes polycategories in our sense are called

n-categories, and the cubical ones n-tuple categories. We shall use the one-sorted point of view on an n-category C, identifying it with a set $\mathrm{Mor}C$, equipped with mappings $s_i, t_i : \mathrm{Mor}C \to \mathrm{Mor}C$, $i = 0,1,\dots,n-1$ and the partial compositions $(a,b) \to a \underset{i}{*} b$ defined when $s_i a = t_i b$. We call *i-morphisms* elements $a \in \mathrm{Mor}C$ such that $s_{i-1}a = t_{i-1}a = a$. Objects are just 0-morphisms.

For any two objects x,y of an n-category C, an (n-1)-category $\mathrm{Hom}_C(x,y)$ is defined, whose objects are 1-morphisms in C from x to y.

Intuitively a pasting scheme is an "algebraic expression with indeterminate elements" which can be evaluated in an arbitrary n-category as soon as we have associated, in a compatible way, to the indeterminates in the expression concrete polymorphisms. For example,

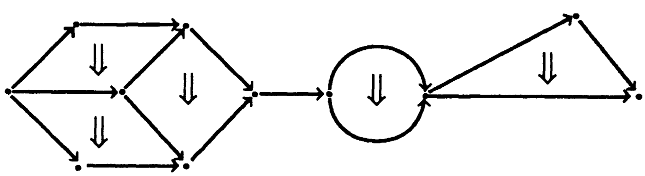

Fig. 1

is a pasting scheme. In [J], a combinatorial theory of pasting schemes was developed. We shall recall some points of the theory of [J]. In fact, our approach to pasting schemes is slightly different from the original one in [J] but in all important cases they are equivalent. A similar approach has been developed in [S2]; see also [P] for the 2-dimensional case.

A *pasting scheme* is a collection $A = (A_i)_{i \geq 0}$ of finite sets such that $A_i = \emptyset$ for $i >> 0$, equipped with binary relations $B_i, E_i \subseteq A_{i+1} \times A_i$. Elements of A_i will be called, somewhat abusively, i-cells of A, cf.§2. These relations must satisfy certain conditions, the most important of which is the following. Let $\mathbb{Z}[A_i]$ be the free abelian group generated by A_i. Define the differential $\partial : A_{i+1} \to A_i$ by the formula $\partial(a) = \partial_+(a) - \partial_-(a)$, where

$$\partial_+(a) = \sum_{b:(a,b)\in B_i} b , \qquad \partial_-(a) = \sum_{b:(a,b)\in E_i} b .$$

Then $\partial\partial$ must be equal to zero, that is,

$$\mathbb{Z}[A] = \{\dots \longrightarrow \mathbb{Z}[A_2] \xrightarrow{\partial} \mathbb{Z}[A_1] \xrightarrow{\partial} \mathbb{Z}[A_0]\}$$

must be a chain complex. This complex determines A as a pasting scheme. Let us inductively define families of operators $\partial(k)$, $\partial_+(k)$, $\partial_-(k)$ from $\mathbb{Z}[A_n]$ to $\mathbb{Z}[A_{n-k}]$ as follows.

For $k = 0$, set $\partial(0) = \partial_+(0) = \partial_-(0) = \mathrm{Id}$.
For $k = 1$, set $\partial(1) = \partial$, $\partial_+(1) = \partial_+$, $\partial_-(1) = \partial_-$.
For $k > 1$, set $\partial(k) = \partial_+(k\text{-}1)\partial(1) = \partial_-(k\text{-}1)\partial(1)$.

Let $a \in A_n$ be some element and $\partial(k)(a) = \sum_{b\in A_{n-k}} n_b b$. Define $\partial_+(k)(a) = \sum_{n_b>0} n_b b$ and similarly for $\partial_-(k)$, so that $\partial(k) = \partial_+(k)\text{-}\partial_-(k)$. The set of data $\{E,B,A\}$ is called a pasting scheme if the multiplicities n_b in all $\partial_+(k)$ and $\partial_-(k)$ are equal to 1. We define relations $E_i^j, B_i^j \subseteq A_j \times A_i$ for $j \geq i$ setting $(a,b) \in E_i^j$ if b enters in the sum for $\partial_-(j\text{-}i)(a)$. Similarly for B_i^j. Therefore, we can say that a pasting scheme is a based chain complex of a particular kind. We shall use this description in the sequel. We shall note $\dim A$, the dimension of A, i.e. the maximum of i such that A_i is non-empty. If $a \in A_i$ then we shall write $\dim a = i$. If $a \in A_{i+1}$, then we set $B_i(a) = \{b \in A_i : (a,b) \in B_i\}$. Similarly for E_i.

If A is a pasting scheme and $a \in A_m$, then we denote by $R(a)$ the set of all $b \in A_i$, $i \leq m$ such that there exists a sequence $a = a_1, a_2, \dots, a_{m-i} = b$, in which for each j the pair (a_j, a_{j+1}) lies either in E_{m-j+1} or in B_{m-j+1}. Geometrically $R(a)$ is to be thought of as the set of cells lying in the closure of a (cf. §2 below).

The really important notion is the notion of a *composable pasting scheme*, that is a scheme which is, in Johnson's terminology, loop free and well-formed. These conditions eliminate the following types of behaviour:

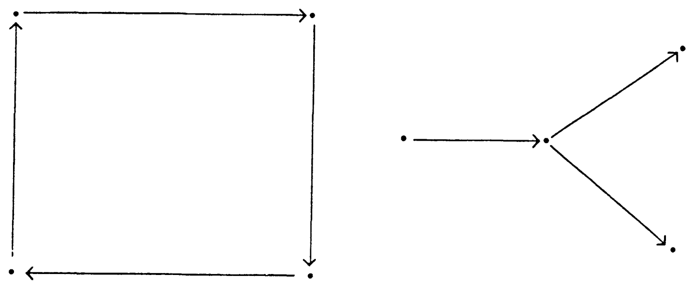

Fig. 2

For a composable pasting scheme A of dimension n, an n-category $\mathrm{Cat}(A)$ is defined [J]. Its polymorphisms are composable subpasting schemes (in a natural sense) in A, and the compositions are given by the union. In particular, for any composable pasting scheme A of dimension n we have composable subpasting schemes $s_i A, t_i A \subset A$ of dimension i. Explicitly,

$$(s_{n-1}A)_i = A_i - E_i^n(A_n) \ , \ (t_{n-1}A)_i = A_i - B_i^n(A_n) \ ,$$

and further s_i, t_i are defined by iterating this construction. A *realisation* of a composable pasting scheme A in an n-category C is an n-functor $\mathrm{Cat}(A) \longrightarrow C$. The "resulting polymorphism" of such a realisation is the value of this functor on $A \in \mathrm{Mor}\,\mathrm{Cat}(A)$. As shown in [J], $\mathrm{Cat}(A)$ is freely generated (in the sense of Street [S]) by polymorphisms of the type $R(a)$, $a \in A_i$, $i \geq 0$, which, in this case, are composable subpasting subschemes. Therefore to define a realisation of A it suffices to associate, in a compatible way, to each element A_i some i-morphism of C.

§2. GEOMETRIC REALISATIONS OF PASTING SCHEMES. STRUCTURES OF PASTING SCHEMES ON CONVEX POLYTOPES.

Most of pasting schemes arising in practise come from some geometric objects, e.g. polytopes. This induces an idea to consider the geometric realisation of a pasting scheme as a cellular complex.

Let A be a pasting scheme. The set $A = \cup A_i$ is partially ordered by the relation R.

Definition 2.1. The *geometric realisation* $|A|$ of a pasting scheme A is the nerve of the category associated to the poset (A,R).

Therefore, $|A|$ is a simplicial complex, whose p-dimensional simplices correspond to chains $x_0Rx_1R...Rx_p$, $x_i \neq x_j$, where yRx means that $y \in R(x)$. To any m and $a \in A_m$ we associate a closed subcomplex $[a] \subset |A|$ whose p-simplices correspond to chains $x_0Rx_1R...Rx_p$, where x_pRa.

Theorem 2.2. *If* A *is a composable pasting scheme of dimension* n, *then:*

a) $|A| - |s_{n-1}A| - |t_{n-1}A|$ *is homeomorphic to a disjoint union of several open* n-*balls.*

b) *For each* m *and* $a \in A_m$ *the subcomplex* $[a]$ *is homeomorphic to a closed* m-*ball. Therefore the subcomplexes of the form* $[a]$, $a \in A_m$, $m \geq 0$, *form a cellular decomposition of* $|A|$.

In general, it is very difficult to decide, whether a given CW-complex is homeomorphic to a ball, or is a topological manifold, because this amounts to recognising a sphere among other manifolds. For a 3-sphere this is a classical problem. The success in our situation comes from considering additional structure on the complex: the grouping of the cells lying on the boundary of a given cell, to "beginning" and "end".

Let $M \subset \mathbb{R}^n$ be a bounded convex polytope of dimension n, and

$$p = \{\mathbb{R}^n \xrightarrow{p_{n,n-1}} \mathbb{R}^{n-1} \longrightarrow \dots \longrightarrow \mathbb{R}^2 \xrightarrow{p_{2,1}} \mathbb{R}\}$$

be a system of affine projections such that any k-dimensional facet of M projects injectively to $\mathbb{R}^k$ (we shall call a system of projections with this property *admissible*). We shall suppose that all $\mathbb{R}^i$ are equipped with their standard orientations. Then the fibres of $p_{k,k-1}$ become oriented lines. Denote the composite projection $\mathbb{R}^n \longrightarrow \mathbb{R}^k$ by p_k.

Let $A_k(M)$ be the set of k-dimensional facets of M. Define on $A(M) = \cup A_k(M)$ a structure of a pasting scheme. Let $\Gamma \in A_k(M)$, $\Delta \in A_{k-1}(M)$, $\Delta \subset \Gamma$. Consider the image $p_k(\Gamma) \subset \mathbb{R}^k$. Let $H : \mathbb{R}^k \longrightarrow \mathbb{R}$ be an affine-linear function such that $H|_{p_k(\Delta)} = 0$, $H|_{p_k(\Gamma)} \geq 0$. Say that $\Delta \in B_k(\Gamma)$ (resp. $\Delta \in E_k(\Gamma)$) if $H(t) \longrightarrow +\infty$ (resp. $H(t) \longrightarrow (-\infty)$) when t tends to the infinity along a fibre of $p_{k,k-1} : \mathbb{R}^k \longrightarrow \mathbb{R}^{k-1}$ in positive direction:

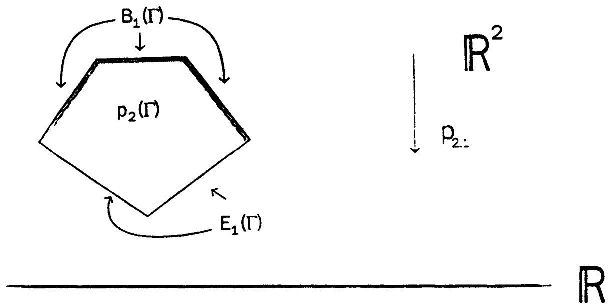

Fig. 3

We shall call this pasting scheme $A(M,p)$.

Theorem 2.3. *If* $M \subset \mathbb{R}^n$ *is a bounded n-dimensional polytope and* $p = \{\mathbb{R}^n \xrightarrow{p_{n,n-1}} \mathbb{R}^{n-1} \longrightarrow \dots \longrightarrow \mathbb{R}^2 \xrightarrow{p_{2,1}} \mathbb{R}\}$ *is an admissible system of projections, then* $A(M,p)$ *is a composable pasting scheme.*

Example 2.4. Let $M = \Delta^n$ be an n-dimensional simplex, and $\partial_j : A_k(\Delta^n) \longrightarrow A_{k-1}(\Delta^n)$ be the standard simplicial operators, $j = 0,1,\dots,k$. Namely, denote vertices of Δ^n by $(0),(1),\dots,(n)$. Then each facet is determined by a subset $\sigma \subset \{0,\dots,n\}$, which we write in the increasing order: $\sigma = \{\sigma_0 < \dots < \sigma_k\}$. Then $\partial_j\sigma = \{\sigma_0 < \dots < \hat{\sigma}_j < \dots < \sigma_k\}$. The standard structure of pasting scheme on Δ^n, considered in [S1], starts from the usual differential $\partial = \Sigma(-1)^i\partial_i$ in the chain complex of Δ^n. Therefore, for $\sigma \in A_k(\Delta^n)$, we have

$$B_{k-1}(\sigma) = \{\partial_j\sigma, j \text{ is even}\}, \quad E_{k-1}(\sigma) = \{\partial_j\sigma, j \text{ is odd}\}.$$

The corresponding n-category $Cat(\Delta^n)$ was called by Street the n-th oriental.

Let us give an interpretation of this structure of pasting scheme by means of projections. Fix $n + 1$ real numbers $t_0 > \dots > t_n \in \mathbb{R}$. Define $n + 1$ points

$$v_j = (t_j, t_j^2, \dots, t_j^n) \in \mathbb{R}^n, \ j = 0,\dots,n .$$

These points are in general position since the determinant of the corresponding matrix is the classical Vandermonde determinant. Therefore the convex hull of v_j is a n-simplex which we consider with the given numeration of vertices. Consider the projections $p_{i,i-1} : \mathbb{R}^i \longrightarrow \mathbb{R}^{i-1}$ which forgets

the last co- ordinate.

Theorem 2.5. *The structure of pasting scheme on* Δ^n *given by the above projections coincides with the combinatorially defined structure used by Street.*

Example 2.6. Let

$$I^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n;\ 0 \le x_i \le 1\}$$

be the n-dimensional cube and $\partial_i^p : A_k(I^n) \to A_{k-1}(I^n)$, $i = 1,2,\dots,k$, $p = 0,1$ be the standard cubic boundary operators, see [K]. Explicitly, facets of I^n have the form

$$F(X,Y) = \{x \in I^n : x_i = 0 \text{ for } i \in X \ ,\ x_i = 1 \text{ for } i \in Y\}$$

for $X,Y \subset \{1,\dots,n\}$, $X \cap Y = \emptyset$. Let

$$k = \dim F(X,Y) = n - \mathrm{Card}(X) - \mathrm{Card}(Y)$$

and $a_1 < \dots < a_k$ be all elements of $\{1,\dots,n\} - X - Y$. Then

$$\partial_i^0 F(X,Y) = F(X \cup \{a_i\},Y) \ ,\quad \partial_i^1 F(X,Y) = F(X,Y \cup \{a_i\},Y) \ .$$

The differential in the chain complex of I^n is given by $\partial = \Sigma(-1)^{i+p}\partial_i^p$. Therefore, we introduce the relations B_k, $E_k \subset A_{k+1}(I^n) \times A_k(I^n)$ by setting

$$B_k(F) = \{\partial_i^p F, i+p \text{ is even}\} \ ,\ E_k(F) = \partial_i^p F, i+p \text{ is odd}\} \ .$$

Theorem 2.7. *The relations* B_k, E_k *define on* $A(I^n)$ *the structure of a composable pasting scheme.*

We can deduce this theorem from the other description of this structure of pasting scheme. Fix real numbers $t_1 >> t_2 >> ,,, >> t_n$, where $>>$ means "sufficiently greater than". Define vectors $v_j = (t_j, t_j^2, \dots, t_j^n) \in \mathbb{R}^n$ as above and realize the cube as the parallelotope with vertices $\sum_{i \in J} v_i$, where J runs over all subsets of $\{1,\dots,n\}$. Define projections $p_{i,i-1} : \mathbb{R}^i \to \mathbb{R}^{i-1}$ as above, by forgetting the last coordinate. This defines on I^n some structure of a composable pasting scheme.

Theorem 2.8. *The two described structures of pasting scheme on* I^n *coincide.*

§3. HIGHER ORDERS ASSOCIATED TO A COMPOSABLE PASTING SCHEME.

Categories of the form Cat(A) , A being a composable pasting scheme, possess remarkable properties of order, which we shall now describe.

Definition 3.1.

a) A 1-category C with a finite number of morphisms is called *ordered,* if the relation $\mathrm{Hom}(x,y) \neq \emptyset$ on the set Ob C is a partial order, and $\mathrm{Hom}(x,x)$ is always a singleton. A category is called *strictly ordered,* if it is ordered and Ob C has unique maximal and minimal elements.

b) Suppose that for $k < n$ the notion of a (strictly) ordered k-category is defined. Say that an n-category C is *(strictly) ordered* if:
the relation $\mathrm{Hom}(x,y) \neq \emptyset$ on Ob C is a partial order (with unique maximal and minimal elements);
for any $x,y \in \mathrm{Ob}\, C$ the (n-1)-category $\mathrm{Hom}_C(x,y)$ is (strictly) ordered, and $\mathrm{Hom}(x,x)$ is a singleton n-category.

If C is a strictly ordered n-category, then we can define a strictly ordered (n-1)-category

$$\Omega C = \mathrm{Hom}_C(x_{min} , x_{max}) ,$$

where x_{min} , $x_{max} \in \mathrm{Ob}\, C$ are maximal and minimal elements. So we can form $\Omega^2 C = \Omega\Omega C$ etc.

Theorem 3.2 *Let* A *be an* n*-dimensional composable pasting scheme. Then* Cat(A) *is a strictly ordered* n*-category.*

So, to each composable pasting scheme A we associate a hierarchy of posets $X_k = \mathrm{Ob}\, \Omega^k \mathrm{Cat}(A)$. There are natural surjective maps

$$\{\text{maximal chains in } X_k\} \longrightarrow X_{k+1} .$$

Definition 3.3. We call the *higher Stasheff order* S(n,k) the poset $\mathrm{Ob}\, \Omega^k \mathrm{Cat}(\Delta^n)$.

Examples 3.4.

a) S(n,1) is the set of all subsets of an n-element set, (i.e. of vertices of an n-cube) partially ordered by inclusion. It has 2^n elements.

b) Elements of S(n,2) are identified with triangulations of a planar convex (n+1)-gon which we shall denote

$M(n+1,2)$. Namely, number the vertices by $0,1,\ldots,n$ in circular order. Let T be a triangulation of $M(n+1,2)$. Lift each triangle of T with vertices i,j,k, to the corresponding triangle in Δ^n. It is clear that we thus obtain all films from $\Omega^2\mathrm{Cat}(\Delta^n)$, cf. [S1].

It is well-known that the triangulations of $M(n+1,2)$ are in bijection with bracketings of n factors. Their number is the Catalan number

$$c_n = (2n - 2)!/(n-1)!(n-1)!(n-1) .$$

These bracketings are vertices of an (n-3)-dimensional polytope constructed by J. Stasheff [Sta], which explains our terminology.

Denote by $M(n+1,k) = p_k(\Delta^n)$ the image of the simplex under the projection to $\mathbb{R}^k$ defined in the example 2.4. In other words, $M(n+1,k)$ is the convex hull of $n+1$ points lying on the Veronese curve in $\mathbb{R}^k$ given by $\{(t,t^2,\ldots,t^k),\ t \in \mathbb{R}\}$. It is classically called the cyclic polytope and is of importance in general theory of convex polytopes, since its face numbers possess some extremal properties, see [Gru] and references therein.

Theorem 3.4. *Elements of the poset* $\Omega^k\mathrm{Cat}(\Delta^n)$ *are in bijection with triangulations of the cyclic polytope* $M(n+1,k)$ *which do not add new vertices.*

Remark 3.5. It would be interesting to construct a natural polytope with the set of vertices $S(n,k)$, thus generalising the Stasheff polytope. In fact, in [GZK1-2] for any convex polytope $Q \subset \mathbb{R}^k$ and any set $A \subset \mathbb{R}^k$ containing all vertices of Q, a new convex polytope $P(Q,A) \subset \mathbb{R}^A$ was defined, whose vertices are in bijection with those triangulations of Q with vertices in A, which are regular, i.e. admit a strictly convex piecewise-linear function. This polytope was called the secondary polytope for (Q,A), cf. also [BFS].

Unfortunately, we do not know whether all triangulations of $M(n+1,k)$ are regular. It seems that the answer is negative.

§4. FREE N-CATEGORY GENERATED BY A N-CUBE AND HIGHER BRUHAT ORDERS.

In the course of study of higher-dimensional generali-

sations of the Yang-Baxter equation, Yu.I. Manin and V.V. Schechtman introduced in [MS 1-3] posets $B(n,k)$ called the higher Bruhat orders. The set $B(n,1)$ is the symmetric group S_n with its weak Bruhat order, and $B(n,k+1)$ is a certain quotient of the set of maximal chains in $B(n,k)$. In [MS 1-3] various connections of $B(n,k)$ with geometry were indicated. Among them are the connection with configurations of hyperplanes in $\mathbb{R}^k$ in general position and the structure of the convex closure of a generic orbit of S_n in $\mathbb{R}^n$. We shall not recall here the original definition of $B(n,k)$ but instead formulate our interpretation. Consider the cube I^n with the structure of pasting scheme introduced in §2.

Theorem 4.1. *There is an isomorphism of posets*

$$B(n,k) \cong \mathrm{Ob}\ \Omega^k \mathrm{Cat}(I^n) .$$

By using mutations of elements of higher Bruhat orders (analogues of multiplications of permutations by transpositions), in [MS3] a $(n-1)$-category $\mathfrak{S}_n$ was defined, whose set of objects is the symmetric group S_n.

Theorem 4.2. *There is an isomorphism of* $(n-1)$*-categories*

$$\mathfrak{S}_n \cong \Omega \mathrm{Cat}(I^n) .$$

From this theorem we easily deduce one conjecture of [MS 3]. Let $P_n \subseteq \mathbb{R}^n$ be the $(n-1)$-dimensional permutohedron. By definition, P_n is the convex hull of the orbit of a point $(x_1 > \dots > x_n) \in \mathbb{R}^n$ under the natural action of the group S_n. Each face of P_n is isomorphic to a product of several permutohedra of smaller dimension, and some are single permutohedra, see [B], [Mi]. We call them indecomposable faces.

Theorem 4.3. *There is a natural bijection between indecomposable* k*-morphisms of* $\mathfrak{S}_n$ *and indecomposable* k*-dimensional faces of* P_n .

The proof is based on the fact that P_n can be viewed as the "derived polytope" of I^n, see §5.

Consider the projection $p_k : I^n \longrightarrow \mathbb{R}^k$ introduced in §2. Denote $Z(n,k)$ its image. It is natural to call this polytope the cyclic zonotope, since by zonotopes one usually means polytopes which are affine images of a cube (every polytope is clearly the affine image of a simplex).

Theorem 4.4. *Elements of* B(n,k) *are in bijection with subcomplexes (i.e. closed subsets which are unions of facets)* $\Sigma \subset I^n$ *such that* $p_k : \Sigma \to Z(n,k)$ *is a bijection.*

For such Σ the images of facets of Σ form a cubillage of Z(n,k) (analogue of triangulation). In particular, to the maximal and minimal elements of B(n,k) correspond subcomplexes Σ which are $s_k(I^n)$ and $t_k(I^n)$. To visualize these subcomplexes (or their images under p_k) one has to draw the cyclic configuration of n affine hyperplanes in $\mathbb{R}^k$:

$$1+t_i x_1+\ldots+t_i^k x_k = 0 \ , \ i = 1,\ldots,n \ .$$

Then we form the cell decomposition of $\mathbb{R}^k$ dual to the decomposition induced by this configuration. This new decomposition is in fact a cubillage and is combinatorially equivalent to $p_k(s_k(I^n))$. For example, $s_2(I^5)$ looks as follows:

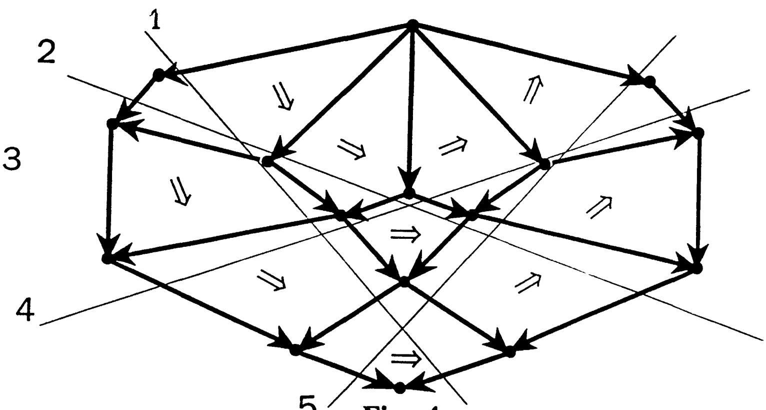

Fig. 4

More generally, let $\Sigma \subseteq I^n$ represent an element of B(n,k). Consider the cell decomposition of Z(n,k), dual to the cubillage induced by $p_k(\Sigma)$. If we look at its (k-1)-dimensional skeleton, we obtain a configuration of n polyhedral hypersurfaces in Z(n,k). These hypersurfaces intersect each other as if they were hyperplanes in general position. In other words, they define an oriented matroid [FL], [BS]. Let us recall necessary definitions.

Definition 4.5. An *oriented matroid* is a system $M = (E,\mathcal{E},*)$, where E is a finite set, $* : E \to E$,

$x \longrightarrow x^*$ is a fixed point-free involution, $\mathcal{C} \subseteq 2^E$ is a family of subsets (called positive circuits) satisfying the conditions:

(i) If $S \in \mathcal{C}$ and $T \subset S$, then $T = S$.

(ii) If $S \in \mathcal{C}$ and $S^* = \{x^*, x \in S\}$, then $S^* \in \mathcal{C}$ and $S \cap S^* = \emptyset$.

(iii) If $S,T \in \mathcal{C}$, $x \in S \cap T^*$, $S \neq T^*$, then there is $C \in \mathcal{C}$ such that $C \subset (S \cup T) - \{x,x^*\}$.

A basic example is given by the set E of non-zero vectors in a real vector space such that for $x \in E$ we have $(-x) \in E$. Define $x^* = -x$ for $x \in E$. Define $\mathcal{C}$ to consist of subsets $C \subset E$ minimal such that:

a) $C \cap C^* = \emptyset$

b) There are $\alpha_s \in \mathbb{R}_+$, $s \in C$ such that $\sum_{s \in S} \alpha_s s = 0$.

Not any oriented matroid is realisable, i.e. comes from a system of vectors as above.

From the "dual" point of view elements of an oriented matroid would represent half-spaces in $\mathbb{R}^n$ arising as complements to hyperplanes of an (imaginary, non-existent in general) configuration. Instead of half-spaces containing 0, one can imagine hemispheres in S^{n-1}, the unit sphere.

One of the main results of [FL] is that oriented matroids correspond to configurations formed by not necessary genuine hemispheres, but by so-called pseudo-hemispheres, that is, by subcomplexes in S^{n-1} homeomorphic to discs and invariant under the involution. This is achieved by "geometric realisation" similar to our construction in §2. Such a configuration may, however, be not stretchable.

Definition 4.6. An oriented matroid $M = (E,\mathcal{C},*)$ is said to have *type* $F(n,k)$, if $\mathrm{card}(E) = 2n$, for each $s \in \mathcal{C}$ we have $\mathrm{card}(C) = k+1$, and for each subset $X \subset E$, $\mathrm{card}(X) = k+1$ there is a decomposition $X = Y \cup Z$, $Y \cap Z = \emptyset$ such that $Y \cup Z^* \in \mathcal{C}$.

Intuitively, such a matroid should represent a configuration of n hyperplanes in $\mathbb{R}^k$ in general position. Not every matroid of type $F(n,k)$ is realisable. In the paper of Ringel [R] there is an example of a non-realisable oriented matroid of type $F(9,3)$, see also [BS].

Definition 4.7. The cyclic oriented matroid of rank k on n directions is the oriented matroid $C(n,k)$ in which E consists of symbols $\delta_1,\delta_1^*,\dots,\delta_n,\delta_n^*$, the involution $*$

interchanges δ_i and δ_i^* and $\mathcal{C}$ is formed by subsets $Z_I = \{\delta_i, i \in I,\ i$ is even, $\delta_i^*,\ i \in I,\ i$ is odd$\}$ and Z_I^* for all $(k+1)$-element subsets $I \subset \{1,\ldots,n\}$.

The oriented matroid $C(n,k)$ is realisable by configuration of hyperplanes dual to the vertices of the cyclic polytope (see §2).

A cell of an oriented matroid M is, by definition, a cell of the cell decomposition of the sphere induced by the configuration of pseudo- hemispheres corresponding to M . Cells can be defined in a purely combinatorial way, see [FL].

Definition 4.8. A *marking* of an oriented matroid M is a complete flag $Z = (Z_0 \subset \ldots \subset Z_{r-1})$ of cells of M . (Here $r-1$ is the dimension of the sphere, i.e. r is the rank of M .)

Example. Define a marking of the cyclic oriented matroid $C(n,k)$ which we shall call the standard one. Let us view elements δ_i, δ_i^* as hemispheres in S^{k-1} . Then set

$$Z_i^{st} = \bigcap_{j=0}^{k-2+i} (\delta_j \cap \delta_j^*) \cap \Big(\bigcap_{j=k-1+i}^{n-1} \delta_j \Big) ,$$

where we set $\delta_0 = \delta_n$.

If B is a cell of an oriented matroid M , then denote by $S(B)$ the unique pseudosphere in S^{r-1} of dimension $\dim(B)$ which is the intersection of some pseudohemispheres of the configuration. By $M|_{S(B)}$ we denote the oriented matroid of rank $\dim(B) + 1$ defined by the configuration of pseudohemispheres induced on $S(B)$.

Theorem 4.9. *The set* $B(n,k)$ *is in bijection with the set of marked oriented matroids* (M,Z) *of rank* $k+1$ *such that*:

(i) M *has the type* $F(n+1\ , k+1)$.

(ii) *The restriction* $M|_{S(Z_{k-1})}$ *is isomorphic to the cyclic oriented matroid* $C(n,k)$.

(iii) *The marking of* $C(n,k) = M|_{S(Z_{k-1})}$ *is the standard one.*

For $k = 3$ any oriented matroid of type $F(n+1,3)$ admits a marking satisfying (ii) - (iii). This corresponds to the numeration of affine (pseudo)-lines in $\mathbb{R}^n$ by the increase of the slopes. For $k>3$ such a marking is not

always possible.

Using this theorem we can easily disprove the conjecture from [MS2] that $B(n,2)$ classifies the combinatorial types of arrangements of n lines in general position in $\mathbb{R}^2$, none of which is parallel to the fixed line. To do this, we can take the Ringel's example [R], [BS] of non-stretchable configuration of 9 pseudo-lines in $\mathbb{R}^2$, thus obtaining an element of $B(9,2)$ which cannot be represented by a configuration of lines.

In general, one can construct from an arbitrary marked oriented matroid a composable pasting scheme.

The connection between the simplex Δ^n and the cube I^{n-1}(see §.5 below) yields a natural surjective (n-1)-functor $Cat(I^{n-1}) \to \Omega Cat(\Delta^n)$. This leads to a connection between higher Bruhat and Stasheff orders.

Theorem 4.10. *For each* n,k *there exists a monotone surjective map* $f : B(n,k) \to S(n+1,k+1)$ *such that for each* $a,b \in B(n,k)$ *we have* $a \le b$ *if and only if* $f(a) \le f(b)$.

It follows from this property that f takes the maximum and minimum of $B(n,k)$ to the maximum and minimum of $S(n+1,k+1)$.

§5 TWO PROBLEMS

5.1 The derived pasting scheme. If A is a composable pasting scheme of dimension n then the (n-1)-category $\Omega Cat(A)$ in general is not free, i.e. not of the form $Cat(B)$ for some composable pasting scheme B. In fact, the 2-dimensional associativity in $Cat(A)$ produces non-trivial relations in $\Omega Cat(A)$. It is natural to look for some "free cover" of $\Omega Cat(A)$ which would have the form $Cat(B)$. In this case, vertices of B should correspond to maximal paths in A. It is natural to call such a B (if it exists) the derived pasting scheme of A and denote it ΩA. Let us define the graded poset ΩA, setting $(\Omega A)_k$ to be the set of all sequences $(a_0, \ldots, a_r)$, $r \ge 0$ of cells of A such that:

1) $(s_0(a_0) = s_0(A)$, $t_0(a_r) = t_0(A)$, and for $i = 0,\ldots,r-1$, $s_0(a_{i+1}) = t_0(a_i)$.

2. $$\sum_{i=0}^{r} (\dim a_i - 1) = k .$$

We say that $(b_0,\dots,b_s) \le (a_0,\dots,a_r)$ if there is a sequence

$$0 = i_0 < i_1 < i_2 < \dots < i_r < i_{r+1} = s$$

such that for any $\nu \in \{0,1,\dots,r\}$ and any $j \in [i_\nu, i_{\nu+1}]$ we have $b_j \in R(a_\nu)$.

This poset should be the poset of cells of the pasting scheme ΩA we are looking for. The sequences of cells satisfying the condition 1) can be called 0-composable. In the same way we can define posets $\Omega_p A$ for $p \ge 1$. By definition, a k-dimensional cell of $\Omega_p A$ is a composable pasting subscheme $T \subset A$ such that:

1) $s_{p-1}(T) = s_{p-1}(A)$, $t_{p-1}(T) = t_{p-1}(A)$.

1′) The polymorphism in Cat(A) represented by T can be obtained from "elementary" polymorphisms $R(a)$, corresponding to cells of A , by using only operations $*_0,\dots,*_{p-1}$.

2) $\sum_{a \in T_{max}} \max(\dim(a)-p,0) = k$, where T_{max} is the set of cells of T that are maximal (with respect to R) . For $p = 1$ we obtain the previous definition.

Problem. Investigate the possibility of endowing the graded posets $\Omega_p A$ with the structure of composable pasting schemes such that the order corresponds to the relation R .

There are some cases when this construction works well. The most important cases are:

$$\Omega(\Delta^n) = I^{n-1} , \ \Omega(I^n) = P_n ,$$

but for the permutohedron P_n the construction behaves badly. The role of the permutohedron as an approximation to the space of paths in the cube was known to R J Milgram [Mi]. Also $\Omega_2(\Delta^n)$ is the (poset of faces of) the Stasheff polytope [Sta]. Similarly, the $\Omega_p(\Delta^n)$ should be the "higher Stasheff polytopes" and should be connected with the secondary polytope of a cyclic polytope [GZK1-2], [BFS].

5.2 Enumeration problem. Calculate the cardinality of the sets $B(n,k)$ and $S(n,k)$. For $S(n,k)$ these are "higher" analogs of the classical Catalan numbers.

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