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MORE ON GRAPHIC TOPOSES

by F. William LAWVERE

RÉSUMÉ. Un monoïde est graphique s'il satisfait à l'identité de Schützenberger-Kimura $aba = ab$. Dans cet article on continue l'étude des topos, dits topos graphiques, engendrés par les objets ayant un monoïde d'endomorphismes de ce type. On démontre que dans un topos graphique, Ω est un cogénérateur.

In [Boulder AMS] and [Iowa AMAST] I began the study of a special class of combinatorial toposes. The general defining property of these special *graphic* toposes is that they are generated by those objects whose endomorphism monoids are finite and satisfy the Schützenberger-Kimura identity $aba = ab$; since this includes all localic (and even all "one-way" i.e. sites in which all endomorphism monoids are *trivial*) examples, to emphasise the special "graphic" features, some results are stated for the case when a *single* such object suffices, in particular, when the topos is hyperconnected. Monoids satisfying the Schützenberger-Kimura identity will also be called "graphic monoids" for short; the finiteness assumption implies that they have enough constants. The other main consequence of the finiteness assumption is that the toposes under consideration are actually *presheaf* toposes, so that the projectivity of the generators will be used in calculations without further comment.

The goals of this study are two-fold: to illustrate in a "simple" context some topos-theoretic constructions which are thus far difficult to calculate in general, and to arrive eventually, via geometric realization, at automatic display of pictures of hierarchical structures such as "hypercard", library catalogues, etc. (The actions of non-commuting idempotents are relevant to the latter problem in that $x{\cdot}a$ can be interpreted as "the part of x reflecting a" , rather than merely "the part of x *in* a" as with commutative intersection). In this note I describe some results of this study which were obtained after the Bangor meeting.

Among the easy propositions stated at the meeting which emphasize the very special nature of a graphic monoid M , are the facts that every left ideal is actually a two-sided ideal and that the lattice of all these is stable with respect to the usual right action on all truth-values (= right ideals); moreover, there is a lattice isomorphism between the two-sided ideals and the essential subtoposes of the topos of all applications ($=_{\mathrm{def}}$ right actions) of M, with $X \cdot S \subset X$ defining the "S-dimensional skeleton" of any X, where S is a given left ideal of M. If the Aufhebung of S is defined to be the smallest $T \geq S$ such that every $X \cdot S$ is a T-sheaf, then the Aufhebung of $\emptyset$ is T_0 = the set of all constants of M, while the Aufhebung of T_0 is T_1 = the smallest left ideal which is connected as a *right* M-set (see [Iowa] Prop.1); $X \cdot T_1$ has the same components as X itself, justifying picturing it as the "one-dimensional" skeleton of X. The putative picture of a general X is the interlocking union of the pictures of singular figures of the generating form. A significant three-dimensional generating form is the "taco" which pictures the eight-element graphic monoid of all those endofunctors of "any" category $\mathcal{A}$ obtained by considering all composites in any twice aufgehobene adjoint analysis of $\mathcal{A}$

$$\mathcal{A} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{B} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{C} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} 1$$

such as for example, an essential analysis of graphic toposes

$$S^{M^{op}} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \text{T-sheaves} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \text{S-sheaves} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} 1$$

with every S-skeleton a T-sheaf, and the empty M-set an S-sheaf.

Since the set $\mathcal{D}$ of all subtoposes of a topos $\mathcal{X}$ is a co-Heyting algebra, we can consider it as a closed category with $T \longrightarrow S$ meaning $T \geq S$ and $\otimes$ meaning $\cup$. For a combinatorial topos all subtoposes are essential and thus admit a skeleton functor, so we can define a covariant functor

$$\mathcal{X} \xrightarrow[\text{display}_0]{} \text{cocomplete } \mathcal{D}\text{-enriched categories}$$

by sending X to the set $\underline{X}$ of all subobjects of X (using direct image on morphisms) with the enrichment

$$\underline{X}(A,B) = \text{the smallest } S \text{ such that } A \cup (X \cdot S) \geq B$$

and in particular

$$\underline{X}(\emptyset,B) = \dim(B) .$$

Here we called the functor "display état 0" , because although it specifies important information about the desired picture, the machine may still not know how to draw the latter, unless we specify the pictures of the generating forms; the examples worked out in [Iowa AMAST] show that these basic pictures may be triangles, but may also be squares or lozenges, that they may be closed, but may also be partly open, as for example the "taco" which has two two-dimensional closed sides, but is open on top.

One example where the display is well understood is the topos of reflexive directed graphs, which are the applications of the three-element graphic monoid Δ_1 with two constants whose generating form is $\boxed{\bullet \longrightarrow \bullet}$. Since this monoid may be considered to be the theory of graphs, a precise justification for the term "graphic" in the more general case is provided by the following

Proposition 1. *Within the category of all monoids, the smallest equational variety containing the single three-element example* Δ_1 *is the category of all monoids satisfying identically* $aba = ab$.

Proof It suffices to show that for each n , the free monoid F_n in the subcategory satisfying the identity can be embedded in a product of copies of Δ_1 , $F_n \hookrightarrow {\Delta_1}^{I_n}$ as monoids, where I_n is a set. The canonical choice for I_n is $I_n = (F_n,\Delta_1)$, the set of monoid homomorphisms which by freeness is Δ_1^n as a set, for then there is a canonical homomorphism $F_n \to {\Delta_1}^{\Delta_1^n}$ given by double-dualization, which to each generating letter of F_n assigns the corresponding projection $\Delta_1^n \to \Delta_1$. It can be shown that this canonical homomorphism is injective, i.e. that if $u \neq w$ in F_n , then there exists an $x \in \Delta_1^n$ such that $u(x) \neq w(x)$ in Δ_1 . Here we use the fact that the elements u,w are *words without repetitions* in the n letters, and that $w(x)$ = the first of the two constants of Δ_1 which occurs in the list $x \circ w$.

Note that while maps of applications *in* a fixed graphic topos never increase dimension (as is expressed in the functorality of $\mathrm{display}_0$ above), by contrast maps between graphics often do; for example, the "taco" is a homomorphic image of F_6 , but any free graphic monoid is one–dimensional.

Although every topos is the fixed part of a graphic topos (indeed one of the form $\mathcal{S}^{\Delta_1^{op}}/X$) for some left-exact idempotent functor on the latter, the graphic toposes are themselves special in many respects. This is born out by the following proposition which generalizes a remark made by Borceux many years ago. Recall that in any topos, Ω is an internal cogenerator in the sense that any object injects into some function space Ω^X ; however, to get an actual cogenerator we must be able to replace the exponent by a discrete one, and this usually does not happen.

Proposition 2. (Gustavo Arenas) *In any graphic topos,* Ω *is a cogenerator.*

Proof First note that if $A \rightrightarrows X$ can be distinguished by any map $X \to \Omega$, then they can be distinguished by (the characteristic map of) some principal subobject (image of some singular figure $B \to X$) . In the graphic case we can even take $B = A$; for if the endomorphisms of A satisfy the graphic identity and x_1,x_2 are two figures in X of form A , consider the characteristic map $[z]$ of the image of $z = x_1$ or $z = x_2$. If $[z]x_1 = [z]x_2$ for both these z , then there exist endomorphisms a,b so that both

$$x_1a = x_2 ,$$
$$x_2b = x_1$$

which implies

$$x_1 = x_2b = x_1ab = x_1aba = x_2ba = x_1a = x_2 .$$

The computation of the Aufhebung relation on essential subtoposes has not yet been carried out in many important examples such as algebraic or differential geometry. Even for graphic toposes this computation may be difficult. However, in the latter case it can at least be reduced to a question involving congruence relations on ideals of the monoid itself:

Proposition 3. *If* $S \leq T$ *are left ideals in a graphic monoid* M , *then in the associated graphic topos* $\mathcal{S}^{M^{op}}$, *one* *has the condition "every* $X \cdot S$ *is a* T*-sheaf" if and only if for every congruence relation* $\equiv$ *on* T **as a right M-set** *one has the implication*

$$\forall\ t \in T\ \exists\ s \in S\ [t \equiv s] \Longrightarrow \exists\ s_0 \in S\ \forall\ t \in T\ [t \equiv s_0 t]\ .$$

Construing the elements of the lattice $\mathcal{D}$ of essential subtoposes as refined *dimensions*, one would also like to *add* them. Since the infima in $\mathcal{D}$ do not always agree [Kelly-Lawvere] with intersections, it may be necessary in general to replace $\mathcal{D}$ with $(2^{\mathcal{D}})^{op}$ to get a workable theory. The idea is that the sum $S_1 * S_2$ of two dimensions should be the smallest T for which

$$X \cdot S_1 \times Y \cdot S_2 \subseteq (X \times Y) \cdot T$$

holds for the skeleta of all X,Y in the topos. (It is usually only for dimension 0 , $T = T_0$, that $() \cdot T$ preserves products). However, for graphics the infima do agree with intersections and moreover we have

Proposition 4. *If* S_1,S_2,T *are left ideals in a graphic monoid, then the above relation holds if and only if*

$$\forall\ s_1 \in S_1\ ,\ s_2 \in S_2\ \exists\ t \in T\ [s_1 t = s_1\ ,\ s_2 t = s_2]\ .$$

Moreover, if T,T' *both have this property relative to* S_1,S_2 *then so does* $T \cap T'$ *; thus the intersection of all such* T *gives a well- defined "sum" of* $S_1 * S_2$.

Proof The necessity of the condition follows from considering X,Y to be principal right ideals. Conversely, if it holds and $x \in X \cdot S_1$, $y \in Y \cdot S_2$ for some arbitrary M-applications X,Y with s_1,s_2 witnessing that fact, i.e. $xs_1 = x$, $ys_2 = y$, then choosing t as in the condition gives that $\langle x,y \rangle$ is in $(X \times Y) \cdot T$ because

$$xt = xs_1 t = xs_1 = x\ ,$$

$$yt = ys_2 t = ys_2 = y\ .$$

If $u \in T'$ (for some other such T') also fixes a given pair s_1,s_2 that $t \in T$ fixes, then $tu \in T \cap T'$ since these are ideals, and obviously

$$s_i tu = s_i u = s_i$$

showing that $T \cap T'$ serves as well or better.

One has $T_0 * S = S$ for all S, but clarification is needed on the

Question. What is the relation between the "successor" $S*T_1$ and the Aufhebung S' of S for graphic toposes? Recall that Michael Zaks (unpublished) showed for the simplicial topos that while these two constructions give equal results for $S = T_0, T_1, T_2$, by contrast for $S \geq T_3$ one has instead that the doubled dimension $S*S = S'*T_1$ for S' the (smallest) Aufhebung of S, i.e. (since dimensions reduce to natural numbers together with $\pm\infty$ in the simplicial case) that $n' = 2n - 1$ for $n \geq 3$.

REFERENCES

F.W.LAWVERE, Qualitative distinctions between some toposes of generalized graphs, *Proceedings of AMS Boulder 1987 Symposium on Category Theory and Computer Science, Contemporary Mathematics* 92 (1989) 261-299.

G.M. KELLY & F.W. LAWVERE, On the Complete Lattice of Essential Localizations, *Bull. Société Mathématique de Belgique,* XLI (1989) 289-319.

F.W. LAWVERE, Display of Graphics and their Applications, as Exemplified by 2-Categories and the Hegelian "Taco", *Proceedings of the First International Conference on Algebraic Methodology and Software Technology, The University of Iowa,* (1989) 51-75.

(There is an error in the version of the latter paper distributed at the Bangor Meeting: Free graphic monoids do not act faithfully on their constants.)

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