V. Koubek
J. Sichler

On Priestley duals of products

Cahiers de topologie et géométrie différentielle catégoriques, tome 32, n° 3 (1991), p. 243-256

<http://www.numdam.org/item?id=CTGDC_1991__32_3_243_0>
ON PRIESTLEY DUALS OF PRODUCTS

by V. KOUBEK and J. SICHLER

RéSUMÉ. Cet article présente les espaces de Priestley représentant les produits et les produits fibrés de (0,1)-treillis distributifs et de double p-algèbres.

1. INTRODUCTION

The well-known Priestley duality [9], a contravariant equivalence of the category P of compact totally order disconnected spaces to the category D of distributive (0,1)-lattices, has become an essential tool for structural and categorical investigations of varieties of algebras with reducts in D. Its applications produced a fairly extensive list of subcategories of D representing such varieties, and also led to catalogues of Priestley duals of numerous algebraic concepts, such as those by Priestley [11], or Davey and Duffus [5].

Since any variety V of algebras with reducts in D is closed under the formation of Cartesian products, a category P(V) contravariantly equivalent to V is closed under coproducts, and a natural question of characterizing these coproducts arises.

A straightforward argument shows that the Priestley dual P(K) of a product $K = \coprod\{K_i \mid i \in I\}$ contains a copy of the sum $Q = \sum\{P(K_i) \mid i \in I\}$ of Priestley duals $P(K_i)$ of its components as a dense ordered subspace. While always totally order disconnected, the ordered topological space Q need not be compact, in which case P(K) must be a proper compactification of Q which is, up to an isomorphism, the 'maximal' compactification M(Q) of Q.

Since the Priestley dual M(Q) of a product K of Boolean algebras $K_i$ is the unordered Stone-Čech compactification $\beta Q$ of Q, one may be tempted to conjecture that $\beta Q$ is also the underlying space of the dual M(Q) of the product $K = \coprod\{K_i \mid i \in I\}$ of a set $\{K_i \mid i \in I\}$ of distributive (0,1)-lattices. This, however, is false. In general, any transitive extension of the order of Q compatible with the topology of $\beta Q$ is only a preorder on $\beta Q \setminus Q$, and hence needs to be factored out to obtain the Priestley dual M(Q) of K.

An alternate approach is adopted here. We say that an object P of P is a Priestley compactification of an ordered topological space Q whenever Q is a dense order subspace of P, characterize these compactifications, and then investigate the special case in which Q is a sum of Priestley spaces. We show that Priestley compactifications of these sums represent weak direct products (in 3.5), characterize the Priestley dual M(Q) of the full direct product, and also collections of lattices for which $\beta Q$ is the underlying space of M(Q). We also describe Priestley duals of ultraproducts. This and all other results are presented in Section 3.

The support of the NSERC is gratefully acknowledged by both authors.
2. PRIESTLEY SPACES

First we review the essentials of Priestley's duality for distributive \((0,1)\)-lattices and distributive double \(p\)-algebras.

A triple \((X, \tau, \leq)\) is called an \textit{ordered topological space} whenever \((X, \tau)\) is a topological space and \((X, \leq)\) is a poset. For any \(Z \subseteq X\) define

\[
(Z) = \{z \in X \mid \exists z \in Z \; z \leq z\} \quad \text{and} \quad \bar{Z} = \{z \in X \mid \exists z \in Z \; z \geq z\}.
\]

A set \(Z \subseteq X\) is \textit{decreasing} if \((Z) = Z\), \textit{increasing} if \(\bar{Z} = Z\), and \textit{clopen} if it is both closed and open in \((X, \tau)\). A \textit{convex set} is an intersection of a decreasing set and an increasing one.

An ordered topological space \((X, \tau, \leq)\) is \textit{totally order disconnected} whenever \(x \nless y\) in \(X\) implies the existence of a clopen decreasing set \(Y \subseteq X\) such that \(y \in Y\) and \(x \notin Y\). A totally order disconnected ordered topological space is called a \textit{Priestley space} if and only if it is compact.

To any distributive \((0, 1)\)-lattice, Priestley \([9]\) assigns an ordered topological space \(P(L) = (X, \tau, \leq)\) in which \(X\) is the set of all prime filters of \(L\), \(z \leq y\) if and only if \(y \subseteq z \subseteq L\), and the topology \(\tau\) has an open subbasis\(S\) of clopen sets. Thus \(P(L)\) is compact and totally order disconnected, \([9]\). For any \((0, 1)\)-homomorphism \(f : L \rightarrow L'\) of distributive \((0, 1)\)-lattices \(L, L'\), the inverse image mapping \(f^{-1} : P(L') \rightarrow P(L)\) is continuous and order preserving. Setting \(P(f) = f^{-1}\) thus gives rise to a contravariant functor \(P : D \rightarrow P\) of the category \(D\) of all \((0, 1)\)-homomorphisms of distributive \((0, 1)\)-lattices into the category \(P\) of all continuous order preserving mappings of Priestley spaces.

Inclusion ordered clopen decreasing subsets of a Priestley space \(P = (X, \tau, \leq)\) form a distributive \((0, 1)\)-lattice \(D(P)\), and the inverse image map \(g^{-1} : D(P) \rightarrow D(P')\) of a \(P\)-morphism \(g : P' \rightarrow P\) is a lattice \((0, 1)\)-homomorphism. Thus \(D(g) = g^{-1}\) completes a definition of a contravariant functor \(D : P \rightarrow D\).

The functors \(P\) and \(D\) determine Priestley's duality as follows.

**Theorem 2.1** (Priestley \([9, 10]\)). The composite functors \(P \circ D : P \rightarrow P\) and \(D \circ P : D \rightarrow D\) are naturally equivalent to the respective identity functors of their domains.

A \(D\)-morphism \(f : L \rightarrow L'\) is surjective if and only if \(P(f)\) is a homeomorphism and order isomorphism of \(P(L')\) onto a closed order subspace of \(P(L)\), and \(f\) is one-to-one if and only if \(P(f) : P(L') \rightarrow P(L)\) is surjective.

For a Priestley space \((X, \tau, \leq)\), let \(\text{Max}(X)\) and \(\text{Min}(X)\) respectively denote the set of all elements which are maximal or minimal in \((X, \leq)\), and let \(\text{Ext}(X) = \text{Max}(X) \cup \text{Min}(X)\) be the set of all extremal members of \((X, \leq)\). For any \(Y \subseteq X\), set \(\text{Max}(Y) = [Y] \cap \text{Max}(X), \text{Min}(Y) = [Y] \cap \text{Min}(X)\) and \(\text{Ext}(Y) = \text{Max}(Y) \cup \text{Min}(Y)\).
Min(Y). In any Priestley space, the sets $\text{Max}(x) = \text{Max}\{x\}$ and $\text{Min}(x) = \text{Min}\{x\}$ are nonvoid for every $x \in X$.

Recall that a distributive $(0,1)$-lattice $L$ is a distributive double $p$-algebra provided that for every $x \in L$ it contains a largest element $x^*$ such that $x \wedge x^* = 0$, and a smallest element $x^+$ satisfying $x \vee x^+ = 1$. Homomorphisms of these algebras are all $D$-morphisms preserving the two unary operations thus defined. Following is a well-known characterization of Priestley duals of distributive double $p$-algebras, [11].

**Theorem 2.2.** Let $f : L \rightarrow L'$ be a $D$-morphism and let $g = P(f) : P(L') \rightarrow P(L)$ be its Priestley dual. Then:

1. $L$ is a distributive double $p$-algebra if and only if $[Y]$ is clopen for every clopen increasing subset $Y$ of $P(L)$ and $[Z]$ is clopen for every clopen decreasing set $Z$;
2. a mapping $f$ is a double $p$-algebra homomorphism if and only if $g(\text{Max}(x)) = \text{Max}(g(x))$ and $g(\text{Min}(x)) = \text{Min}(g(x))$ for every element $x$ of $P(L')$.

A Priestley space satisfying 2.2(1) is called a dp-space, and a continuous order preserving mapping $g$ for which 2.2(2) holds is a dp-map.

Elements $a$ and $b$ of a poset $(X, \leq)$ are connected whenever there exists a finite sequence $a = x_0, x_1, \ldots, x_n = b$ such that $x_{i-1}$ is comparable to $x_i$ for each $i \in \{1, \ldots, n\}$. Classes of the resulting equivalence are called order components of $(X, \leq)$. Since $\text{Max}(x) \neq \emptyset \neq \text{Min}(x)$ for every element $x$ of a Priestley space $P = (X, \tau, \leq)$, a subset $Y$ of $X$ is a component of $P$ if and only if $\text{Ext}(Y)$ is a component of the subposet $\text{Ext}(X)$ of $X$.

The proposition below summarizes some useful properties of Priestley spaces.

**Proposition 2.3.** Let $P = (X, \tau, \leq)$ be a Priestley space and let $cT$ denote the $\tau$-closure of $T \subseteq X$. Then:

1. for any closed disjoint subsets $Y$ and $Z$ there exists a clopen $A \subseteq X$ such that $Z \subseteq A$ and $Y \subseteq X \setminus A$; if, in addition, $Y \cap (Z) = \emptyset$ then $A$ may be chosen to be decreasing; consequently,
2. the sets $[Y]$ and $[Z]$ are closed whenever $Y \subseteq X$ is closed; hence
3. a union $U$ of order components of $P$ is closed if $\text{Max}(U)$ or $\text{Min}(U)$ is closed;
4. $c(T) \subseteq (cT)$ and $c(T) \subseteq [cT]$ for any $T \subseteq X$;
5. the Boolean algebra $C(P)$ of all clopen subsets of $P$ is generated by the lattice $D(P)$ of all clopen decreasing subsets of $P$;
6. if $D(P)$ is a double $p$-algebra, then $\text{Max}(X)$ and $\text{Min}(X)$ are closed sets.

According to 2.1, congruences of a distributive $(0,1)$-lattice $L$ are in one-to-one correspondence to closed order subspaces of $P(L)$, see also [11]: clopen decreasing subsets $A, B$ represent $\Theta$-congruent elements of $L$ exactly when $A \cap Z = B \cap Z$ for the closed subposet $Z$ of $P(L)$ corresponding to the congruence $\Theta$. A closed subset $Z$ of a dp-space $P(L)$ corresponds to a congruence of a distributive double $p$-algebra $L$ if and only if $\text{Ext}(Z) \subseteq Z$, that is, when $Z$ is a closed c-set [4] or [8].
Let $P = (X, \tau, \preceq)$ be a Priestley space. By 2.3(5), every $\tau$-clopen $A \subseteq X$ can be written in the form $A = \bigcup\{A_i \setminus B_i \mid i \in \{1, \ldots, n\}\}$, where $A_i, B_i \subseteq X$ are clopen and decreasing for all $i \in \{1, \ldots, n\}$; since $A_i \cap B_i$ is $\tau$-clopen and decreasing, we may assume that $A_i \subseteq B_i$ for every $i = 1, \ldots, n$. In other words, every $A \in C(P)$ is the union of finitely many clopen convex sets $C_i = A_i \setminus B_i$ with $A_i \subseteq B_i$. Let $\text{Gen}(A)$ denote the least number of clopen convex sets whose union is $A$. The natural $\text{Comp}(P) = \sup\{\text{Gen}(A) \mid A \in C(P)\}$ will be called the complexity of the Priestley space $P$. We say that the Boolean algebra $C(P)$ of all clopen sets of $P$ is uniformly generated whenever $\text{Comp}(P)$ is finite.

It seems clear that the complexity of $P$ will depend on the length of chains contained in $P$. Let $A \in C(P)$. A chain $x_0 < x_1 < \ldots < x_{2k}$ of $P$ is a characteristic chain of $A$ whenever

1. $x_j \in A$ if and only if $j$ is even, and
2. the length of any chain of $P$ satisfying (1) is at most $2k$.

Any $A \in C(P)$ possesses a characteristic chain, and all characteristic chains of $A$ have equal length, say $2k$. Since no convex subset of $A$ may contain two distinct elements of a characteristic chain, it follows that $\text{Gen}(A) \geq k + 1$.

For any $A \in C(P)$, let $T_i = T_i(A)$ consist of all $t \in X$ such that $t = x_i$ in some characteristic chain $x_0 < x_1 < \ldots < x_{2k}$ of $A$. Clearly, the sets $T_0, \ldots, T_{2k}$ are pairwise disjoint. We claim that every $T_i$ is closed. To see this, note that $T_i \cap (T_j)$ is closed for $i \leq j$, in which case $T_i \subseteq (T_j)$ and $T_j \subseteq (T_i)$. But then 2.3(4) implies that $cT_i \subseteq c(T_j) \subseteq (cT_j)$ and $cT_j \subseteq [cT_i]$ and, because $A$ is clopen, $cT_i \subseteq A$ for all even $i$, while $cT_i \subseteq X \setminus A$ when $i$ is odd. Therefore, for every $t \in cT_i$ there is a characteristic chain $x_0 < x_1 < \ldots < x_{2k}$ of $A$ with $x_i = t$ and, consequently, each $T_i$ is closed.

**Lemma 2.4.** If $P = (X, \tau, \preceq)$ is a Priestley space and $A \in C(P)$ has a characteristic chain of length $2k$, then $\text{Gen}(A) = k + 1$.

**Proof:** We proceed by induction on $k$.

If $k = 0$, then $T_0(A) = A$, the clopen set $A$ is convex, and $\text{Gen}(A) = 1$ follows trivially.

Let $k \geq 1$ and suppose that any clopen $B$ whose characteristic chains have the length $2k - 2$ can be written in the form $B = \bigcup\{D_i \mid i \in \{0, \ldots, k - 1\}\}$, where $D_i$ is clopen convex and $T_{2i}(B) \subseteq D_i$ for all $i \in \{0, \ldots, k - 1\}$. Since the sets $T_j(A), T_j(A)$ are closed for $j \in \{0, \ldots, 2k\}$ and because $T_j(A) \cap [T_{2k}(A)] = \emptyset$ for all $j < 2k$, by 2.3(1), we obtain a clopen increasing set $I$ such that $\bigcup\{T_j(A) \mid j < 2k\} \subseteq X \setminus A$ and $T_{2k}(A) \subseteq I$. Characteristic chains of the clopen set $A \setminus I$ are of length $2k - 2$, and $T_{2i}(A) \subseteq T_{2i}(A \setminus I)$ for all $i \in \{0, \ldots, k - 1\}$. By the induction hypothesis, there exist clopen convex sets $D_0, \ldots, D_{k-1}$ such that $T_{2i}(A \setminus I) \subseteq D_i$ and $A \setminus I = \bigcup\{D_i \mid i \in \{0, \ldots, k - 1\}\}$. Characteristic chains of the clopen set $A \setminus D_0$ are of length $2k - 2$, and the induction hypothesis provides clopen convex sets $C_0, \ldots, C_{k-1}$ such that $T_{2i}(A \setminus D_0) \subseteq C_i$ and $A \setminus D_0 = \bigcup\{C_i \mid i \in \{0, \ldots, k - 1\}\}$. The sets $E_0 = D_0$ and $E_{i+1} = C_i$ for $i \in \{0, \ldots, k - 1\}$ are clopen and convex, and $A = \bigcup\{E_j \mid j \in \{0, \ldots, k\}\}$. From $T_{2i+2}(A) \subseteq T_{2i}(A \setminus D_0)$ it follows that
T_{2j}(A) \subseteq E_j \text{ for all } j \in \{0, \ldots, k\}. \text{ This shows that } \text{Gen}(A) \leq k + 1. \]

**Lemma 2.5.** Any chain of length $2k$ in a Priestley space $P$ is a characteristic chain of some $A \in C(P)$.

**Proof:** Let $x_0 < x_1 < \ldots < x_{2k}$ be a chain in $P$. Since $P$ is totally order disconnected, for every $i \in \{0, \ldots, 2k - 1\}$ there exists a clopen decreasing set $A_i$ such that $x_i \in A_i$ and $x_{i+1} \notin A_i$. Define $A_{-1} = \emptyset$ and $A_{2k} = P$. Then $C_i = A_{2i} \setminus A_{2i-1}$ is clopen and convex, and $x_0 < x_1 < \ldots < x_{2k}$ is a characteristic chain of $A = \bigcup\{C_i \mid i \in \{0, \ldots, k\}\}$. □

**Corollary 2.6.** The Boolean algebra $C(P)$ of all clopen sets of a Priestley space $P = (X, \tau, \leq)$ is uniformly generated if and only if the poset $(X, \leq)$ has a finite height.

**Remark 2.7.** Adams and Beazer [1] show that chains of a Priestley space $P = P(L)$ have at most $n$ elements if and only if for any chain $a_0 \leq a_1 \leq \ldots \leq a_{n-1}$ of elements of the distributive $(0,1)$-lattice $L$ there exist $a'_0, a'_1, \ldots, a'_{n-1} \in L$ such that $a_0 \land a'_0 = 0$, $a_i \lor a'_i = a_{i+1} \land a'_{i+1}$ for $0 \leq i < n - 1$, and $a_n-1 \lor a'_{n-1} = 1$.

Hence $C(P)$ is uniformly generated if and only if the lattice $L = D(P)$ satisfies the Adams–Beazer condition for some finite $n$.

Let $Con(L)$ denote the congruence lattice of a distributive $(0,1)$-lattice $L$. A congruence $\Psi$ of $L$ is compact if $\Psi \leq \bigvee\{\Theta_i \mid i \in I\}$ holds in $Con(L)$ only when $\Psi \leq \bigvee\{\Theta_j \mid j \in J\}$ for some finite $J \subseteq I$. The lattice $Con(L)$ is distributive, complete, and each of its members is a join of compact congruences. The least congruence $\theta(a, b) \in Con(L)$ containing the pair $\{a, b\} \subseteq L$ of distinct elements of $L$ is compact and, because compact elements form a join semilattice, any join $\bigvee\{\theta(a_j, b_j) \mid j = 1, \ldots, n\}$ of finitely many principal congruences $\theta(a_j, b_j)$ is compact.

Let $L \subseteq P(L)$ be the closed set representing a compact congruence $\Psi$ of $L$, and let $\{U_i \mid i \in I\}$ be an open covering of $P(L) \setminus L$. Then $P(L) \setminus U_i$ represents a $\Theta_i \in Con(L)$ for each $i \in I$ and $K \supseteq \bigcap\{P(L) \setminus U_i \mid i \in I\}$, that is, $\Psi \leq \bigvee\{\Theta_i \mid i \in I\}$. From the compactness of $\Psi$ we obtain that $P(L) \setminus K \subseteq \bigcup\{U_j \mid j \in J\}$ for some finite $J \subseteq I$, so that $P(L) \setminus K$ is compact and hence closed. This shows that compact congruences of $L$ are represented by clopen subsets of $P(L)$. Since the set $P(L) \setminus K$ is closed, it represents the complement $\Psi'$ of $\Psi$ in $Con(L)$. Hence all compact members of $Con(L)$ are complemented, see also Hashimoto [7].

Since $Con(L)$ is dually isomorphic to the inclusion ordered poset of all closed subsets of $P(L)$, any subset of $P(L)$ representing a complemented member of $Con(L)$ must be clopen.

For clopen decreasing sets $A \subseteq B \subseteq P(L)$, let $\Phi = \Phi(A, B) \in Con(L)$ be represented by the clopen convex set $B \setminus A$; thus $(U, V) \in \Phi$ if and only if $U \cap (B \setminus A) = \forall \gamma(B \setminus A)$. Then $(\theta, A) \in \Phi$ and $(B, P(L)) \in \Phi$, so that $\theta(0, A) \vee \theta(B, 1) \leq \Phi$ in $Con(L)$. On the other hand, for any $(U, V) \in \Phi$ we obtain $(U \cup A) \cap B = (V \cup A) \cap B$, that is, $(U \cup A, V \cup A) \in \theta(B, 1)$; from $(U, U \cup A), (V, V \cup A) \in \theta(0, A)$ it then follows
that $(U, V) \in \theta(0, A) \lor \theta(B, 1)$. Therefore $\theta(0, A) \lor \theta(B, 1) = \emptyset$ and, consequently, $\theta(A, B) \lor \emptyset = \theta(0, 1)$ is the unit of $Con(L)$. Hence $\theta(A, B) \land \emptyset = (\theta(A, B) \land \theta(0, A)) \lor (\theta(A, B) \lor \theta(B, 1))$. If $(U, V) \in \theta(A, B) \land \theta(0, A)$ or $(U, V) \in \theta(A, B) \lor \theta(B, 1)$, then $U = V$ (see Grätzer [6], p.89). This shows that $\Phi(A, B)$ is the complement $\theta(A, B)'$ of the principal congruence $\theta(A, B)$ for any $A \subseteq B$.

Let $\Psi \in Con(L)$ be the congruence represented by a clopen set $K \subseteq P(L)$. Then the clopen set $P(L) \setminus K$ can be written as $P(L) \setminus K = \bigcup\{B_j \setminus A_j \mid j = 1, \ldots, n\}$ with clopen decreasing $A_j \subseteq B_j \subseteq P(L)$. If $\Phi_j \in Con(L)$ denotes the congruence represented by the clopen convex set $B_j \setminus A_j$ for $j = 1, \ldots, n$, then $\Psi' = \bigwedge \{\Phi_j \mid j = 1, \ldots, n\}$, so that $\Psi = \bigvee \{\Phi_j' \mid j = 1, \ldots, n\} = \bigvee \{\theta(A_j, B_j) \mid j = 1, \ldots, n\}$ is a join of finitely many principal congruences. This completes the proof of the claim below.

**Proposition 2.8.** Let $K \subseteq P(L)$ be a closed set representing $\Psi \in Con(L)$, that is, let $\Psi$ consist of all $(U, V) \in L^2$ with $U \cap K = V \cap K$. Then the following are equivalent:

1. $\Psi$ is a compact congruence;
2. $\Psi$ has a complement in $Con(L)$;
3. $\Psi$ is a join of finitely many principal congruences;
4. $K$ is clopen.

Moreover, a compact $\Psi \in Con(L)$ is a join of at most $n$ principal congruences if and only if $n \geq \text{Gen}(P(L) \setminus K)$.

### 3. Priestley compactifications and Priestley duals of products

**Definition 3.1.** Let $P = (X, \tau, \leq)$ and $Q = (Y, \nu, \preceq)$ be ordered topological spaces. We say that $P$ is a Priestley compactification of $Q$ whenever $P$ is a Priestley space containing $Q$ as a dense ordered subspace, that is, whenever

1. $(Y, \nu)$ is a dense subspace of $(X, \tau)$, and
2. the partial orders $\preceq$ and $\leq$ coincide on $Y$.

Thus every Priestley space is its own Priestley compactification.

For any ordered topological space $Q = (Y, \nu, \preceq)$, let $C(Q)$ denote the Boolean algebra of all $\nu$-clopen subsets of $Y$, and let $D(Q)$ be the $(0,1)$-sublattice of $C(Q)$ formed by all decreasing members of $C(Q)$. We say that a $(0,1)$-sublattice $L$ of $C(Q)$ creates the order of $Q$ provided

$$y_0 \preceq y_1 \text{ if and only if } y_1 \in A \text{ implies } y_0 \in A \text{ for all } A \in L.$$ 

Thus any sublattice $L \subseteq C(Q)$ creating the order of $Q$ is, in fact, a sublattice of $D(Q)$, and the Boolean algebra $B(L) = [L]_{C(Q)}$ generated within $C(Q)$ by $L$ is an open basis of a topology $\nu$ in which $Q = (Y, \nu, \preceq)$ is totally order disconnected.

In particular, if $Q$ is a Priestley space, then $B(D(Q)) = C(Q)$ by 2.3(5) and, since $Q$ is totally order disconnected, $D(Q)$ creates the order of $Q$. 
Lemma 3.2. Let \( P = (X, \tau, \leq) \) be a Priestley compactification of an ordered topological space \( Q = (Y, \nu, \leq) \), and let \( \varphi : C(P) \rightarrow C(Q) \) be the mapping defined by \( \varphi(V) = Y \cap V \) for every \( V \in C(P) \). Then \( \varphi \) is an embedding of the Boolean algebra \( C(P) \) into \( C(Q) \), and the \((0,1)\)-sublattice \( L = \varphi(D(P)) \) of \( D(Q) \) has the following properties:

1. \( P \) is the Priestley space \( P(L) \) of \( L \),
2. \( L \subseteq D(Q) \) creates the order of \( Q \), and
3. members of \( B(L) = [L]_{C(Q)} \) form an open basis of \( Q \).

Proof: The mapping \( \varphi \) is one-to-one on \( C(P) \) because \( Q \) is dense in \( P \). It is also clear that \( \varphi \) preserves all Boolean operations. Since \( \leq \) coincides with the restriction of \( \leq \) to \( Q \), it follows that \( \varphi \) maps \( D(P) \) isomorphically onto a \((0,1)\)-sublattice \( L \) of \( D(Q) \); thus, by 2.1, the Priestley space \( P \) is, in fact, the Priestley dual \( P(L) \) of \( L \).

If \( y_0 \not\leq y_1 \) in \( Q \), then \( y_0 \not\leq y_1 \) in \( P \) and, because the latter space is totally order disconnected, for some \( A \in D(P) \) we have \( y_1 \in A \) and \( y_0 \in X \setminus A \). But then \( \varphi(A) \subseteq L \subseteq D(Q) \) is such that \( y_1 \in \varphi(A) \) and \( y_0 \in Y \setminus \varphi(A) \). Therefore \( L \) creates the order of \( Q \).

To prove (3), assume that \( G \subseteq Y \) is \( \nu \)-closed and \( y \in Y \setminus G \). Then there exists a \( \tau \)-closed \( F \subseteq X \) such that \( G = Y \cap F \). Since \( \{y\} \) is \( \tau \)-closed, 2.3(1) implies the existence of an \( A \in C(P) \) with \( F \subseteq A \) and \( y \in X \setminus A \). Hence \( B = \varphi(A) \) is \( \nu \)-clopen, \( G \subseteq B \) and \( y \in Y \setminus B \). But \( B \in B(L) \) because \( D(P) \) generates \( C(P) \) and \( L = \varphi(D(P)) \). Points and closed subsets of \( Q \) are thus separated by members of \( B(L) \), so that \( B(L) \) is an open basis of \( Q \).

Let \( Q = (Y, \nu, \leq) \) be an ordered topological space and let \( L \) be a \((0,1)\)-sublattice of \( C(Q) \) such that \( L \) creates the order of \( Q \) and \( B(L) \) is an open basis of \( Q \).

Next we aim to prove a converse of Lemma 3.2 by showing that the Priestley dual \( P(L) \) of any such \( L \subseteq C(Q) \) is a Priestley compactification of \( Q \).

Let \( e : L \rightarrow B(L) \) denote the inclusion homomorphism of \( L \) into the Boolean algebra \( B(L) \subseteq C(Q) \) generated by \( L \), and let \( F(B(L)) \) be the set of all prime filters of \( B(L) \). If \( \sigma \) is the topology on \( F(B(L)) \) whose open basis is formed by all sets

\[
\text{cl}(B) = \{ x \in F(B(L)) \mid B \in B(L), \}
\]

then \( (F(B(L)), \sigma) = P(B(L)) \) is the Stone space of \( B(L) \).

For every prime filter \( y \) of \( L \) there exists a unique prime filter \( x \in F(B(L)) \) such that \( y = L \cap x \), so that the \( \mathbf{P} \)-morphism \( P(e) : P(B(L)) \rightarrow P(L) \) dual to the inclusion \( e : L \rightarrow B(L) \) is a continuous bijection, and hence a homeomorphism, of \( P(B(L)) \) onto the (unordered) underlying compact Hausdorff space of \( P(L) \). For any \( x_0, x_1 \in F(B(L)) \) set \( x_0 \leq x_1 \) exactly when \( x_1 \cap L \subseteq x_0 \cap L \). Then \( \leq \) is a partial order under which \( (F(B(L)), \sigma, \leq) \) becomes an ordered space homeomorphic and also order isomorphic to the Priestley space \( P(L) \) of \( L \). We need only show that \( Q = (Y, \nu, \leq) \) is a dense ordered subspace of \( (F(B(L)), \sigma, \leq) \).

For any \( B \in B(L) \) we have \( \text{cl}(F(B(L)) \setminus B) = \{ x \in F(B(L)) \mid B \notin x \} \) because each \( x \in F(B(L)) \) is a prime filter of \( B(L) \). It follows that \( \text{cl}(B) \cup \text{cl}(Y \setminus B) = F(B(L)) \) and \( \text{cl}(B) \cap \text{cl}(Y \setminus B) = \emptyset \), so that \( \text{cl}(B) \) is \( \sigma \)-clopen for every \( B \in B(L) \).
Let $A \in L$ and $z_1 \in \text{cl}(A)$. If $z_0 \leq z_1$ then $A \subseteq z_1 \subseteq z_0$, that is, $z_0 \in \text{cl}(A)$, so that $\text{cl}(A)$ is decreasing for every $A \in L$.

For every $y \in Y$ define $p(y) = \{B \in B(L) \mid y \in B\}$. Since $p(y)$ is a prime filter of $B(L)$, this defines a mapping $p : Y \rightarrow F(B(L))$. Since $L$ creates the order $\leq$ of $Q$, $p(y_1) \cap L \subseteq p(y_0) \cap L$ if and only if $y_0 \leq y_1$. This shows that $p$ is one-to-one and that $p(y_0) \leq p(y_1)$ is equivalent to $y_0 \leq y_1$ for all $y_0, y_1 \in Y$.

Since $p(y) \in \text{cl}(B)$ if and only if $y \in B$, it follows that $\text{cl}(B) \cap p(Y) = p(B)$ and hence also $p^{-1}(\text{cl}(B)) = B$ for all $B \in B(L)$. Thus $p$ is continuous. In fact, since $B(L)$ is an open basis of $\nu$, the mapping $p$ is a homeomorphism of $Q$ onto the ordered subspace $p(Y)$ of $(F(B(L)), \sigma, \leq)$.

Let $A \neq \emptyset$ be $\sigma$-open. Since $\{\text{cl}(B) \mid B \in B(L)\}$ is an open basis of $(F(B(L)), \sigma, \leq)$, there exists a $B \in B(L)$ for which $\text{cl}(B)$ is a nonvoid subset of $A$. But then $p(B) = p(Y) \cap \text{cl}(B) \subseteq p(Y) \cap A$ is nonvoid, by the definition of $\text{cl}(B)$. Thus $p(Y)$ is dense in $(F(B(L)), \sigma, \leq)$.

To conclude the proof, we identify $Q$ with its homeomorphic and order isomorphic copy $p(Y)$ dense in $(F(B(L)), \sigma, \leq) \cong P(L)$.

Observe that if $L \subseteq D(Q)$ creates the order of $Q$ and if $B(L)$ forms an open basis of $Q$, then these two properties are inherited by any $(0,1)$-sublattice $K$ of $D(Q)$ containing $L$ and, in particular, by the lattice $D(Q)$ itself. In conjunction with 3.2, these observations yield the result below.

**Theorem 3.3.** Let $Q = (Y, \nu, \leq)$ be an ordered topological space. Then:

1. A Priestley compactification of $Q$ exists if and only if $D(Q)$ creates the order $\preceq$ of $Q$ and $B(D(Q))$ is an open basis of $\nu$;
2. An ordered topological space $P$ is a Priestley compactification of $Q$ if and only if $P = P(L)$ for some $(0,1)$-sublattice $L$ of $D(Q)$ such that $L$ creates the order $\preceq$ of $Q$ and $B(L)$ is an open basis of $\nu$.

Thus, for example, the Stone–Čech compactification $\beta Q$ is a Priestley compactification for any infinite discrete antichain $Q$; in fact, its Priestley compactifications are exactly its (unordered) compactifications. On the other hand, for the naturally ordered discrete set $N$ of all positive integers, the only $(0,1)$-sublattice $L \subseteq D(N)$ creating the order of $N$ is that consisting of $\emptyset, N$ and all initial segments \{1, 2, \ldots, n\} \subseteq N. Hence the one-point compactification of $N$ by a largest element is the only Priestley compactification of $N$.

Returning to general considerations, we now assume that $Q = (Y, \nu, \preceq)$ has a Priestley compactification $P(L)$ dual to a $(0,1)$-sublattice $L$ of $D(Q)$. It follows that the Priestley space

$$M(Q) = P(D(Q)) = (F(D(Q)), \sigma, \leq)$$

is a Priestley compactification of $Q$ as well. Next we show that $M(Q)$ is the 'largest' Priestley compactification of $Q$. 

- 250 -
Theorem 3.4. Let $P(L)$ be a Priestley compactification of $Q = (Y, \nu, \preceq)$, and let $P(K)$ be the Priestley dual of a distributive $(0,1)$-lattice $K$. If $g : Q \rightarrow P(K)$ is a continuous order preserving mapping, then:

1. there is a unique $P$-morphism $P(f) = g' : M(Q) \rightarrow P(K)$ extending $g$;
2. $g$ extends to a $P$-morphism $g'' : P(L) \rightarrow P(K)$ if and only if $f(K) \subseteq L$, in which case $g'' = g' \circ P(e_L)$ with the inclusion $(0,1)$-homomorphism $e_L : L \rightarrow D(Q)$.

Moreover, $M(Q)$ is an ordered Stone-Cech compactification of $Q$ if and only if $D(Q)$ generates $C(Q)$.

Proof: If $g : Q \rightarrow P(K)$ is a continuous order preserving mapping, then $g^{-1}(A) \in D(Q)$ for any clopen decreasing subset $A$ of $P(K)$. From $D(P(K)) \cong K$ it follows that the restriction $f : K \rightarrow D(Q)$ of $g^{-1}$ to $D(P(K))$ is a lattice $(0,1)$-homomorphism. The $P$-morphism $P(f) : M(Q) \rightarrow P(K)$ satisfies $P(f)^{-1}(A) \cap Y = g^{-1}(A)$ for all $A \in D(P(K))$; since $P(K)$ is totally order disconnected, this is possible only when the restriction of $P(f)$ to $Y$ coincides with $g$. In other words, the $P$-morphism $g' = P(f)$ extends $g$. Since $Q$ is dense in $M(Q)$, the extension $g'$ of $g$ is unique.

Now $g' = g'' \circ g_L$ for some continuous order preserving maps $g_L : M(Q) \rightarrow P(L)$ and $g'' : P(L) \rightarrow P(K)$ if and only if $f(K) \subseteq L$. If this is the case then, since $Q$ is dense in $P(L)$, the $P$-morphism $g_L$ is surjective and, in fact, $g_L = P(e_L)$ for the inclusion $(0,1)$-homomorphism $e_L : L \rightarrow D(Q)$. This demonstrates (1) and (2).

The unordered reduct $Q_0 = (Y, \nu)$ of $Q$ is a completely regular $T_1$-space. Its Stone-Cech compactification $\beta Q_0$ is thus one of its Priestley compactifications and, consequently, the identity mapping $id_Y$ of $Y$ extends to a continuous mapping $h : M(Q_0) \rightarrow \beta Q_0$. Since $M(Q_0)$ compactifies $Q_0$, there is also a continuous $h' : \beta Q_0 \rightarrow M(Q_0)$ extending $id_Y$ and, because $Y$ is dense in either space, $h$ is a homeomorphism with the inverse $h'$. Thus $M(Q_0) = (F(C(Q)), \beta)$ with the Stone-Cech topology $\beta$.

The inclusion mapping of $Q_0$ into $M(Q)$ is continuous and order preserving, and $Q_0$ is dense in $M(Q)$. Hence there exists a unique continuous extension $k : M(Q_0) \rightarrow M(Q)$ of $id_Y$. The mapping $k$ is one-to-one if and only if $F(B(D(Q))) = F(C(Q))$; since $k$ is surjective, it is a homeomorphism if and only if $C(Q)$ is generated by $D(Q)$. Therefore the $\beta$-compactification of $Q_0$ can be partially ordered to become a Priestley compactification of $Q$ if and only if the lattice $D(Q)$ generates $C(Q)$.

Next we turn our attention to Priestley duals of products.

Let $Q_i = (Y_i, \nu_i, \preceq_i)$ be nonvoid Priestley spaces for all $i \in I \neq \emptyset$, and denote $Q = \Sigma\{Q_i | i \in I\} = (Y, \nu, \preceq)$ their sum; that is, the partial order $\preceq$ is the union of $\preceq_i$ and the collection of all $\nu_i$-open sets forms an open basis of $\nu$. Since $A \in D(Q)$ if and only if $A \cap Y_i \in D(Q_i)$ for all $i \in I$ and because each $Q_i$ is a Priestley space, the lattice $D(Q)$ creates the order of $Q$ and the Boolean algebra $B(D(Q))$ forms an open basis of $Q$. Thus the Priestley compactification $M(Q)$ of $Q$ exists, by 3.3(1). The sum $g : Q \rightarrow P(K)$ of any collection of $P$-morphisms $g_i : Q_i \rightarrow P(K)$ with

- 251 -
\( i \in I \) is a continuous order preserving mapping and hence, by 3.4(1), it extends uniquely to a \( P \)-morphism \( g' : M(Q) \to P(Q) \). This, of course, means that \( M(Q) \) is dual to the product \( \Pi \{ D(Q_i) \mid i \in I \} \) of the nontrivial distributive \((0,1)\)-lattices \( D(Q_i) \). In particular, \( Q_i \) is a closed order subspace of \( M(Q) = P(\Pi \{ D(Q_i) \mid i \in I \}) \) for every \( i \in I \).

We show that Priestley compactifications of the sum \( Q = \Sigma \{ Q_i \mid i \in I \} \) are Priestley spaces of certain subdirect products of lattices \( D(Q_i) \). For example, if \( I \) is infinite then a one-point compactification \( Q \cup \{ \emptyset \} \) of \( Q \) such that \( \emptyset > y \) for all \( y \in Q \) is the Priestley space of the lower weak direct product of all \( D(Q_i) \) with \( i \in I \) - that is, the \((0,1)\)-sublattice \( L \subseteq K = \Pi \{ D(Q_i) \mid i \in I \} \) consisting of the unit 1 in \( K \) and of all \( \kappa \in K \) for which \( \{ i \in I \mid \kappa(i) > 0 \} \) is finite. On the other hand, if \( I \) is finite then the only Priestley compactification of the sum \( Q \) is the Priestley dual \( Q = \Sigma \{ Q_i \mid i \in I \} \) of the product \( \Pi \{ D(Q_i) \mid i \in I \} \).

Let \( K_i = P(Q_i) \) be a distributive \((0,1)\)-lattice with more than one element for each \( i \in I \neq \emptyset \), and let \( L \) be a \((0,1)\)-sublattice of the product \( K = \Pi \{ K_i \mid i \in I \} \). For any \( J \subseteq I \), let \( \pi_J \in Con(L) \) consist of all pairs \( (\lambda, \lambda') \in L^2 \) such that \( \lambda(j) = \lambda'(j) \) for all \( j \in J \). We say that \( L \) is a weak direct product of the set \( \{ K_i \mid i \in I \} \) if and only if, for every finite subset \( J \) of \( I \), the congruence \( \pi_J \) is complemented and \( L/\pi_J \cong \Pi \{ K_j \mid j \in J \} \).

**Proposition 3.5.** A Priestley space \( P(L) \) is a Priestley compactification of the sum \( Q = \Sigma \{ Q_i \mid i \in I \} \) if and only if \( L \) is a weak direct product of \( \{ K_i \mid i \in I \} \).

**Proof:** If \( P(L) \) is a Priestley compactification of \( Q = \Sigma \{ Q_i \mid i \in I \} \), then there is a surjective \( P \)-morphism \( h : M(Q) \to P(L) \) extending the identity mapping of \( Q \) onto itself, by 3.4; therefore \( Q_i = h(Q_i) \) is compact and hence closed in \( P(L) \) for every \( i \in I \). But then \( Q_i \cup c(Q \setminus Q_i) = c(Q) = P(L) \) because \( Q \subset P(L) \) is dense. Furthermore, since \( Q \) is a subspace of \( P(L) \) and because \( Q_i \) is open in \( Q \), it follows that \( Q_i \cap c(Q \setminus Q_i) = \emptyset \) in \( P(L) \). Thus \( Q_i \subset P(L) \) is clopen for every \( i \in I \), and so \( Q_i = \bigcup \{ Q_j \mid j \in J \} \) for every finite \( J \subseteq I \). By 2.8, the congruence \( \pi_J \) represented by \( Q_J \) has a complement, while \( L/\pi_J \cong \Pi \{ K_j \mid j \in J \} \) follows from the fact that \( Q_J \) is a closed order subspace of \( P(L) \).

Conversely, let \( L \) be a \((0,1)\)-sublattice of \( K \) such that \( L/\pi_J \cong \Pi \{ K_j \mid j \in J \} \) and \( \pi_J \in Con(L) \) is complemented for every finite \( J \subseteq I \). In particular, \( L/\pi_i \cong K_i = D(Q_i) \) for each \( i \in I \). By 2.1, there exists an order isomorphism and homeomorphism \( g_i : Q_i \to P(L) \) for each \( i \in I \) and, consequently, a continuous order preserving joint extension \( g : Q \to P(L) \) of all \( g_i : Q_i \to P(L) \). For distinct \( i, j \in I \), the hypothesis gives \( L/\pi_{i,j} \cong K_i \times K_j \) which implies that \( g \) maps the sum \( Q_i + Q_j \) onto its order copy in \( P(L) \). Therefore \( g \) is an order isomorphism of \( Q \) into \( P(L) \). Since \( \pi_{i,j} \in Con(L) \) has a complement, the set \( g(Q_i) \subseteq P(L) \) is clopen by 2.8, and hence the copy \( g(Q) \) of \( Q \) is a subspace of \( P(L) \). By 3.4, there exists a \( P \)-morphism \( h : M(Q) \to P(L) \) extending \( g \); the mapping \( h \) is surjective because \( L \) is a \((0,1)\)-sublattice of \( D(M(Q)) = \Pi \{ K_i \mid i \in I \} \), see 2.1. But then the copy \( g(Q) \subseteq P(L) \) of \( Q \) is dense in \( P(L) \) because \( Q \) is dense in \( M(Q) \). Altogether, \( g(Q) \) is a dense subspace of \( P(L) \) that is order isomorphic to \( Q \). ■
Remark 3.6. If $L$ is a weak direct product of $\{K_i \mid i \in I\}$ then, for any finite subset $J$ of $I$, the complement $\pi'_J$ of $\pi_J \in \text{Con}(L)$ is the congruence $\pi_{I\setminus J}$. To see this, select a $j \in J$ and note that $\pi_j \vee \pi_i = 1 \in \text{Con}(L)$ for any $i \in I \setminus \{j\}$ because $Q_i \cap Q_j = \emptyset$ in $P(L)$. From the distributivity of $\text{Con}(L)$ it then follows that $\pi_i \geq \pi'_J$ for each $i \neq j$, and hence also $\bigwedge \{\pi_i \mid i \neq j\} \geq \pi'_J$. On the other hand, $\bigwedge \{\pi_i \mid i \neq j\} \land \pi_i = 0 \in \text{Con}(L)$ because $L$ is a sublattice of $\prod \{K_i \mid i \in I\}$, so that $\bigwedge \{\pi_i \mid i \neq j\} \leq \pi'_J$ because $\text{Con}(L)$ is distributive.

Proposition 3.7. Let $Q_i = P(K_i)$ be the Priestley space of a nontrivial distributive $(0, 1)$-lattice $K_i$ for each $i \in I \neq \emptyset$, and let $Q = \Sigma \{Q_i \mid i \in I\}$. Then the Priestley dual $M(Q)$ of the product $K = \prod \{K_i \mid i \in I\}$ is an ordered $\beta$-compactification of $Q$ if and only if the chain lengths of all but finitely many component spaces $Q_i$ are uniformly bounded by a finite cardinal $n$.

Proof: Let $J \subseteq I$ be finite and let all chains of every $Q_i$ with $i \in I \setminus J$ have length at most $n$. To prove that $\beta Q$ is the underlying space of $M(Q)$, in view of the last claim in 3.4 we need only show that the Boolean algebra $C(Q)$ of all clopen sets is generated by $D(Q)$.

Let $C \in C(Q)$ be arbitrary. Since $Q_i$ is a Priestley space and because $C \cap Q_i \in C(Q_i)$ for every $i \in I$ there exists an integer $n_i$ such that $C \cap Q_i = \bigcup \{A_{k,i} \setminus B_{k,i} \mid k = 1, \ldots, n_i\}$ with $A_{k,i}, B_{k,i} \in D(Q_i)$, by 2.3(5). From 2.6 it follows that there exist sets $A_{k,i}, B_{k,i} \in D(Q_i)$ such that $n_i \leq m_0$ for all $i \in I \setminus J$ and a finite $m_0$. If $m_1 = \max \{n_i \mid j \in J\}$ and $m = \max \{m_0, m_1\}$, we may write $C \cap Q_i = \bigcup \{A_{k,i} \setminus B_{k,i} \mid k = 1, \ldots, m\}$. Set $A_k = \bigcup \{A_{k,i} \mid i \in I\}$ and $B_k = \bigcup \{B_{k,i} \mid i \in I\}$. Then $A_k, B_k$ are clopen decreasing for $k = 1, \ldots, m$ and $C = \bigcup \{A_k \setminus B_k \mid k = 1, \ldots, m\}$. Therefore $D(Q)$ generates $C(Q)$, as claimed.

Conversely, if the order condition fails, then there exists a one-to-one countably infinite sequence $i(1), i(2), \ldots$ such that $Q_{i(n)}$ contains a chain of length $2n$ for $n = 1, 2, \ldots$. By 2.5 and 2.4, there exist $C_{i(n)} \in C(Q_{i(n)})$ such that $C_{i(n)} = \bigcup \{A_{k} \setminus B_k \mid k = 1, \ldots, m\}$ for some $A_k, B_k \in D(Q_{i(n)})$ only when $m \geq n + 1$. Since $Q = \Sigma \{Q_i \mid i \in I\}$, the set $C = \bigcup \{C_{i(n)} \mid n = 1, 2, \ldots\}$ is clopen in $Q$, yet lies outside the Boolean algebra generated by $D(Q)$.

Remark 3.8. Adams and Beazer [2] show that the congruences of a distributive $(0, 1)$-lattice $L$ are $(n + 1)$-permutable if and only if all chains of $P(L)$ have at most $n$ elements. Hence 3.7 can be reformulated as follows: the Priestley dual of a product $\prod \{K_i \mid i \in I\}$ is an ordered $\beta$-compactification of $\Sigma \{P(K_i) \mid i \in I\}$ if and only if there exists some finite $n$ such that all but finitely many lattices $K_i$ have $(n + 1)$-permutable congruences.

Remark 3.9. Since any product of distributive double $p$-algebras is a distributive double $p$-algebra, the Priestley compactification $M(Q)$ of the sum $Q = \Sigma \{P(K_i) \mid i \in I\}$ of $dp$-spaces is the dual of the double $p$-algebra $K = \prod \{K_i \mid i \in I\}$, and the inclusion $Q_i \hookrightarrow M(Q)$ is a $dp$-map for every $i \in I$. Therefore 3.7 remains valid in the category of all $dp$-maps between $dp$-spaces. According to Beazer [3], a distributive double $p$-algebra $L$ has $n$-permutable congruences if and only if any
chain in its dp-space $P(L)$ has at most $n + 1$ elements, and the claim below follows immediately.

**COROLLARY 3.10.** Let $Q_i = P(K_i)$ be the Priestley space of a nontrivial distributive double $p$-algebra $K_i$ for each $i \in I \neq \emptyset$, and let $Q = \Sigma \{Q_i \mid i \in I\}$. Then the Priestley dual $M(Q)$ of the product $K = \Pi \{K_i \mid i \in I\}$ is an ordered $\beta$-compactification of $Q$ if and only if the chain lengths of all but finitely many component spaces $Q_i$ are uniformly bounded by a finite cardinal $n$. This is the case exactly when all but finitely many algebras $K_i$ have $n$-permutable congruences.  

To describe Priestley duals of ultraproducts, let $Q_i = P(K_i)$ be the Priestley dual of a nontrivial distributive $(0,1)$-lattice or a double $p$-algebra $K_i$ for each $i \in I \neq \emptyset$, and let $M(Q)$, where $Q = \Sigma \{Q_i \mid i \in I\}$, be the Priestley dual of $K = \Pi \{K_i \mid i \in I\}$.

Since $Q_i \subseteq Q \subseteq M(Q)$ is clopen in $M(Q)$ for every $i \in I$, the mapping $e : Q \rightarrow \beta I$ into the unordered Stone-Čech compactification $\beta I$ of the discrete space $I$ defined by $e(q) = i$ for all $q \in Q_i$ is continuous and order preserving, and satisfies $e(Ext(p)) = Ext(e(p))$ for all $p \in M(Q)$. Since $I$ is dense in $\beta I$, from 3.4 we obtain the existence of a continuous surjective extension $h : M(Q) \rightarrow \beta I$ of $e$; the $\mathcal{D}$-morphism $\psi = D(h)$ embeds the Boolean algebra $2^I$ canonically into $K$. Of course, $h(Ext(p)) = Ext(h(p))$ follows from the fact that $\beta I$ is unordered.

For any ultrafilter $u$ on $I$, let $\phi_u : K \rightarrow K/\theta_u$ denote the canonical surjective homomorphism from $K$ to the ultraproduct $K/\theta_u$. Thus $\phi_u(\kappa) = \phi_u(\kappa')$ if and only if $E(\kappa, \kappa') = \{i \in I \mid \kappa(i) = \kappa'(i)\} \in u$. Let $\phi_u \circ \psi = \mu_u \circ \epsilon_u$ be a decomposition such that $\mu_u$ is one-to-one and $\epsilon_u$ is surjective. For any $\lambda \in 2^I$ exactly one of the sets $\lambda^{-1}\{0\}$, $\lambda^{-1}\{1\}$ belongs to $u$, so that $\epsilon_u$ maps $2^I$ onto the two-element Boolean algebra $2 = \{0, 1\}$, and $\epsilon_u(\lambda) = 1$ if and only if $\lambda^{-1}\{1\} \in u$. Furthermore, these four morphisms form a pushout. To see this, let $\phi : K \rightarrow L$ and $\mu : 2 \rightarrow L$ satisfy $\phi \circ \psi = \mu \circ \epsilon_u$ and let $\kappa, \kappa' \in K$ be such that $E(\kappa, \kappa') \in u$. If $\lambda \in 2^I$ is given by $\lambda^{-1}\{1\} = E(\kappa, \kappa')$, then $\phi(\psi(\lambda)) = \mu(\epsilon_u(\lambda)) = \mu(1) = 1$ and $\kappa \wedge \psi(\lambda) = \kappa' \wedge \psi(\lambda)$ in $K$, so that $\phi(\kappa) = \phi(\kappa') \wedge \phi(\psi(\lambda)) = \phi(\kappa \wedge \psi(\lambda)) = \phi(\kappa' \wedge \psi(\lambda)) = \phi(\kappa')$. This shows that $\theta_u$ is contained in the kernel of $\phi$. Hence $\phi = \phi' \circ \phi_u$ for some $\mathcal{D}$-morphism $\phi'$. But then $\phi' \circ \mu_u = \mu$ follows from the fact that $\epsilon_u$ is surjective, and the four $\mathcal{D}$-morphisms in $\phi_u \circ \psi = \mu_u \circ \epsilon_u$ do, indeed, constitute a pushout. Therefore the diagram formed by their Priestley duals is a pullback in which $P(\epsilon_u) : \{1\} \rightarrow \beta I$ is given by $P(\epsilon_u)(1) = u$. Thus the closed order subspace $h^{-1}\{u\}$ of $M(Q)$ is the Priestley dual $P(K/\theta_u)$ of the ultraproduct $K/\theta_u$.

This concludes the proof of the claim below.

**PROPOSITION 3.11.** Let $\{K_i \mid i \in I\}$ be a nonvoid set of nontrivial distributive $(0,1)$-lattices or double $p$-algebras, and let $h : P(K) \rightarrow \beta I$ be the Priestley dual of the canonical embedding $e : 2^I \rightarrow K$ of the Boolean algebra $2^I$ into the product $K = \Pi \{K_i \mid i \in I\}$. Then, for any ultrafilter $u$ on $I$, the closed order subspace $h^{-1}\{u\} \subseteq P(K)$ is the Priestley dual $P(K/\theta_u)$ of the ultraproduct $K/\theta_u$.

It is clear that Proposition 3.11 applies also to all varieties of distributive $(0,1)$-lattices with operators — such as varieties of $p$-algebras and of (double) Heyting algebras.
REMARK 3.12. It is easily verified that D-morphisms (or double p-algebra homomorphisms) \( \varphi_i : L \rightarrow K_i \) determine a subdirect product \( L \) of lattices (or double p-algebras) \( K_i \) with \( i \in I \) if and only if each \( P(\varphi_i) \) is a homeomorphism and an order isomorphism (and a dp-map) onto a closed order subspace (or a closed c-set) of \( P(L) \), and the union of images of all \( P(\varphi_i) \) is dense in \( P(L) \).

EXAMPLES AND OBSERVATIONS. While there are many minimal weak direct products of distributive lattices, there is only one minimal weak direct product in the category of distributive double p-algebras. We use Priestley compactifications to illustrate these points.

For instance, if \( Q \) is the sum of infinitely many two-element chains \( Q_i = \{0_i, 1_i\} \) with \( i \in I \), then its one-point compactification \( R = Q \cup \{w\} \) in which \( \{w\} \cap Q_i = \{1_i\} \) for each \( i \in I \) is the dual of a minimal weak direct product of three-element chains \( D(Q_i) \), by 3.3 and 3.5. Yet any singleton \( \{1_i\} \) is a clopen increasing set for which \( \{1_i, 0_i, w\} \) is not open; hence, according to 2.2(1), the Priestley space \( R \) is not the dual of any distributive double p-algebra.

For an example of another kind, consider the two-point extension \( S = Q \cup \{z, u\} \) of the sum \( Q \) as above, in which \( z \leq s \leq u \) for all \( s \in S \), while \( \{z\} \) compactifies \( \text{Min}(Q) \) and \( \{u\} \) compactifies \( \text{Max}(Q) \). Then \( S \) is the dual of a double Stone algebra, and also a minimal Priestley compactification of \( Q \). No insertion of \( Q_i \) into \( S \) is a dp-map, however; as a result, \( R \) is not the dual of any weak direct product of three-element double Stone algebras \( D(Q_i) \).

These two examples indicate that dp-spaces of weak direct products of distributive double p-algebras must satisfy additional requirements.

Assume that \( Q = \Sigma\{Q_i \mid i \in I\} \) is the sum of arbitrary nontrivial dp-spaces \( Q_i \) and that \( L \) is a weak direct product of algebras \( K_i = D(Q_i) \) in the category of distributive double p-algebras. Then, as in 3.5 and 3.6, \( Q \) is dense in \( P(L) \) and the order subspaces \( Q_i \) and \( P(L) \setminus Q_i = c(\{Q \setminus Q_i\}) \) form a clopen decomposition of \( P(L) \) for each \( i \in I \). Since these sets also represent distributive double p-algebra congruences, it follows that \( Q_i \) and \( P(L) \setminus Q_i \) are clopen c-sets.

If \( p \in P(L) \) satisfies \( p \leq q \) for some \( q \in Q_i \), then there exists an \( m \in \text{Max}(Q_i) \subseteq \text{Max}(P(L)) \) such that \( p \leq m \). But then \( \text{Min}(p) \subseteq \text{Min}(m) \). Since \( m \) belongs to the c-set \( Q_i \), we have \( \text{Min}(p) \subseteq \text{Min}(Q_i) \) and then, because \( P(L) \setminus Q_i \) is a c-set, we conclude that \( p \in Q_i \). Together with a dual argument, this shows that \( Q \) is a union of order components of \( P(L) \), and explains the findings of the two preceding examples.

Observe that the set \( P(L) \setminus Q = \cap\{P(L) \setminus Q_i \mid i \in I\} \) is a closed union of order components of \( P(L) \).

Let \( Q \cup \{v\} \) be the one-point compactification of \( Q \) in which \( v \) is incomparable to any member of \( Q \). It is easily seen that \( Q \cup \{v\} = P(L_0) \) is the Priestley dual of a distributive double p-algebra \( L_0 \subseteq \Pi\{K_i \mid i \in I\} \) which consists of all \( \lambda \) satisfying \( \lambda(i) = 0 \) for all but finitely many \( i \in I \) or \( \lambda(i) = 1 \) for all but finitely many \( i \in I \).

Since the mapping \( h : P(L) \rightarrow P(L_0) \) defined by \( h(q) = q \) for all \( q \in Q \) and \( h(p) = v \) for all \( p \in P(L) \setminus Q \) is, clearly, a dp-map, this shows that \( L_0 \) is the unique minimal weak direct product in the class of all distributive double p-algebras.
References


Keywords. distributive (0,1)-lattice, Heyting algebra, distributive double p-algebra, Priestley duality

1980 Mathematics subject classifications: 06E15

MFF KU
Malostranské nám. 25
118 00 Praha 1
Czechoslovakia

Department of Mathematics
University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

- 256 -