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NOTES ON KURATOWSKI-MRÓWKA THEOREMS 
IN POINT-FREE CONTEXT

by A. PULTR \textsuperscript{1} and A. TOZZI \textsuperscript{1}

Introduction

The famous Kuratowski theorem characterizes compact spaces \( X \) by the fact that for each \( Y \) the projection \( X \times Y \to Y \) is closed. Precisely, Kuratowski \cite{7} proved that, in the realm of metric spaces, if \( X \) is compact then the projection \( \pi_Y \) is closed for any metric space \( Y \), Bourbaki \cite{1} proved the same property in the category of Hausdorff spaces and Mrówka \cite{9} established the converse property so that a space \( X \) is compact iff the projection \( \pi_Y \) is closed for any space \( Y \). Similarly, by results of Noble \cite{10}, Vaughan \cite{11} and Giuli \cite{3}, a space \( X \) is \( \alpha \)-compact (\( [\alpha, \beta] \)-compact) iff the projection \( \pi_Y : X \times Y \to Y \) is closed for each \( \alpha \)-discrete (\( \alpha \)-discrete with character \( \beta \)) space \( Y \). In this paper we consider these phenomena in the pointfree context.

In particular, we prove the Kuratowski theorem in the form that a locale \( A \) is compact iff the natural projection \( A \times B \to B \) is closed for each locale \( B \). For general \( \alpha \) the result we present is not so satisfactory: We are able to prove only that a locale \( A \) is \( \alpha \)-compact iff \( A \times B \to B \) is closed for all \( \alpha \)-discrete spatial \( B \). Thus, the question whether in this case \( A \times B \to B \) is closed for all \( \alpha \)-discrete \( B \) remains open. Still, the result answers the question whether there is a class \( \mathcal{C}(\alpha) \) of locales such that the \( \alpha \)-compactness is characterized by the closedness of the projections \( A \times B \to B \) with \( B \in \mathcal{C}(\alpha) \). More-

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over, the negative part of the statement presented, namely that for an $A$ which
is not $\alpha$-compact there is a spatial $\alpha$-discrete $B$ such that $A \times B \to B$ is not
closed, is in fact stronger that the respective part of the desired statement.

Finally, we consider a "dual" of the Kuratowski's characterization, namely
the question as to which $B$ have the property that the projections $A \times B \to B$
are closed for all $A$. In the classical context this characterizes the quasidiscrete
spaces, hence, in the regular case, the discrete spaces, not a very colourful class. In
the pointfree context, however, this requirement characterizes the complete Boolean
algebras in among regular frames, which is perharps more interesting.

Only basic knowledge of category theory (as e.g. in the introductory chapters
of [8]) is assumed. All the necessary facts of pointfree topology are presented in
Section 1. For more detail, the reader can consult, e.g., [5].

1. Preliminaries

1.1. Basic conventions: The cardinality of a set $X$ will be indicated
as $|X|$. The identity mapping of a set (object) $X$ onto itself will be denoted
by $id_X$ or simply $id$. If $p_i : X_1 \times X_2 \to X_i$ is a (categorial) product, the
morphisms $p_i$ will be referred to as the natural projections, similarly the coprod-
uct morphisms $X_i \to X_1 \oplus X_2$ as the natural injections.

If $X$ is a partially ordered set and $x \in X$, $\uparrow x = \{y | x \leq y\}$.

1.2. Frames and locales: A frame is a complete lattice satisfying the distribu-
tive law $(\bigvee a_i) \wedge b = \bigvee (a_i \wedge b)$. If $X$ is a topological space, the lattice

$$\Omega(X)$$

of all open sets of $X$ is a frame. Another example is a Boolean alge-
bra. If $A, B$ are frames, a (frame) homomorphism $\phi : A \to B$ is
a mapping preserving all joins and finite meets. If $f : X \to Y$ is
a continuous mapping, $\Omega(f) : \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}(U)$ is a frame homomorphism. If $A, B$ are Boolean algebras, the frame
homomorphisms $A \to B$ coincide with complete Boolean homomorphisms.
Denote by $\text{Frm}$ the category of frames and frame homomorphisms. The
correspondence $\Omega$ above constitutes a contravariant functor

$$\Omega : \text{Top} \to \text{Frm}.$$ 

The dual of $\text{Frm}$ is called the category of locales and denoted by $\text{Loc}$.
This makes $\Omega$ a covariant functor. In the "localic point of view" one thinks of frames
(locales) as generalized spaces; technically, however, we will stay in \( \text{Frm} \). Thus,
e.g., statements on products of generalized spaces will appear as statements on
coproducts of frames. A frame (locale) is said to be \textit{spatial} if it is isomorphic
to \( \Omega(X) \).

The top (bottom) of \( A \) will be denoted by \( 1_A \) or simply \( 1 \) \( (0_A \)
or \( 0 \)). The two-element Boolean algebra will be denoted by \( \mathbf{2} \).

A cover of a frame \( A \) is a subset \( U \subseteq A \) such that \( \bigvee U = 1 \).

1.3. \textbf{Regularity:} The pseudocomplement of \( a \in A \) is
\[
a^* = \bigvee \{ x | x \land a = 0 \}.
\]
By the distributivity, \( a^* \land a = 0 \); hence, \( a^* \) is the largest element
meeting \( a \) in zero. We have
\[
(\bigvee a_i)^* = \bigwedge a_i^*
\]
but the other De Morgan formula does not generally hold. We write
\[
a \triangleleft b \quad \text{for} \quad a^* \lor b = 1.
\]
A frame \( A \) is said to be \textit{regular} if for each \( a \in A \), \( a = \bigvee \{ x | x \triangleleft a \} \).
Note that, trivially, each Boolean algebra is regular.

1.4. \textbf{Sublocales:} A sublocale (cf [4]) of a frame \( A \) is a surjective
homomorphism \( \phi : A \rightarrow B \) (as, e.g., \( \Omega(j) \) for an embedding \( j : Y \subseteq X \) of a
space). Sublocales are, obviously, in a one-one correspondence (up to isomorphism)
with congruences (with respect to general joins and finite meets) on \( A \) and will
be often dealt with as such. Note that:
\[
\text{(CR)} \quad \text{for regular} \ A \ , \ \text{a congruence is determined}
\quad \text{by the set of elements congruent to} \ 1.
\]
Closed sublocales are those given by the congruences
\[
x \sim y \iff x \lor a = y \lor a \quad (a \text{ fixed}).
\]
If \( \phi : A \rightarrow B \) is a homomorphism and \( \gamma : B \rightarrow C \) a sublocale, the
image of \( \gamma \) under \( \phi \), denoted \( \phi[\gamma] \) is given by the congruence
\[
x \sim y \iff \gamma \phi(x) = \gamma \phi(y).
\]
1.5. Closed homomorphisms: A homomorphism $\phi : A \rightarrow B$ is said to be closed if the image of a closed sublocale is always closed. That is, if for each $b \in B$ there is an $a = \psi(b)$ such that

$$x \lor \psi(b) = y \lor \psi(b) \iff \phi(x) \lor b = \phi(y) \lor b.$$ 

This condition can be easily rewritten to

(CLOSED) \hspace{1cm} \phi(x) \leq y \lor \phi(z) \Rightarrow x \leq \phi_+(y) \lor z

where $\phi_+$ is the right Galois adjoint to $\phi$. In the regular case it can be reduced (using (CR) above) to

$$\phi(x) \lor y = 1 \Rightarrow x \lor \phi_+(y) = 1.$$ 

1.6. Coproducts (details see, e.g. [2] or [5]) :

The coproducts of frames $A, B$ will be denoted by

$$A \overset{i_A}{\rightarrow} A \oplus B \overset{i_B}{\leftarrow} B.$$ 

One uses the symbol $a \oplus b$ for $i_A(a) \land i_B(b)$. We will need the following two facts:

(1) $A \oplus B$ is join-generated by the elements $a \oplus b$,

(2) if $a \oplus b \leq a \oplus c$ and $a \neq 0$ then $b \leq c$

If $\phi_i : A_i \rightarrow B_i$ ($i = 1, 2$) are homomorphisms, we write

$$\phi_1 \oplus \phi_2 : A_1 \oplus A_2 \rightarrow B_1 \oplus B_2$$

for the homomorphism given by $(\phi_1 \oplus \phi_2) \circ i_{A_i} = i_{B_i} \circ \phi_i$. Obviously,

$$(\phi_1 \oplus \phi_2)(a_1 \oplus a_2) = \phi_1(a_1) \oplus \phi_2(a_2).$$

The coproduct $A \oplus 2$ can be identified with $A$ (then, $a \oplus 1$ is $a$, and $a \oplus 0 = 0$).

The functor $\Omega$ does not generally preserve products (in frame point of view, does not send products to coproducts). The natural homomorphism

$$\mu : \Omega(X_1) \oplus \Omega(X_2) \rightarrow \Omega(X_1 \times X_2),$$
determined by $\mu \circ \iota = \Omega(p_1)$, obviously satisfies the formula

$$\mu(\bigvee_{j \in J} a_j \oplus b_j) = \bigcup_{j \in J} a_j \times b_j.$$ 

Thus, it is always onto.

1.7. Closed injections: Recall 1.5. We easily see that the natural injection $\iota : B \to A \oplus B$ is closed iff

for each $u \in A \oplus B$, there is a $b \in B$ such that

$$1 \oplus b \leq u$$

and $(1 \oplus v \leq u \lor (1 \oplus w) \Rightarrow v \leq b \lor w).$

For regular $B$, this reduces to

$$1 \oplus b \leq u$$

and $(u \lor (1 \oplus w) = 1 \Rightarrow b \lor w = 1)$.

2. A characterization of $\alpha$-compact frames

2.1. In this and the following sections, $\alpha$ is a regular cardinal. Recall that a frame $A$ is $\alpha$-compact if each cover of $A$ has a subcover of cardinality $< \alpha$. A space is $\alpha$-discrete if any intersection of $< \alpha$ open sets is an open set.

2.2. Construction: Let $A$ be a frame which is not $\alpha$-compact. Fix a cover $\mathcal{U}$ such that there is no subcover of cardinality $< \alpha$. Define a space $X$ on $A$ as the underlying set with $\Omega(X) = B$ consisting of the $M \subseteq A$ such that

if $1 \in M$ then $\uparrow \bigvee K \subseteq M$ for some $K \subseteq \mathcal{U}$, $|K| < \alpha$.

2.3. Lemma. In $A \oplus B$ define $c = \bigvee \{u \oplus \uparrow u | u \in \mathcal{U}\}$. Then $(1_A \oplus (A \setminus \{1\})) \lor c = 1_A \oplus 1_B$, and $1 \oplus M \leq c$ only for $M = \emptyset$. 

Proof: Put $z = (1 \oplus (A \setminus \{1\})) \lor c$. For $u \in \mathcal{U}$ we have $u \oplus 1 = u \oplus (A \setminus \{1\}) \lor u \oplus \uparrow u \leq z$. As $\mathcal{U}$ is a cover, $1 = \bigvee \mathcal{U} \oplus 1 \leq z$.

Now consider, for $x \in A$, the homomorphisms $\xi_x : B \to 2$ defined by $\xi_x(M) = 1$ iff $x \in M$. We have

$$(id \oplus \xi_x)(u \oplus \uparrow u) = u \oplus \xi_x(\uparrow u) = \begin{cases} u & \text{if } u \leq x, \\ 0 & \text{otherwise}. \end{cases}$$
Thus,

\[(id \oplus \xi_x)(c) = \bigvee \{u | u \leq x\} \leq x.\]

Each non-void open \( M \) contains an \( x \neq 1 \). Then

\[(id \oplus \xi_x)(1 \oplus M) = 1 \not\leq x\]

and hence \((1 \oplus M) \not\leq c.\)

**2.4. Corollary.** Let \( A \) not be \( \alpha \)-compact. Then there is a space \( X \) such that

1. \( X \) has only one non-isolated point,
2. \( X \) is \( \alpha \)-discrete, and
3. the natural injection \( \Omega(X) \to A \oplus \Omega(X) \) is not closed.

**2.5. Theorem.** A frame \( A \) is \( \alpha \)-compact iff for each \( \alpha \)-discrete space \( X \) the natural injection \( \Omega(X) \to A \oplus \Omega(X) \) is closed.

**Proof:** Let \( X \) be \( \alpha \)-discrete. For \( x \in X \) consider the \( \xi_x : \Omega(X) \to 2 \)
with \( \xi_x(u) = 1 \iff x \in u \). Take \( y = \bigvee a_i \oplus b_i \) in \( A \oplus \Omega(X) \) and put

\[M = \{x \in X | (id \oplus \xi_x)(y) = 1\} .\]

Thus, for \( x \in M \) we have \( 1 = (id \oplus \xi_x)(y) = \bigvee \{a_i | x \in b_i\} \) and, by \( \alpha \)-compactness, there is a \( K(x) \subseteq J , |K(x)| < \alpha \), such that

\[\bigvee_{K(x)} a_i = 1 \quad \text{and} \quad b(x) = \bigcap_{K(x)} b_i \ni x .\]

Put \( b = \bigvee_{x \in M} b(x) \). Since obviously for \( a_i \in K(x) \) we have \( a_i \oplus b(x) \leq y \), we infer \( 1 \oplus b(x) \leq y \) and consequently \( 1 \oplus b \leq y \). Now let \( 1 \oplus v \leq y \lor (1 \oplus w) \).

If \( x \in v \) we have \( 1 = (id \oplus \xi_x)(1 \oplus w) = (id \oplus \xi_x)(y) \lor \xi_x(w) \). Hence either \( x \in M \subseteq b \) or \( x \in w \). Thus, \( v \leq b \lor w \).

If \( A \) is not \( \alpha \)-compact, consider the \( X \) from 2.4.

**2.6 Remark.** By 2.4.(1), of course, the "testing class" for the \( \alpha \)-compactness in 2.5 can be reduced to the \( \alpha \)-discrete spaces with at most one non-isolated point.
3. Pointfree Kuratowski Theorem

3.1. We say that a frame \( A \) satisfies the unit decomposition property (briefly, UD) if for each frame \( B \) and each decomposition of the unit

\[
1_{\Delta B} = \bigvee\{a_i \oplus b_i | i \in J\}
\]

the system \( \{\bigwedge\{b_i | i \in K\} | K \subseteq J \text{ such that } \bigvee\{a_i | i \in K\} = 1\} \) is a cover of \( B \). If this is required only for the frames \( B \) from a class \( C \) we speak on the property \( \text{UDC} \).

3.2. Lemma. Let \( A \) satisfy UD and let

\[
1 \oplus v \leq \bigvee\{a_i \oplus b_i | i \in J\}.
\]

then \( \bigvee_{K} \{\bigwedge_{i \in K} b_i | K \subseteq J \text{ such that } \bigvee_{i \in K} a_i = 1\} \geq v \).

Proof: Consider the sublocale \( q : B \rightarrow [0, v] \) (the interval between 0 and \( v \)) given by \( q(x) = x \wedge v \). Then

\[
1 \leq (id \oplus q)(\bigvee_{J} a_i \oplus b_i) = \bigvee_{J} a_i \oplus q(b_i)
\]

so that by UD

\[
\bigvee_{K} \{\bigwedge_{i \in K} b_i \wedge v | K \subseteq J \text{ such that } \bigvee_{i \in K} a_i = 1\} = v (= 1_{[0,v]}) ,
\]

that is, \( v \wedge \bigvee_{K} \{\bigwedge_{i \in K} b_i | K \subseteq J \text{ such that } \bigvee_{i \in K} a_i = 1\} = v \).

3.3. Lemma. Let \( A \) be compact, \( B \) arbitrary non trivial (that is, \( 1_B \neq 0_B \)). Let

\[
1_{\Delta B} = \bigvee\{a_i \oplus b_i | i \in J\}.
\]

Then there exists a finite \( K \subseteq J \) such that \( \bigvee_{K} a_i = 1 \) and \( \bigwedge_{K} b_i \neq 0 \).


3.4. Proposition. Each compact frame satisfies UD.

Proof: Let \( A \) be compact, \( 1_{\Delta B} = \bigvee\{a_i \oplus b_i | i \in J\} \). Put \( U = \{\bigwedge_{K} b_i | K \subseteq J, K \text{ finite such that } \bigvee_{K} a_i = 1\} \), \( c = \bigvee U \). Consider the congruence

\[ x \sim y \text{ iff } x \vee c = y \vee c \]
Suppose $U$ is not a cover. Then $\overline{B} = B/\sim$ is not trivial. Let $q : B \to \overline{B}$ be the sublocale homomorphism. We have

$$1_{A\oplus \overline{B}} = (id \oplus q)(\bigvee a_i \oplus b_i) = \bigvee a_i \oplus q(b_i).$$

Thus, by 3.3, there is a finite $K \subseteq J$ such that $\bigvee_K a_i = 1$ and $q(\bigwedge_K b_i) = \bigwedge_K q(b_i) \neq 0$. By the definition of $q$, $(\bigwedge b_i) \vee c \neq c$ which is a contradiction since $\bigwedge_K b_i \in U$.

3.5. Theorem: The following statements are equivalent:

(a) $A$ is compact,
(b) $A$ satisfies UD,
(c) For each frame $B$ the natural injection $B \to A \oplus B$ is closed,
(d) for each space $X$ with at most one non-isolated point the natural injection $\Omega(X) \to A \oplus \Omega(X)$ is closed.

Proof: (a)⇒(b) is proved in 3.4. (b)⇒(c): Recall 1.7. Consider $u = \bigvee \{ a_i \oplus b_i \mid i \in J \}$ in $A \oplus B$. Put $b = \bigvee \{ \bigwedge_K b_i \mid K \subseteq J \text{ such that } \bigvee_K a_i = 1 \}$. Since in each individual case $1 \oplus \bigwedge b_i \leq u$, we have $1 \oplus b \leq u$. Let $1 \oplus v \leq u \vee (1 \oplus w)$. By 3.2, $b \vee w \geq v$.

(c)⇒(d) is trivial and (d)⇒(a) by 2.5.

4. $\alpha$-Discrete-frames

4.1. Recall the following simple characteristics of $\alpha$-discrete spaces (see, e.g., [10] cor.2.3; the proof however, can be left to the reader as an easy exercise):

Proposition: A space $Y$ is $\alpha$-discrete iff for each discrete $X$ with $|X| < \alpha$ the natural projection $X \times Y \to Y$ is closed.

4.2. We introduce the following condition

$$D(\alpha) : \text{If } |J| < \alpha \text{ then } \bigwedge_j a_i \vee b = \bigwedge_j (a_i \vee b).$$

We have

Theorem: A frame $B$ satisfies $D(\alpha)$ iff for each discrete $X$ with $|X| < \alpha$ the natural injection $B \to \Omega(X) \oplus B$ is closed.
Proof: Let the injection be closed. Consider \( \{a_i\}_J \subseteq B \) with \( |J| < \alpha \), and endow \( J \) with discrete topology. Let \( b \) be in \( B \). Put \( y = \bigvee \{ \{i\} \oplus a_i | i \in J \} \). By 1.7 we have an \( a \in B \) such that

\[
1 \oplus a \leq y \text{ and } (1 \oplus v \leq y \lor (1 \oplus b) \implies v \leq a \lor b) .
\]

Thus in particular \( \{i\} \oplus a = (1 \oplus a) \land ((i) \oplus 1) \leq y \land ((i) \oplus 1) = \{i\} \oplus a_i \), hence (recall 1.6(2)) \( a \leq a_i \) and finally

\[
a \leq \bigwedge a_i .
\]

Consider \( v = \bigwedge_J (a_i \lor b) \). Since \( \{i\} \oplus v \leq \{i\} \oplus a_i \lor \{i\} \oplus b \), we have \( 1 \oplus v \leq y \lor (1 \oplus b) \) and hence \( \bigwedge (a_i \lor b) \leq a \lor b \leq (\bigwedge a_i) \lor b \). As the opposite inequality is trivial, \( D(\alpha) \) holds.

On the other hand let \( B \) satisfy \( D(\alpha) \) and let \( X \) be discrete, \( |X| < \alpha \). Consider an element \( y \in \Omega(X) \oplus B \). Put \( a_x = \bigvee \{ c | \{x\} \oplus c \leq y \} \), \( a = \bigwedge_{x \in X} a_x \). As \( \{x\} \oplus a \leq y \) for all \( x \), we have \( 1 \oplus a \leq y \). Now let \( 1 \oplus b \leq y \lor (1 \oplus w) \). That is,

\[
1 \oplus v \leq \bigvee_X \{x\} \oplus a_x \lor \bigvee_X \{x\} \oplus w = \bigvee_X \{x\} \oplus (a_x \lor w)
\]

and meeting both sides with \( \{x\} \oplus 1 \) we infer that \( v \leq (a_x \lor w) \), by \( D(\alpha) \), \( v \leq a \lor w \).

4.3. Proposition: Let \( Y \) be a space. Then \( Y \) is \( \alpha \)-discrete iff \( \Omega(Y) \) satisfies \( D(\alpha) \).

Proof: As soon as we have realized that, for a discrete \( X \), \( |X| < \alpha \), \( \Omega(X \times Y) = \Omega(X) \oplus \Omega(Y) \), the statement will follow from 4.1 and 4.2.

Consider the \( \mu \) from 1.6. We have to show that it is one-one, that is, that \( \bigcup_i a_i \times b_i = \bigcup_i a'_i \times b'_i \) implies \( \bigvee a_i \oplus b_i = \bigvee a'_i \oplus b'_i \). For \( x \in X \) put \( b(x) = \bigcup \{ b_i | x \in a_i \} \). If \( \bigcup_i a_i \times b_i = \bigcup_i a'_i \times b'_i \), we have also \( b(x) = \bigcup_i b'_i | x \in a'_i \) and obtain

\[
\bigvee a_i \oplus b_i = \bigvee_i \bigvee x \in a_i \{x\} \oplus b_i = \bigvee \{x\} \oplus b(x) = \bigvee a'_i \oplus b'_i .
\]

4.4. Proposition 4.3 justifies proclaiming a frame \( \alpha \)-discrete if it satisfies \( D(\alpha) \).

It should be noted that for \( T_0 \)-spaces the statement of 4.3 is immediate: if \( Y \) is not \( \alpha \)-discrete, we have an instance of open \( u_i \subseteq Y, i \in J, |J| < \alpha \) such
that $\bigwedge_{i \neq x} u_i \subseteq \bigcap_{i \neq x} u_i$. Consider $x \in \bigcap_{i} u_i - \bigwedge_{i} u_i$. We have $\bigwedge(u_i \lor (Y - \{x\})) = Y$ while $x \notin (\bigwedge_{i} u_i) \lor (Y - \{X\})$.

4.5. We are so far unable to tell whether the positive part of 2.5 can be extended to general frames (as in Section 3 for $\alpha = \omega_0$) in case of general $\alpha$. That is, we do not know whether, if $A$ is $\alpha$-compact, the injection $B \rightarrow B \oplus A$ is closed for all $\alpha$-discrete frames. In the following two paragraphs we will show that this cannot be decided by a simple modification of the techniques from 2.5 and Section 3. Note, in particular, that the proof of 4.6 will be virtually the same as that of 2.5.

4.6. Proposition: Let $C(\alpha)$ be the class of all spatial $\alpha$-discrete frames. Then $A$ is $\alpha$-compact iff it satisfies UDC($\alpha$).

Proof: If $A$ is not $\alpha$-compact, take the $B$ from 2.2. By the proof of 2.3, the system corresponding to the decomposition $1 = (u \oplus (A - \{1\})) \lor \bigvee \{u \uparrow u \mid u \in U\}$ does not cover the element 1.

On the other hand, let $A$ be $\alpha$-compact, $B$ $\alpha$-discrete and $1 = \forall a_i \oplus b_i$. In the notation of the proof of 2.5 we have $y = 1$, hence $M = X$ and $\{\bigwedge_{i} \{b_i\} \mid x \in X\}$ is a cover. ♦

4.7. The class $C(\alpha)$ in 3.6 cannot be replaced by that of all $\alpha$-discrete frames, not even for $\alpha = \omega_1$. Consider the real line $\mathbb{R}$, $A = \Omega(\mathbb{R})$ and $B$ the Boolean algebra of regular open subset of $\mathbb{R}$. $B$ is $\alpha$-discrete for all $\alpha$. The decomposition

$$1_{A \oplus B} = \bigvee_{r \in \mathbb{Z}} \bigvee_{s \in \mathbb{Z}} (r, r + 2) \oplus \left( \frac{s}{|r| + 1}, \frac{s + 1}{|r| + 1} \right)$$

($\mathbb{Z}$ the set of integers), if $\bigvee_{K} a_i = 1$ we always have $\bigwedge_{K} b_i = 0$, however.

5. Complete Boolean algebras

5.1. We say that a complete lattice is completely distributive if meets distribute over general joins (as in frames), and also joins distribute over general meets. Theorem 4.2 immediately yields

Corollary: A frame is completely distributive iff for each discrete $X$ the natural injection $B \rightarrow \Omega(X) \oplus B$ is closed.
5.2. The following is well known, but easier to prove than quote:

**Proposition:** A regular frame is completely distributive iff it is a complete Boolean algebra.

**Proof:** Obviously, complete Boolean algebras are completely distributive. On the other hand, let \( B \) be regular completely distributive. For \( a \in B \) we have \( a = \bigvee \{ x | x^* \lor a = 1 \} \), hence \( a^* = \bigwedge \{ x^* | x^* \lor a = 1 \} \), and finally \( a^* \lor a = \bigwedge \{ x^* \lor a | x^* \lor a = 1 \} = 1 \).

5.3. **Proposition:** Let \( B \) be a complete Boolean algebra and \( A \) an arbitrary frame. Then each homomorphism \( \varphi : B \rightarrow A \) is closed.

**Proof:** Recall 1.5; Let \( \varphi(x) \lor y = 1 \). Meeting both sides with \( \varphi(x^*) \) we obtain \( \varphi(x^*) \land y = \varphi(x^*) \) and hence \( \varphi(x^*) \leq y \). Thus, \( x^* \leq \varphi_+(y) \) and hence \( x \lor \varphi_+(y) \geq x \lor x^* = 1 \).

5.4. **Theorem:** Let \( B \) be a regular frame. Then the following statements are equivalent:

(a) \( B \) is a complete Boolean algebra,

(b) every homomorphism \( \varphi : B \rightarrow A \) is closed,

(c) for each \( A \) the natural injection \( B \rightarrow B \oplus A \) is closed,

(d) for each atomic complete Boolean algebra \( A \) the natural injection \( B \rightarrow B \oplus A \) is closed.

**Proof:** (a) \( \Rightarrow \) (b) by 5.3, (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) is trivial, and (d) \( \Rightarrow \) (a) follows from 5.1.

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While preparing this article for publication we learned that the fact that for a compact locale \( A \) and a general \( B \) the projection \( A \times B \rightarrow B \) is closed was also proved independently by Vermeulen and Xiang Dong in so far unpublished papers.

**References**