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MODEL-THEORETIC CHARACTERISATIONS OF CONVENIENCE PROPERTIES IN TOPOLOGICAL CATEGORIES

by Vaclav VAJNER

Résumé. Quelques types importants de catégories topologiques (par exemple, les catégories topologiques qui sont cartésiennes fermées, ou les catégories universellement topologiques) sont caractérisés comme catégories des modèles qui correspondent aux théories (c'est-à-dire, foncteurs) spéciales prenant leurs valeurs dans des catégories de treillis complets. Ces caractérisations ont une forme commune : en tout cas, la théorie associée au type donné de catégorie transforme les carrés cartésiens en diagrammes commutatifs d'une espèce bien déterminée.

0 Introduction

By a result essentially due to O. Wyler ([12], [13]), every topological category (over an arbitrary base category \mathcal{X}) can be considered as a category of models corresponding to a functor (called a *theory*) from \mathcal{X} into a category of complete lattices.

We extend this correspondence to the so-called "convenient" topological categories, i.e., topological categories which admit additional structure. We accomplish this by establishing "preservation" properties of a given theory T which are necessary and sufficient for a particular convenience property to be lifted from a suitable base category \mathcal{X} to the associated category of T -models.

Specifically, the various types of convenient topological categories are characterised by means of complete lattice-valued (resp. frame-valued) theories which send pullback diagrams into commutative diagrams satisfying special order-theoretic conditions. The characterisation result for the universally topological categories also extends an earlier result of H. Herrlich (see 5.6 of [7]), which characterised universally topological functor-structured categories in terms of set-valued theories.

1 Preliminaries

A *concrete category* over a fixed base category \mathcal{X} consists of a pair (\mathcal{A}, U) , where $U : \mathcal{A} \rightarrow \mathcal{X}$ is a faithful and *amnesic* (meaning, any \mathcal{A} -isomorphism f is an \mathcal{A} -identity whenever $U(f)$ is an \mathcal{X} -identity) functor. A *concrete functor* $F : (\mathcal{A}, U) \rightarrow (\mathcal{B}, V)$ between concrete categories over \mathcal{X} is a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfying $U = V \cdot F$.

If (\mathcal{A}, U) is concrete over \mathcal{X} , and $X \in \mathcal{X}$, the *fibre* of X with respect to U , denoted by $U^{-1}[X]$, is defined to be the class of all \mathcal{A} -objects A for which $UA = X$. If, for each X in \mathcal{X} , $U^{-1}[X]$ is a set, then (\mathcal{A}, U) is said to be *small-fibred* (or, *fibre-small*). For each X in \mathcal{X} , $U^{-1}[X]$ can be ordered as follows : $A \leq B$ (A is *finer than* B) iff there exists an \mathcal{A} -morphism $a : A \rightarrow B$ such that $U(a) = id_X$. (The amnescticity of U ensures that \leq is in fact a partial ordering.)

Given (\mathcal{A}, U) over \mathcal{X} , a *U-morphism* is a pair (f, A) , where $f : UA \rightarrow X$ is an \mathcal{X} -morphism and A an \mathcal{A} -object. A *U-sink* on X consists of a pair (X, S) , where $X \in \mathcal{X}$ and $S = (f_i : UA_i \rightarrow X)_I$ is a family of U -morphisms indexed by some class I . A *U-source* $(f_i : X \rightarrow UA_i)_I$ is dually defined. (\mathcal{A}, U) is said to be *finally complete* provided that every U -sink has a unique *U-final lift*; (\mathcal{A}, U) is called *topological* if it is finally complete and small-fibred. For more details concerning these notions, the reader may consult [1] or [5].

The following terminology is primarily based on that of [12] and [13].

A (*topological*) *theory* on a category \mathcal{X} is a functor $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$, where \mathbf{CSLatt} denotes the category of complete lattices and supremum-preserving maps. For each object X of \mathcal{X} , TX is a complete lattice, the associated order referred to as the *finer than* relation. Every topological theory T on a category \mathcal{X} induces a concrete category over \mathcal{X} , called the category of *T-models* and *T-morphisms*, denoted by $\text{Mod}(T)$, as follows : the objects of $\text{Mod}(T)$ are pairs (X, α) , where X is an \mathcal{X} -object and $\alpha \in TX$; an \mathcal{X} -morphism $f : X \rightarrow Y$ is a *T-morphism* $f : (X, \alpha) \rightarrow (Y, \beta)$ provided $Tf(\alpha) \leq \beta$. Composition in $\text{Mod}(T)$ is lifted from \mathcal{X} . Observe, given $f : X \rightarrow Y$ and $\alpha \in TX$, that $f : (X, \alpha) \rightarrow (Y, Tf(\alpha))$ is final in $\text{Mod}(T)$. The associated forgetful functor $U_T : \text{Mod}(T) \rightarrow \mathcal{X}$ is defined by : $U_T(X, \alpha) = X$, for a *T-model* (X, α) , and $U_T(f) = f$, for a *T-morphism* $f : (X, \alpha) \rightarrow (Y, \beta)$.

Topological categories can be characterised in terms of theories as follows :

1.1 Theorem. ([4], [11], [13]) *The following conditions are equivalent, for a concrete category (\mathcal{A}, U) over \mathcal{X} :*

(1) (\mathcal{A}, U) is topological;

(2) (\mathcal{A}, U) is concretely isomorphic to $\text{Mod}(T)$ for some topological theory $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$. □

For a topological category of form $\text{Mod}(T), T : \mathcal{X} \rightarrow \mathbf{CSLatt}$, it is easily seen that for each $X \in \text{Ob}(\mathcal{X})$, TX is isomorphic to $U_T^{-1}[X] = \{ (X, \tau) \mid \tau \in TX \}$. Hence finality in $\text{Mod}(T)$ can be characterised as follows : a sink $(f_i : (X, \tau_i) \rightarrow (X, \tau))_I$ is final in $\text{Mod}(T)$ iff $\tau = \bigvee_I T f_i(\tau_i)$ in TX . It can also further be verified that for a structured source $(g_i : X \rightarrow (X_i, \tau_i))_I$, the initial structure on X with respect to $(g_i)_I$ is given by $\tau = \bigvee \{ \mu \in TX \mid T g_i(\mu) \leq \tau_i \text{ for each } i \in I \}$.

2 (Concrete) cartesian closedness

2.1 Definition. Let \mathcal{X} be an arbitrary category.

(1) ([3], [7]) A sink $(f_i : X_i \rightarrow X)_I$ in \mathcal{X} is called *regular* if there exists a subset $J \subset I$ such that the canonical morphism $\coprod_J f_j : \coprod_J X_j \rightarrow X$ is a regular epimorphism; \mathcal{X} is said to have *regular sink factorisations* if it is cocomplete and for each sink $(f_i : X_i \rightarrow X)_I$ in \mathcal{X} there exists a monomorphism $m : Y \rightarrow X$ and a regular sink $(e_i : X_i \rightarrow Y)_I$ with $f_i = m \cdot e_i$ for each $i \in I$.

(2) ([3]) If \mathcal{X} has finite products, then regular sinks are said to be *finitely productive* provided that for each regular sink $(f_i : X_i \rightarrow X)_I$ and each \mathcal{X} -object Y , the sink $(f_i \times \text{id}_Y : X_i \times Y \rightarrow X \times Y)_I$ is regular. The finite productivity of final sinks, colimits and regular epimorphisms is defined analogously.

For the remainder of this Section we assume that \mathcal{X} is a cartesian closed category which admits finite limits and regular sink factorisations. Our first goal is to characterise, in theoretic terms, the cartesian closed topological categories. The following result is needed :

2.2 Theorem. ([7]) For a topological category (\mathcal{A}, U) over \mathcal{X} , the following conditions are equivalent :

(1) (\mathcal{A}, U) is cartesian closed;

(2) regular sinks in \mathcal{A} are finitely productive. □

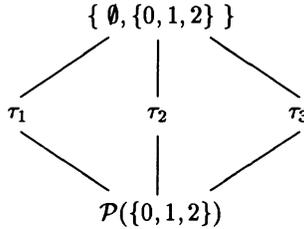
In order to apply the concept of cartesian closedness to categories of the form $\text{Mod}(T), T : \mathcal{X} \rightarrow \mathbf{CSLatt}$, it should be observed that limits and colimits in $\text{Mod}(T)$ are naturally lifted from the base category \mathcal{X} , for example, given T -models (X_1, α) and (X_2, β) , the product $(X_1, \alpha) \times (X_2, \beta)$ in $\text{Mod}(T)$ is given by $(X_1 \times X_2, \alpha \otimes \beta)$, where $\alpha \otimes \beta$ is the initial structure on $X_1 \times X_2$ with respect to the projection source $(p_1 : X_1 \times X_2 \rightarrow (X_1, \alpha), p_2 : X_1 \times X_2 \rightarrow (X_2, \beta))$, i.e., $\alpha \otimes \beta = \bigvee \{ \gamma \in T(X_1 \times X_2) \mid T p_1(\gamma) \leq \alpha, T p_2(\gamma) \leq \beta \}$. Note also that a sink $(f_i : (X_i, \alpha_i) \rightarrow (X, \alpha))_I$ in $\text{Mod}(T)$ is regular iff the underlying sink in \mathcal{X} is regular, and $\alpha = \bigvee_I T f_i(\alpha_i)$.

2.3 Examples. (1) ([7]) The cartesian closed topological categories over the terminal (i.e., one-morphism) category \odot are the frames, that is, the complete lattices in which arbitrary suprema distribute over finite infima.

(2) Define a theory $R : \mathbf{Set} \rightarrow \mathbf{CSLatt}$ as follows : for each set X , put $RX = (\mathcal{P}(X \times X), \subset)$, and for a map $f : X \rightarrow Y$, let $Rf : RX \rightarrow RY$ be defined by the assignment $\rho \mapsto (f \times f)[\rho]$. Then, up to concrete isomorphism, $\mathbf{Mod}(R)$ is the category of binary relations (denoted by \mathbf{Rel}) which is a cartesian closed topological category ([3]).

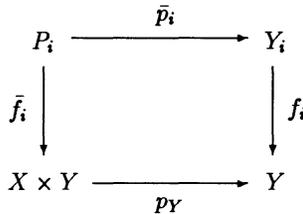
In view of 2.3 (1) above, a natural question to ask is whether for any cartesian closed topological category, each fibre is a frame (notice that each \mathbf{Rel} -fibre is a frame). The negative answer is provided by the following :

2.4 Example. The category \mathbf{Compl} of complemented spaces (a topological space is called *complemented* provided each of its open sets is closed) is a cartesian closed topological subcategory of the category of topological spaces (see [6]), but there exist \mathbf{Compl} -fibres which are not frames. Let $\tau_1 = \{ \emptyset, \{0\}, \{1, 2\}, \{0, 1, 2\} \}$, $\tau_2 = \{ \emptyset, \{2\}, \{0, 1\}, \{0, 1, 2\} \}$, $\tau_3 = \{ \emptyset, \{1\}, \{0, 2\}, \{0, 1, 2\} \}$. Then it can be verified that the lattice



is in fact the \mathbf{Compl} -fibre of $\{0, 1, 2\}$.

2.5 Definition. Let $X \times Y$ be a product in \mathcal{X} , and suppose that for each $i \in I$ the diagram



is a pullback in \mathcal{X} , where $p_Y : X \times Y \rightarrow Y$ is the projection onto Y . We say that a theory $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$ sends these pullbacks into a *product covering family* of commutative diagrams

$$\begin{array}{ccc}
 TP_i & \xrightarrow{T\bar{p}_i} & TY_i \\
 T\bar{f}_i \downarrow & & \downarrow Tf_i \\
 T(X \times Y) & \xrightarrow{Tp_Y} & TY
 \end{array}$$

provided that for each $\alpha_i \in TY_i$ ($i \in I$), and each $\beta \in TX$, there exists $\gamma_i \in TP_i$ with $T\bar{p}_i(\gamma_i) \leq \alpha_i$, and $\bigvee_I T\bar{f}_i(\gamma_i) = \beta \otimes \bigvee_I Tf_i(\alpha_i)$.

2.6 Theorem. *For a topological category (\mathcal{A}, U) over \mathcal{X} , the following conditions are equivalent :*

(1) (\mathcal{A}, U) is cartesian closed;

(2) (\mathcal{A}, U) is concretely isomorphic to $\text{Mod}(T)$ for some theory $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$ sending the pointwise pullback of any regular sink along a projection into a product covering family of diagrams.

Proof. (1) \Rightarrow (2) : Without loss of generality, consider a cartesian closed topological category of the form $\text{Mod}(T)$, $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$. By 2.2 above, regular sinks in $\text{Mod}(T)$ are finitely productive, equivalently, pointwise pullbacks of regular sinks along projections are regular. Consider the pointwise pullback in \mathcal{X} of a regular sink $(f_i : Y_i \rightarrow Y)_I$ along a projection $p_Y : X \times Y \rightarrow Y$, which is of the following form

$$\begin{array}{ccc}
 X \times Y_i & \xrightarrow{\bar{p}_i} & Y_i \\
 id_X \times f_i \downarrow & & \downarrow f_i \\
 X \times Y & \xrightarrow{p_Y} & Y
 \end{array}$$

where each $\bar{p}_i : X \times Y_i \rightarrow Y_i$ is a projection. Let $(\alpha_i)_I$ be a family with $\alpha_i \in TY_i$ for each $i \in I$, and let $\beta \in TX$. The sink $(f_i : (Y_i, \alpha_i) \rightarrow (Y, \bigvee_I Tf_i(\alpha_i)))_I$ is regular

in $\text{Mod}(T)$, hence by the cartesian closedness of $\text{Mod}(T)$, the sink $(id_X \times f_i : (X \times Y_i, \beta \otimes \alpha_i) \rightarrow (X \times Y, \beta \otimes \bigvee_I T f_i(\alpha_i)))_I$ is final in $\text{Mod}(T)$, i.e., $\beta \otimes \bigvee_I (\alpha_i) = \bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i)$; in addition, it is clear that $T\bar{p}_i(\beta \otimes \alpha_i) \leq \alpha_i$ for each $i \in I$. Hence the image under T of the above family of diagrams is product covering.

(2) \Rightarrow (1) : It is sufficient to show that regular sinks in $\text{Mod}(T)$ are finitely productive, equivalently, that regular sinks in $\text{Mod}(T)$ are stable under pullbacks along projections. So, consider a regular sink $(f_i : (Y_i, \alpha_i) \rightarrow (Y, \alpha))_I$ in $\text{Mod}(T)$, and let (X, β) be any T -model. Take the pointwise pullback in $\text{Mod}(T)$ of $(f_i)_I$ along the projection $p_Y : (X \times Y, \alpha \otimes \beta) \rightarrow (Y, \alpha)$, which is of the following form :

$$\begin{array}{ccc}
 (X \times Y_i, \beta \otimes \alpha_i) & \xrightarrow{\bar{p}_i} & (Y_i, \alpha_i) \\
 id_X \times f_i \downarrow & & \downarrow f_i \\
 (X \times Y, \beta \otimes \alpha) & \xrightarrow{p_Y} & (Y, \alpha)
 \end{array}$$

Note that since $(f_i : (Y_i, \alpha_i) \rightarrow (Y, \alpha))_I$ is regular in $\text{Mod}(T)$, the underlying sink in \mathcal{X} is regular, and $\alpha = \bigvee_I T f_i(\alpha_i)$. By the cartesian closedness of \mathcal{X} , the sink $(id_Y \times f_i : X \times Y_i \rightarrow X \times Y)_I$ is regular in \mathcal{X} , so it remains to show that $\beta \otimes \bigvee_I T f_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i)$. Since T sends the pointwise pullback of the regular sink $(f_i : Y_i \rightarrow Y)_I$ along p_Y into a product covering family, it follows that for each $i \in I$ there exists $\gamma_i \in T(X \times Y_i)$ such that $T\bar{p}_i(\gamma_i) \leq \alpha_i$ and $\beta \otimes \bigvee_I T f_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\gamma_i)$. But, it is clear that for each $i \in I$, $\gamma_i \leq \beta \otimes \alpha_i$, hence we have $\bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i) \leq \beta \otimes \bigvee_I T f_i(\alpha_i) = \bigvee_I T(id_X \times f_i)(\gamma_i) \leq \bigvee_I T(id_X \times f_i)(\beta \otimes \alpha_i)$, i.e., the sink $(id_X \times f_i : (X \times Y_i, \beta \otimes \alpha_i) \rightarrow (X \times Y, \beta \otimes \bigvee_I T f_i(\alpha_i)))_I$ is regular in $\text{Mod}(T)$. \square

In [8], a *concretely cartesian closed topological category* is defined to be a cartesian closed topological category (\mathcal{A}, U) for which the forgetful functor U preserves the cartesian structure, i.e., for $A, B \in \mathcal{A}$, $U(B^A) = UB^{UA}$ and $U(ev : A \times B^A \rightarrow B) = ev : UA \times UB^{UA} \rightarrow UB$. Concretely cartesian closed topological categories have been characterised as follows :

2.7 Theorem. ([3]) *For a topological category (\mathcal{A}, U) over \mathcal{X} , the following are equivalent :*

(1) (\mathcal{A}, U) is concretely cartesian closed;

(2) final sinks in \mathcal{A} are finitely productive. \square

Note that even for concretely cartesian closed topological categories, fibres need not be frames. The category **Compl** given in 2.4 is a cartesian closed topological c-category (that is, a topological category in which every one-element set has a trivial fibre), and hence has a concretely cartesian closed topological hull (cf. [8]). It can be checked, using τ_1, τ_2 and τ_3 of 2.4, that the fibre of the set $\{0,1,2\}$ in the concretely cartesian closed topological hull is not a frame.

2.8 Theorem. *Concretely cartesian closed topological categories (\mathcal{A}, U) over \mathcal{X} are precisely those categories which are concretely isomorphic to $\text{Mod}(T)$ for some theory $T : \mathcal{X} \rightarrow \text{CSLatt}$ sending the pointwise pullback of any sink along a projection into a product covering family of diagrams.*

Proof. In view of 2.7, we now consider finite products of arbitrary final sinks instead of regular sinks. Hence the proof of 2.6 may be applied, with “regular sink in \mathcal{X} ” replaced by “arbitrary sink in \mathcal{X} ”, and “regular sink in $\text{Mod}(T)$ ” by “final sink in $\text{Mod}(T)$ ”. □

3 Universally topological categories

For the purposes of this section we assume that any given base category \mathcal{X} is finitely complete.

3.1 Definition. ([3])

- (1) Let (\mathcal{A}, U) be topological over \mathcal{X} . Final sinks in \mathcal{A} are said to be *universal* provided that for each sink $(a_i : A_i \rightarrow A)_I$ in \mathcal{A} and each \mathcal{A} -morphism $g : B \rightarrow A$, the sink $(b_i : B_i \rightarrow B)_I$, obtained by taking pointwise pullbacks of $(a_i)_I$ along g , is final.
- (2) A topological category with universal final sinks is called *universally topological*.

3.2 Examples. (1) A concrete category (\mathcal{A}, U) over the terminal category \odot is universally topological iff it is a frame.

- (2) **Rel**, the category of binary relations, is universally topological.
- (3) For a functor $F : \mathcal{X} \rightarrow \text{Set}$, let $S(F)$ denote the category with objects pairs (X, α) where X is an \mathcal{X} -object and $\alpha \in FX$, and morphisms $f : (X, \alpha) \rightarrow (Y, \beta)$ those \mathcal{X} -morphisms for which $Ff(\alpha) \subset \beta$. A category of form $S(F)$ is universally topological iff F sends pullbacks into weak pullbacks ([7]).

3.3 Proposition. *Let (\mathcal{A}, U) be concrete over \mathcal{X} . If (\mathcal{A}, U) is universally topological, then each \mathcal{A} -fibre is a frame.*

Proof. Given an \mathcal{X} -object X , let $B \in U^{-1}[X]$, and suppose that $(A_i)_I$ is a family in $U^{-1}[X]$. Recall that $\bigvee_I A_i$ is given by the final lift of the U -sink $(id_X : UA_i \rightarrow X)_I$, so there exists a final identity-carried \mathcal{A} -sink $(a_i : A_i \rightarrow \bigvee_I A_i)_I$. Note also that $B \wedge \bigvee_I A_i$ is given by the initial structure on X with respect to the U -source $(id_X : X \rightarrow UB, id_X : X \rightarrow U(\bigvee_I A_i))$. Since $B \wedge \bigvee_I A_i \leq \bigvee_I A_i$ in $U^{-1}[X]$, there is an \mathcal{A} -morphism $b : B \wedge \bigvee_I A_i \rightarrow \bigvee_I A_i$ such that $U(b) = id_X$. Now, for each $i \in I$, take the pullback of a_i along b :

$$\begin{array}{ccc}
 B \wedge A_i & \xrightarrow{\bar{b}_i} & A_i \\
 \bar{a}_i \downarrow & & \downarrow a_i \\
 B \wedge \bigvee_I A_i & \xrightarrow{b} & \bigvee_I A_i
 \end{array}$$

Each a_i is identity-carried, and b is also identity-carried, hence for each $i \in I$, $U(\bar{a}_i) = U(\bar{b}_i) = id_X$. Since (\mathcal{A}, U) is topological, the pullback of each a_i along b is given by the initial \mathcal{A} -structure with respect to the U -source $(id_X : X \rightarrow UA_i, id_X : X \rightarrow U(B \wedge \bigvee_I A_i))$. Since \bar{a}_i and \bar{b}_i are both identity-carried, this initial structure is $(B \wedge \bigvee_I A_i) \wedge A_i = B \wedge A_i$. Now, since $(a_i)_I$ is final, and (\mathcal{A}, U) is universally topological, $(\bar{a}_i : B \wedge A_i \rightarrow B \wedge \bigvee_I A_i)_I$ is final, i.e., $B \wedge \bigvee_I A_i = \bigvee_I (B \wedge A_i)$. \square

From 3.3 above it follows that if (\mathcal{A}, U) is universally topological, then it is concretely isomorphic over \mathcal{X} to $\text{Mod}(T)$ for some frame-valued theory T . Given such a theory T , one may ask, in addition, whether for each morphism f in \mathcal{X} , Tf preserves finite infima. The negative answer is obtained by looking at **Rel** : recall that **Rel** is concretely isomorphic to the theory R defined in 2.3 (2). It is easy to see that in general, for a map $f : X \rightarrow Y$ and $\rho_1, \rho_2 \in RX$, $(f \times f)[\rho_1 \cap \rho_2] \neq (f \times f)[\rho_1] \cap (f \times f)[\rho_2]$.

We know that universally topological categories are, up to concrete isomorphism, categories of models corresponding to frame-valued theories. Our goal is to determine which such theories characterise these categories. Some additional terminology is required :

3.4 Definition. Let $f : L \rightarrow M$ be a morphism in **CSLatt**. Then

- (1) f is said to *preserve downsets* (alternatively, f is called *downset-preserving*) iff for each $a \in L$, $f(\downarrow a) = \downarrow f(a)$ (i.e., for each $a \in L, b \in M, b \leq f(a) \Rightarrow b = f(c)$ for some $c \in L$ such that $c \leq a$).
- (2) f is called *cover-reflecting* iff for each $a \in L$ and each family $(b_i)_I$ in M , $f(a) \leq \bigvee_I b_i \Rightarrow a \leq \bigvee_I c_i$ for some family $(c_i)_I$ in L such that for each $i \in I$, $f(c_i) \leq b_i$.

3.5 Definition. A commutative diagram

$$\begin{array}{ccc}
 L & \xrightarrow{\bar{g}} & M \\
 \bar{f} \downarrow & & \downarrow f \\
 N & \xrightarrow{g} & K
 \end{array}$$

in **Set** is called a *covering diagram* provided for each $a \in M$ and $b \in N$ with $f(a) = g(b)$ there exists an element $c \in L$ with $\bar{g}(c) = a$ and $\bar{f}(c) = b$. Such an element c is said to *cover* the pair (b, a) .

3.6 Definition. A family of diagrams (over a fixed $g : N \rightarrow K$)

$$\begin{array}{ccc}
 L_i & \xrightarrow{\bar{g}_i} & M_i \\
 \bar{f}_i \downarrow & & \downarrow f_i \\
 N & \xrightarrow{g} & K
 \end{array}$$

in **CSLatt** is called *order-covering* provided it satisfies the following condition : for every family $(a_i)_I$ with $a_i \in M_i$ for each $i \in I$, and $b \in N$ with $g(b) \leq \bigvee_I f_i(a_i)$, there exists a family $(c_i)_I$, with $c_i \in L_i$ for each $i \in I$, such that $b = \bigvee_I \bar{f}_i(c_i)$ and $\bar{g}_i(c_i) \leq a_i$ for every $i \in I$.

Denote by **Frm** the full subcategory of **CSLatt** consisting of all frames and supremum-preserving maps.

3.7 Theorem. For a concrete category (\mathcal{A}, U) over \mathcal{X} , the following conditions are equivalent :

- (1) (\mathcal{A}, U) is universally topological;
- (2) (\mathcal{A}, U) is concretely isomorphic to $\text{Mod}(T)$ for some theory $T : \mathcal{X} \rightarrow \mathbf{Frm}$ sending morphisms into downset-preserving, cover-reflecting maps and pullbacks into covering diagrams;

(3) (\mathcal{A}, U) is concretely isomorphic to $\text{Mod}(T)$ for some theory $T : \mathcal{X} \rightarrow \mathbf{Frm}$ which sends the pointwise pullback of any sink in \mathcal{X} into an order-covering family of diagrams.

Proof. (1) \Rightarrow (3) : Without loss of generality we may consider a universally topological category of the form $\text{Mod}(T)$ for some $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$. By 3.3 above it follows that T is in fact frame-valued. Let $(f_i : X_i \rightarrow X)_I$ be an arbitrary sink in \mathcal{X} , and consider the pointwise pullback $(\bar{f}_i : Y_i \rightarrow Y)_I$ of $(f_i)_I$ along any $g : Y \rightarrow X$ in \mathcal{X} . For each $i \in I$, let $\alpha_i \in TX_i$, and let $\beta \in TY$ with $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$. So $(f_i : (X_i, \alpha_i) \rightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ is final in $\text{Mod}(T)$ and $g : (Y, \beta) \rightarrow (X, \bigvee_I Tf_i(\alpha_i))$ is a T -morphism. Therefore, for the pointwise pullback

$$\begin{array}{ccc} (Y_i, \sigma_i) & \xrightarrow{\bar{g}_i} & (X_i, \alpha_i) \\ \bar{f}_i \downarrow & & \downarrow f_i \\ (Y, \beta) & \xrightarrow{g} & (X, \bigvee_I Tf_i(\alpha_i)) \end{array}$$

in $\text{Mod}(T)$, the sink $(\bar{f}_i : (Y_i, \sigma_i) \rightarrow (Y, \beta))_I$ is final in $\text{Mod}(T)$ by the universal topologicity of $\text{Mod}(T)$, so $\beta = \bigvee_I Tf_i(\sigma_i)$ and $T\bar{g}_i(\sigma_i) \leq \alpha_i$ for all $i \in I$, as required.

(3) \Rightarrow (2) : Let $T : \mathcal{X} \rightarrow \mathbf{Frm}$ be a theory which satisfies the condition in (3), and let $f : X \rightarrow Y$ be a morphism in \mathcal{X} . We first show that $Tf : TX \rightarrow TY$ preserves downsets : let $\alpha \in TX, \beta \in TY$ with $\beta \leq Tf(\alpha)$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{id_Y} & Y \end{array}$$

is a pullback in \mathcal{X} . Since the image of the above diagram under T is order-covering, and $\beta \leq Tf(\alpha)$, there exists $\sigma \in TX$ with $\beta = Tf(\sigma)$ and $\sigma \leq \alpha$, equivalently, $Tf(\downarrow \alpha) = \downarrow Tf(\alpha)$. Further, suppose that $(\alpha_i)_I$ is a family in TY , and let $\beta \in TX$ with $Tf(\beta) \leq \bigvee_I \alpha_i$. For the identity sink $(id_Y : Y \rightarrow Y)_I$, take the pointwise pullback in \mathcal{X} along f :

$$\begin{array}{ccc}
 X & \xrightarrow{(f)_i} & Y \\
 (id_X)_i \downarrow & & \downarrow (id_Y)_i \\
 X & \xrightarrow{f} & Y
 \end{array}$$

The image under T of the above family is an order-covering family of diagrams, hence, since $Tf(\beta) \leq \bigvee_I \alpha_i$, there exists a family $(\sigma_i)_I$ in TX such that $\beta = \bigvee_I \sigma_i$ and, for each $i \in I$, $Tf(\sigma_i) \leq \alpha_i$. So, Tf is also cover-reflecting. Finally, to show that T sends pullbacks into covering diagrams, consider a pullback

$$\begin{array}{ccc}
 P & \xrightarrow{\bar{g}} & X \\
 \bar{f} \downarrow & & \downarrow f \\
 Y & \xrightarrow{g} & Z
 \end{array}$$

in \mathcal{X} . Let $\alpha \in TX, \beta \in TY$ such that $Tf(\alpha) = Tg(\beta)$. Since the image under T of the above diagram is order-covering (over Tg), and $Tf(\alpha) = Tg(\beta)$, there exists $\sigma \in TP$ with $\beta = T\bar{f}(\sigma)$ and $T\bar{g}(\sigma) \leq \alpha$. But the image of the above diagram is also order-covering over Tf , hence there exists $\gamma \in TP$ with $\alpha = T\bar{g}(\gamma)$ and $T\bar{f}(\gamma) \leq \beta$. Now, $T\bar{f}(\gamma \vee \sigma) = T\bar{f}(\gamma) \vee T\bar{f}(\sigma)$ (since $T\bar{f}$ preserves suprema), and $T\bar{f}(\gamma) \vee T\bar{f}(\sigma) = \beta$, since $T\bar{f}(\gamma) \leq \beta = T\bar{f}(\sigma)$. By analogous reasoning, we obtain $T\bar{g}(\gamma \vee \sigma) = \alpha$. So the structure $\gamma \vee \sigma$ covers the pair (β, α) .

(2) \Rightarrow (1) : It is sufficient to prove that $\text{Mod}(T)$ is universally topological. We begin by showing that pullbacks of final maps in $\text{Mod}(T)$ are final : given a final T -morphism $f : (X, \alpha) \rightarrow (Z, Tf(\alpha))$ and a T -morphism $g : (Y, \beta) \rightarrow (Z, Tf(\alpha))$, consider the pullback in $\text{Mod}(T)$ of f along g :

$$\begin{array}{ccc}
 (P, \sigma) & \xrightarrow{\bar{g}} & (X, \alpha) \\
 \bar{f} \downarrow & & \downarrow f \\
 (Y, \beta) & \xrightarrow{g} & (Z, Tf(\alpha))
 \end{array}$$

(In the above diagram, $\sigma = \bigvee \{ \mu \in TP \mid T\bar{f}(\mu) \leq \beta, T\bar{g}(\mu) \leq \alpha \}$.) We have $Tg(\beta) \leq Tf(\alpha)$, hence, since Tf preserves downsets, $Tg(\beta) = Tf(\gamma)$ for some $\gamma \in TX$ with $\gamma \leq \alpha$. Since the diagram

$$\begin{array}{ccc}
 TP & \xrightarrow{T\bar{g}} & TX \\
 T\bar{f} \downarrow & & \downarrow Tf \\
 TY & \xrightarrow{Tg} & TZ
 \end{array}$$

is a covering diagram, the pair (β, γ) can be covered by some $\delta \in TP$, i.e., $T\bar{g}(\delta) = \gamma$ and $T\bar{f}(\delta) = \beta$ for some $\delta \in TP$. But clearly $\delta \leq \sigma$ since σ is the initial structure on P with respect to the structured source $(\bar{g} : P \rightarrow (X, \alpha), \bar{f} : P \rightarrow (Y, \beta))$, hence $\beta = T\bar{f}(\delta) \leq T\bar{f}(\sigma) \leq \beta$, i.e., $\beta = T\bar{g}(\sigma)$, and $\bar{f} : (P, \sigma) \rightarrow (Y, \beta)$ is final, as required. Now let $(f_i : (X_i, \alpha_i) \rightarrow (X, \alpha))_I$ be final in $\text{Mod}(T)$ (i.e., $\alpha = \bigvee_I Tf_i(\alpha_i)$) and suppose $g : (Y, \beta) \rightarrow (X, \alpha)$ is a T -morphism. Taking pointwise pullbacks in $\text{Mod}(T)$,

$$\begin{array}{ccc}
 (Y_i, \sigma_i) & \xrightarrow{\bar{g}_i} & (X_i, \alpha_i) \\
 \bar{f}_i \downarrow & & \downarrow f_i \\
 (Y, \beta) & \xrightarrow{g} & (X, \alpha)
 \end{array}$$

we must show that $(\bar{f}_i : (Y_i, \sigma_i) \rightarrow (Y, \beta))_I$ is final, i.e., that $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$. Since $T\bar{f}_i(\sigma_i) \leq \beta$ for each $i \in I$, $\bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$. Since Tg is cover-reflecting and $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$, there exists a family $(\beta_i)_I$ in TY such that $\beta \leq \bigvee_I \beta_i$ and for each $i \in I$, $Tg(\beta_i) \leq Tf_i(\alpha_i)$. By the frame law, $\beta = \beta \wedge \bigvee_I \beta_i = \bigvee_I (\beta \wedge \beta_i)$. For each $i \in I$, let $\bar{f}_i : (Y_i, \gamma_i) \rightarrow (Y, \beta \wedge \beta_i)$ be the pullback in $\text{Mod}(T)$ of $f_i : (X_i, \alpha_i) \rightarrow (X, Tf_i(\alpha_i))$ along $g : (Y, \beta \wedge \beta_i) \rightarrow (X, Tf_i(\alpha_i))$. We have already shown that final morphisms in $\text{Mod}(T)$ are universal, hence for each $i \in I$, $T\bar{f}_i(\gamma_i) = \beta \wedge \beta_i$, and so $\beta = \bigvee_I T\bar{f}_i(\gamma_i)$. But, for each $i \in I$, $\gamma_i \leq \sigma_i$ (since $\gamma_i = \bigvee_I \{ \mu \in TY_i \mid T\bar{f}_i(\mu) \leq \beta \wedge \beta_i, T\bar{g}_i \leq \alpha_i \}$) and $T\bar{f}_i(\sigma_i) \leq \beta$, so $\beta = \bigvee_I T\bar{f}_i(\gamma_i) \leq \bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$, hence $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$. \square

3.8 Remark. In fact, the above theorem extends the well-known result in [7] which says that a category of form $S(F)$ (see 3.3 (3)) is universally topological iff F

sends pullbacks into weak pullbacks, since one easily sees that in **Set** the covering diagrams are exactly the weak pullbacks.

3.9 Remark. In the discussion following 3.3, **Rel** is presented as an example of a universally topological $\text{Mod}(T)$ for which the associated theory T does not send every \mathcal{X} -morphism $f : X \rightarrow Y$ to a map Tf preserving finite infima (i.e., T is not a theory into the category of frames and *frame* homomorphisms). Consider a universally topological $\text{Mod}(T)$ for which the associated T sends every \mathcal{X} -morphism to a map which preserves finite infima : given any $f : X \rightarrow Y$ in \mathcal{X} , it follows that Tf preserves indiscrete structures (i.e. top elements) and this, together with the fact that Tf preserves downsets, implies that Tf is surjective. Moreover, it can be shown that the image under T of any monomorphism in \mathcal{X} is injective : given a monomorphism $m : X \rightarrow Y$ in \mathcal{X} and $\alpha, \beta \in TX$ with $Tm(\alpha) = Tm(\beta)$, the following diagram is a pullback in $\text{Mod}(T)$:

$$\begin{array}{ccc}
 (X, \alpha \wedge \beta) & \xrightarrow{id_X} & (X, \alpha) \\
 id_X \downarrow & & \downarrow m \\
 (X, \beta) & \xrightarrow{m} & (X, Tm(\alpha) = Tm(\beta))
 \end{array}$$

It follows that $\beta = \alpha \wedge \beta = \alpha$, since $m : (X, \alpha) \rightarrow (Y, Tm(\beta))$ and $m : (X, \alpha) \rightarrow (Y, Tm(\alpha))$ are both final T -morphisms. Hence, for each monomorphism m in \mathcal{X} , Tm is bijective, so the structure of those universally topological $\text{Mod}(T)$ for which every Tf preserves finite infima can be trivial, for example, for each such T defined on **Set**, $T(\emptyset)$ is isomorphic to TX for every set X .

4 Topological quasitopoi

4.1 Definition. ([3]) A *quasitopos* is a category \mathcal{X} which is cartesian closed, has finite limits and colimits, and in which strong monomorphisms are representable.

In this section we assume that any given base category \mathcal{X} is a quasitopos with regular sink factorisations. We consider those topological categories over \mathcal{X} which are quasitopoi. Analogous to 3.1, regular sinks in a topological category (\mathcal{A}, U) are said to be *universal* if the pointwise pullback of any regular sink in \mathcal{A} along an arbitrary \mathcal{A} -morphism is regular.

4.2 Theorem. ([3]) *For a topological category (\mathcal{A}, U) over \mathcal{X} , the following conditions are equivalent :*

(1) (\mathcal{A}, U) is a quasitopos;

(2) regular sinks in \mathcal{A} are universal. □

4.3 Examples. ([3]) (1) **Conv**, the category of convergence spaces, is a quasitopos over **Set**.

(2) The category **Rere** of reflexive relations is a quasitopos over **Set**.

4.4 Proposition. *If a topological category (\mathcal{A}, U) is a quasitopos, then every \mathcal{A} -fibre is a frame.*

Proof. Analogous to the corresponding proof for 3.3, since identity-carried sinks are trivially regular. □

4.5 Theorem. *For a topological category (\mathcal{A}, U) over \mathcal{X} , the following conditions are equivalent :*

(1) (\mathcal{A}, U) is a quasitopos;

(2) (\mathcal{A}, U) is concretely isomorphic to $\text{Mod}(T)$ for some theory $T : \mathcal{X} \rightarrow \mathbf{Frm}$ which sends the pointwise pullback of any regular sink into an order-covering family of diagrams.

Proof. (1) \Rightarrow (2) : Without loss of generality, consider a quasitopos of form $\text{Mod}(T)$, $T : \mathcal{X} \rightarrow \mathbf{CSLatt}$. By 4.4 above, T is frame-valued. Let $(f_i : X_i \rightarrow X)_I$ be a regular sink in \mathcal{X} , and let $(\bar{f}_i : P_i \rightarrow Y)_I$ be the pointwise pullback of $(f_i)_I$ along a morphism $g : Y \rightarrow X$ in \mathcal{X} . For each $i \in I$, let $\alpha_i \in TX_i$ and let $\beta \in TY$ with $Tg(\beta) \leq \bigvee_I Tf_i(\alpha_i)$. So, $(f_i : (X_i, \alpha_i) \rightarrow (X, \bigvee_I Tf_i(\alpha_i)))_I$ is regular in $\text{Mod}(T)$ and $g : (Y, \beta) \rightarrow (X, \bigvee_I Tf_i(\alpha_i))$ is a T -morphism. Therefore, for the pointwise pullback

$$\begin{array}{ccc}
 (P_i, \sigma_i) & \xrightarrow{\bar{g}_i} & (X_i, \alpha_i) \\
 \bar{f}_i \downarrow & & \downarrow f_i \\
 (Y, \beta) & \xrightarrow{g} & (X, \bigvee_I Tf_i(\alpha_i))
 \end{array}$$

in $\text{Mod}(T)$, the sink $(\bar{f}_i : (P_i, \sigma_i) \rightarrow (Y, \beta))_I$ is regular in $\text{Mod}(T)$ by 4.2 above, so $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$, and $T\bar{g}_i(\sigma_i) \leq \alpha_i$ for all $i \in I$, as required.

(2) \Rightarrow (1) : We must show that regular sinks in $\text{Mod}(T)$ are universal. So, let $(f_i : (X_i, \alpha_i) \rightarrow (X, \bigvee_I T f_i(\alpha_i)))_I$ be a regular sink, and take the pointwise pullback in $\text{Mod}(T)$ of $(f_i)_I$ along an arbitrary T -morphism $g : (Y, \beta) \rightarrow (X, \bigvee_I T f_i(\alpha_i))$, as in the previous diagram. The induced sink $(\bar{f}_i : P_i \rightarrow Y)_I$ is regular in \mathcal{X} (since \mathcal{X} is a quasitopos), so it is sufficient to show that $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$. Note that the family of diagrams

$$\begin{array}{ccc}
 TP_i & \xrightarrow{T\bar{g}_i} & TX_i \\
 T\bar{f}_i \downarrow & & \downarrow Tf_i \\
 TY & \xrightarrow{Tg} & TZ
 \end{array}$$

is order-covering, and since $Tg(\beta) \leq \bigvee_I T f_i(\alpha_i)$, for every $i \in I$ there exists $\gamma_i \in TP_i$ with $\beta = \bigvee_I T\bar{f}_i(\gamma_i)$ and $T\bar{g}_i(\gamma_i) \leq \alpha_i$. Hence, for each $i \in I$, $T\bar{f}_i(\gamma_i) \leq \beta$, so we have $\gamma_i \leq \sigma_i$ for all $i \in I$ since each $\sigma_i = \bigvee \{ \mu \in TP_i \mid T\bar{f}_i(\mu) \leq \beta, T\bar{g}_i(\mu) \leq \alpha_i \}$. Clearly $\bigvee_I T\bar{f}_i(\sigma_i) \leq \beta$, hence, we have $\bigvee_I T\bar{f}_i(\sigma_i) \leq \beta = \bigvee_I T\bar{f}_i(\gamma_i) \leq \bigvee_I T\bar{f}_i(\sigma_i)$, i.e., $\beta = \bigvee_I T\bar{f}_i(\sigma_i)$, and so the sink $(\bar{f}_i : (P_i, \sigma_i) \rightarrow (Y, \beta))_I$ is regular in $\text{Mod}(T)$. \square

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