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SHEAVES ON A QUANTALE

by Francis BORCEUX and Rosanna CRUCIANI

Résumé. La notion de faisceau sur un locale est bien connue et donne lieu à diverses présentations équivalentes: préfaisceaux sur le locale possédant une propriété de recollement, ensembles munis d'une égalité à valeurs dans le locale, locales étalés sur le locale de base. La notion de *quantale* est apparue ces dernières années en se présentant comme une généralisation de la notion de locale, particulièrement bien adaptée aux besoins de l'algèbre non commutative. Diverses notions de faisceau sur un quantale ont déjà été proposées, sans pour autant pouvoir généraliser les situations connues dans le cas d'un locale. Dans le présent travail, nous affinons ces tentatives et proposons, comme dans le cas des locales, trois notions donnant lieu à des catégories équivalentes: faisceau sur un quantale, ensemble muni d'une égalité à valeurs dans un quantale et quantale étalé au-dessus d'un quantale de base.

It is a common practice, when studying some rings or algebras, to define a topological space called the "spectrum" of the algebraic gadget and recapturing some essential properties of it. Most often, the original ring or algebra can be seen as the set of global sections of some "good" sheaf on the spectrum. It is a matter of fact that such processes are often less adequate in the non commutative case. A general explanation can be given to this: the open subsets of the spectrum correspond generally to some suitable ideals of the original algebraic gadget, the union of open subsets being related to the sum of ideals and the intersection of open subsets to the product of ideals. And the intersection of open subsets is by nature commutative, a property which is generally not shared by the multiplication of ideals in a non commutative context.

It is an obvious observation that when defining sheaves on a topological space, just the lattice of open subsets plays a role. This is a complete lattice where arbitrary joins distribute over finite meets, i.e. what has been called a locale (cf. [7]). C.J. Mulvey suggested that in the non-commutative case, the good notion of spectrum should be obtained from that of a locale by replacing the (of course commutative) binary meet operation with an arbitrary non commutative multiplication. This is what we call a quantale.

One of the most famous spectral constructions is that leading to the Gelfand duality for commutative $\mathbb{C}^*$-algebras. The lattice of open subsets of the spectrum is

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isomorphic to that of closed right ideals of the $\mathbb{C}^*$-algebra, the intersection of open subsets corresponding to the closure of the multiplication of ideals. Therefore the original definition of a (right) quantale has been designed with in mind the lattice of closed right ideals of a non commutative $\mathbb{C}^*$-algebra (cf. [4], [9]).

The notion of a sheaf on a locale $\mathcal{L}$ admits several equivalent presentations: a contravariant functor $F: \mathcal{L} \longrightarrow \mathcal{SET}$ with suitable glueing conditions; a locale $\mathcal{F} \longrightarrow \mathcal{L}$ étalé over $\mathcal{L}$ (i.e. locally isomorphic to $\mathcal{L}$); a set A provided with an equality $[\cdot = \cdot]: A \times A \longrightarrow \mathcal{L}$ with values in $\mathcal{L}$. Those various approaches have been generalized to the case of a quantale (cf [4], [2], [1]), but unfortunately the categories obtained in this way are no longer equivalent.

The definition of a right quantale $\mathcal{Q}$, used in the previous papers, was of course unsymmetric since the top element was a unit on the right, but not on the left. The second author made the observation that most of what had been done in those papers could be performed dropping as well the distributy on the left of arbitrary suprema over the multiplication. This slight weakening has the striking effect to force a lot of stability conditions for the new notion of a quantale $\mathcal{Q}$, yielding in particular the equivalence of the various categories of sheaves over $\mathcal{Q}$, quantales étalé over $\mathcal{Q}$ and sets provided with a $\mathcal{Q}$-equality. Moreover the intrinsic properties of this category induce the structure of a quantale on each poset of subobjects, the original quantale being recaptured as that of subobjects of the terminal object 1.

1 Quantales

Definition 1.1. *A right quantale is a complete lattice $(Q, \leq)$ provided with a binary multiplication $\&: Q \times Q \longrightarrow Q$ which preserves the partial order $\leq$ in each variable and satisfies the following axioms:*

(Q1) $a\&(b\&c) = (a\&b)\&c$

(Q2) $a\&a = a$

(Q3) $a\&1 = a$

(Q4) $a\& \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a\&b_i)$

where I is an arbitrary indexing set, a, b, c are elements of Q and 1 is the top element of Q.

We recall (cf. [4]) that the notion of a quantale has been specially designed to study the case of the quantale of closed right ideals of a $\mathbb{C}^*$-algebra, where the multiplication is then the closure of the algebraic multiplication of ideals. We recall also that a locale is a complete lattice where arbitrary joins distribute over binary meets; in other words, a locale is a right quantale where the meet operation is the multiplication. It is also proved in [4] that a locale is just a left-right quantale.

Our definition of a right quantale is slightly weaker than the one in [4], [1] or [2]: we do not require that arbitrary joins distribute on the left over the multiplication.

Nevertheless a right quantale has some amount of left distributivity, left unity and even commutativity: our first proposition indicates that all those properties hold as soon as the first factor is fixed.

Proposition 1.2. *In a right quantale, the following relations hold:*

$$\begin{aligned} a\&(1\&b) &= a\&b \\ (a\&\bigvee_{i\in I} b_i)\&c &= \bigvee_{i\in I}(a\&b_i\&c) \\ a\&b\&c &= a\&c\&b \end{aligned}$$

where the notations are as in 1.1.

Proof: The first statement is obvious by associativity. The last one follows immediately from the idempotency

$$a\&(b\&c) = a\&(b\&c)\&(b\&c) \leq a\&(1\&c)\&(b\&1) = a\&c\&b.$$

The second statement is then an immediate consequence of the right distributivity, via the third statement. ■

Next observe a first important stability condition:

Proposition 1.3. *If a is an element of a right quantale Q, the down-segment of a*

$$\downarrow a = \{b \in Q \mid b \leq a\}$$

is again a quantale, for the induced multiplication.

Proof: If $b \leq a$ and $c \leq a$, then $b\&c \leq a\&a = a$; moreover $b\&a \leq b\&1 = b$ while $b\&a \geq b\&b = b$. ■

Using a classical terminology for lattices of ideals, every element a of a right quantale is right-sided since $a\&1 = a$. When moreover $1\&a = a$, the element is said to be 2-sided. We shall denote by 2-Q the sub-poset of 2-sided elements of a quantale Q.

Proposition 1.4. *Given a right quantale Q and elements a, b of Q:*

1. *$\overline{a} = 1\&a$ is the smallest 2-sided element greater than a;*
2. *$a\&b = a \wedge \overline{b}$;*
3. *when b is 2-sided, $a\&b = a \wedge b$;*
4. *the 2-sided elements of Q constitute a locale where the meet operation coincides with multiplication in Q;*
5. *the locale of 2-sided elements of Q is stable in Q under arbitrary meets and arbitrary joins;*

6. *the mappings* $2\text{-}{\downarrow}\overline{a} \rightleftarrows 2\text{-}{\downarrow}a$ *applying* $b \leq \overline{a}$ *on* $a\&b$ *and* $c \leq a$ *on* $\overline{a}\&c$ *are isomorphisms of locales.*

Proof: Everything follows easily from the second statement. First $a\&b \leq a\&a = a$ and $a\&b \leq 1\&b = \overline{b}$; next

$$a \wedge \overline{b} = (a \wedge \overline{b})\&(a \wedge \overline{b}) \leq a\&\overline{b} = a\&b.$$

Observe that the first statement implies the existence of a left adjoint to the inclusion of $2\text{-}Q$ in Q, thus the stability of $2\text{-}Q$ under arbitaray meets. ∎

Since not all elements are 2-sided in general in a right quantale Q, we shall pay a special attention to the case where $b \leq a$, with b a 2-sided element of the quantale ${\downarrow}a$; we shall use the notation $b \ll a$ to indicate this fact and read this "b is 2-sided in a". Observe that for arbitrary elements a, b in Q, the relation $a\&b = b$ implies $b = a\&b \leq a\&1 = a$, so that $b \ll a$ if and only if $a\&b = b$. Obviously, the relation $\ll$ is a poset structure on Q.

One could also consider, in a right quantale Q, the relation $b \preccurlyeq a$, meaning that $b = b\&a$, thus $b \leq \overline{a}$. This is just a preorder on Q since, for example, $a \preccurlyeq \overline{a}$ and $\overline{a} \preccurlyeq a$ for every element a.

When necessary, we shall use the notations $(Q, \leq)$, $(Q, \ll)$ or $(Q, \preccurlyeq)$ to distinguish the three partial order or preorder structures on a right quantale Q. Observe that

Proposition 1.5. *Given a right quantale Q, the quotient poset associated with the preordered set $(Q, \preccurlyeq)$ is isomorphic to the locale $2\text{-}Q$.*

Proof: The quotient identifies a and b when $a \preccurlyeq b$ and $b \preccurlyeq a$; this is indeed equivalent to $\overline{a} = \overline{b}$. ∎

2 Q-sets

Throughout this section, Q is a fixed right quantale. The first way of defining a sheaf on Q is via generators and relations: a sheaf is a set (the set of generators) together with an equality which takes values in Q (the Q-equality of a and b is the biggest level where the generators a and b become equal).

Definition 2.1. *Let Q be a right quantale. A Q-set is a pair $(A, [\cdot,\cdot])$ where A is a set and $[\cdot,\cdot]\colon A \times A \longrightarrow Q$ is a mapping satisfying the following axioms:*

(S1) $\overline{[a = a']} = \overline{[a' = a]}$

(S2) $[a = a']\&[a' = a''] \leq [a = a'']$

where a, a', a'' are elements of A.

Most often, we shall just write A for the Q-set $(A, [\cdot,\cdot])$. For $a \in A$, the element $[a = a] \in Q$ should be thought as the biggest level of Q where a is defined. In this spirit, the element $\varepsilon(A) = \bigvee_{a\in A}[a = a]$ should be thought as the level where A is inhabited or in other words, the support of A.

Proposition 2.2. *Let Q be a right quantale and A a Q-set. The following relations hold*

$$\begin{aligned}[a=a]\&[a=a'] &= [a=a'] \\ [a=a']\&[a'=a'] &= [a=a']\end{aligned}$$

for elements a, a' in A.

Proof:

$$\begin{aligned}[a=a'] &= [a=a']\&[a=a'] = [a=a']\&\overline{[a=a']} \\ &= [a=a']\&\overline{[a'=a]} = [a=a']\&[a'=a] \leq [a=a]\end{aligned}$$

from which

$$[a=a'] = [a=a']\&[a=a'] \leq [a=a]\&[a=a']$$

and thus the equality. The second statement is analogous. ∎

In the same spirit as definition 2.1, a morphism of Q-sets $f: A \longrightarrow B$ consists in precising the biggest level where $f(a)$ becomes equal to b, for $a \in A$ and $b \in B$; axiom (M4) imposes the functionality of this relation and axiom (M5), the fact it is defined everywhere.

Definition 2.3. *Let Q be a right quantale and A, B Q-sets. A morphism of Q-sets $f: A \longrightarrow B$ is a mapping $f: A \times B \longrightarrow Q$ satisfying*

(M1) $\varepsilon(A) \leq \varepsilon(b)$

(M2) $[a=a']\&[fa=b] \leq [fa'=b]$

(M3) $[fa=b]\&[b=b'] \leq [fa=b']$

(M4) $[b=b]\&[fa=b]\&[fa=b']\&[b'=b'] \leq [b=b']$

(M5) $\bigvee_{b\in B}[fa=b] = [a=a]$

(M6) $[fa=b] \leq \overline{[b=b]}$

for elements a, a' in A and b, b' in B.

Straightforward computations, analogous to those developed to prove proposition 2.2, yield the the following results:

Proposition 2.4. *Let Q be a right quantale and $f: A \longrightarrow B$ a morphism of Q-sets. The following relations hold:*

$$\begin{aligned}[a=a]\&[fa=b] &= [fa=b] \\ [fa=b]\&[b=b] &= [fa=b]\end{aligned}$$

for elements $a \in A$ and $b \in B$. ∎

Proposition 2.5. *Let Q be a right quantale. Given two morphims of Q-sets $f: A \longrightarrow B$ and $g: B \longrightarrow C$, the formula*

$$[(g \circ f)a = c] = \bigvee_{b \in B} [fa = b] \& [gb = c]$$

defines a composite morphism of Q-sets $g \circ f: A \longrightarrow C$, where $a \in A$, $b \in B$, $c \in C$. For this composition, the Q-sets and their morphisms become a category where the identity on the Q-set A is just the Q-equality on A. ∎

We write Q-$\mathcal{SET}$ for the category of Q-sets on the right quantale Q.

3 Sub-Q-sets

Again Q is a fixed right quantale. Giving a sub-Q-set of a Q-set A is, intuitively, precising to what extent an element $a \in A$ belongs to the sub-Q-set.

Definition 3.1. *Let Q be a right quantale and A a Q-set. A sub$Q$-set of A is a mapping $s: A \longrightarrow Q$ satisfying*

(s1) $sa \leq \varepsilon(A)$

(s2) $sa \& [a = a'] \leq sa'$

(s3) $(\bigvee_{a \in A} sa) \& sa' \leq sa'$

(s4) $sa \leq sa \& [a = a]$

where a, a' are elements of A.

Again the element $\varepsilon(s) = \bigvee_{a \in A} sa$ appearing in (s3) can be thought as the extent to which the sub-Q-set s is inhabited or, in other words, the support of s. It is immediate from the axioms for a sub-Q-set that in fact equality holds in (s3) and (s4). We write $\mathcal{S}(A)$ for the set of sub-Q-sets of the Q-set A.

Proposition 3.2. *Let A be a Q-set on the right quantale Q. The set $\mathcal{S}(A)$ of sub-Q-sets of A becomes a quantale when equipped with the preorder*

$$s_1 \leq s_2 \Longleftrightarrow \forall a \in A \quad s_1 a \leq s_2 a$$

and the multiplication

$$(s_1 \& s_2)a = s_1 a \& s_2 a$$

where s_1, s_2 are sub-Q-sets of A and $a \in A$.

Proof: If $(s_i)_{i \in I}$ is a family of sub-Q-sets of A, one observes first that the formula

$$(\bigvee_{i \in I} s_i)a = (\bigvee_{i \in I} \varepsilon s_i) \& (\bigvee_{i \in I} s_i a)$$

where $a \in A$ defines the supremum of this family. The rest is straightforward computations. ∎

Corollary 3.3. *With the notations of the previous proposition*

1. $s_1 \wedge s_2$ *is given by* $(s_1 \wedge s_2)a = s_1(a) \wedge s_2(a)$;
2. *the top element of* $\mathcal{S}(A)$ *applies* $a \in A$ *on* $\varepsilon(A)\&[a = a]$;
3. *an element* s *of the quantale* $\mathcal{S}(A)$ *is 2-sided when for every* $a \in A$, $s(a)$ *is 2-sided in* $\varepsilon(A)$. ∎

Sub-$\mathcal{Q}$-sets are very easy to handle in computations. Let us now prove they correspond exactly to the usual notion of subobject in the category $\mathcal{Q}$-$\mathcal{SET}$.

Proposition 3.4. *Let* $f: A \longrightarrow B$ *be a morphism of* $\mathcal{Q}$*-sets with* $\mathcal{Q}$ *a right quantale.* f *is a monomorphims in the category* $\mathcal{Q}$-$\mathcal{SET}$ *when the relation*

$$[fa = b]\&[fa' = b] \leq [a = a']$$

holds for every elements a, a' *in* A *and* b *in* B.

Proof: If f is a monomorphism and a, a' are in A, put

$$q = \bigvee_{b \in B} [fa = b]\&[fa' = b].$$

Consider the $\mathcal{Q}$-set $(\downarrow q, \&)$ and the morphisms $f_a, f_{a'}: \downarrow q \rightrightarrows A$ defined by

$$[f_a x = a"] = x\&[a = a"] \qquad [f_{a'} x = a"] = x\&[a' = a"].$$

From $f \circ f_a = f \circ f_{a'}$ we deduce $f_a = f_{a'}$. Choosing $x = q$ and $a" = a'$ we get $q\&[a = a'] = q$ and therefore

$$[fa = b]\&[fa' = b] \leq q = q\&[a = a'] \leq [a = a']\&[a = a'] = [a = a'].$$

Conversely, if $g, h: C \rightrightarrows A$ are morphisms of $\mathcal{Q}$-sets such that $f \circ g = f \circ h$, for all a, a' in A, b in B and c in C

$$\begin{aligned}
[gc = a'] &= [gc = a']\&[a' = a'] \\
&= [gc = a]\& \bigvee_{b \in B} [fa' = b] \\
&\leq \bigvee_{b \in B} \bigvee_{a \in A} [gc = a]\&[fa = b]\&[fa' = b] \\
&= \bigvee_{b \in B} \bigvee_{a \in A} [hc = a]\&[fa = b]\&[fa' = b] \\
&\leq \bigvee_{a \in A} [hc = a]\&[a = a'] \\
&= [hc = a']
\end{aligned}$$

from which one concludes by symmetry. ∎

Proposition 3.5. *Let $\mathcal{Q}$ be a right quantale and A a $\mathcal{Q}$-set. The poset $\mathcal{S}(A)$ of sub-$\mathcal{Q}$-sets of A is isomorphic to the poset of subobjects of A in the category $\mathcal{Q}$-$\mathcal{SET}$.*

Proof: If $f\colon B \rightarrowtail A$ is a monomorphims in the category $\mathcal{Q}$-$\mathcal{SET}$, we define a sub-$\mathcal{Q}$-set Θf of A by the formula

$$(\Theta f)a = \varepsilon B \,\&\, \bigvee_{b\in B} [fb = a].$$

It is routine to observe that given another monomorphism $h\colon D \rightarrowtail B$, one has $\Theta(f \circ h) \leq \Theta f$; therefore Θ factors through the poset of subobjects and respects the poset structure.

If $f\colon B \rightarrowtail A$ and $g\colon C \rightarrowtail A$ are monomorphisms in $\mathcal{Q}$-$\mathcal{SET}$ such that $\Theta f = \Theta g$, the formulas

$$\begin{aligned}[] [\alpha b = c] &= \bigvee_{a\in A} [fb = a] \,\&\, [gc = a] \\ [\beta c = b] &= \bigvee_{a\in A} [gc = a] \,\&\, [fb = a] \end{aligned}$$

for $b \in B$ and $c \in C$ define two inverse isomorphisms $\alpha, \beta\colon B \rightleftarrows C$ in $\mathcal{Q}$-$\mathcal{SET}$, expressing the fact that f and g define the same subobject of A. Thus Θ is injective.

Θ is also surjective. Given a sub-$\mathcal{Q}$-set $s \in \mathcal{S}(A)$, the formulas

$$[a = a']_s = sa \,\&\, [a = a'] \qquad [f_s a = a'] = sa \,\&\, [a = a']$$

define a monomorphism in $\mathcal{Q}$-$\mathcal{SET}$

$$f_s\colon (A, [\cdot,\cdot]_s) \rightarrowtail (A, [\cdot,\cdot])$$

applied on s by Θ.

Finally, with the notations just defined, observe that if $s_1 \leq s_2$ are sub-$\mathcal{Q}$-sets of A, the formula

$$[ha = a'] = s_1 a \,\&\, [a = a'] \,\&\, s_2 a'$$

for a, a' in A, defines a morphism $h\colon A_{s_1} \longrightarrow A_{s_2}$ such that $f_{s_2} \circ h = f_{s_1}$. Therefore Θ reflects the poset structure. ■

4 Complete $\mathcal{Q}$-sets

Let us first observe the obvious result:

Proposition 4.1. *Let $\mathcal{Q}$ be a right quantale and $a \in A$ an element of a $\mathcal{Q}$-set A. The assignment*

$$A \longrightarrow \mathcal{Q} \;\; ; \;\; x \mapsto [a = x]$$

is a sub-$\mathcal{Q}$-set of A. ■

A sub-Q-set as in 4.1 satisfies clearly the next definition for being a "singleton":

Definition 4.2. *Let Q be a right quantale and A a Q-set. A singleton of A is a sub-Q-set s of A satisfying the additional requirement*

(s5) $sa \& sa' \leq \overline{[a = a']}$

for all elements a, a' of A.

Definition 4.3. *Let Q be a right quantale. A Q-set A is complete when every singleton on A has the form $[a = \cdot]$ for a unique element $a \in A$.*

The interest of complete Q-sets lies in the fact that morphisms to a complete Q-set can be represented by actual mappings ... and each Q-set is isomorphic to a complete one!

Proposition 4.4. *Let Q be a right quantale and A, B two Q-sets, with B complete. There exists a bijection between:*

1. *the morphisms of Q-sets $f\colon A \longrightarrow B$;*
2. *the actual mappings $\varphi\colon A \longrightarrow B$ satisfying the two conditions*

$$[a = a] = [\varphi a = \varphi a] \qquad [a = a'] \leq [\varphi a = \varphi a']$$

for all elements a, a' in A.

This bijection is compatible with the composition.

Proof: Given f and an element $a \in A$, $[fa = \cdot]$ is a singleton on B, thus is represented by a unique element of B which we choose as φa. Conversely given φ one defines f as the assignment

$$A \times B \longrightarrow Q \ ; \ (a, b) \mapsto [\varphi a = b]$$

where this last bracket is the Q-equality on B. ∎

Proposition 4.5. *Let Q be a right quantale. Every Q-set is isomorphic to a complete Q-set.*

Proof: For a given Q-set A, write $\widehat{A}$ for the set of singletons on A. $\widehat{A}$ becomes itself a Q-set when provided with the equality

$$[s = s'] = \bigvee_{a \in A} (sa \& sa')$$

for all singletons s, s'.

To prove this new Q-set $\widehat{A}$ is complete, choose a singleton $\sigma\colon \widehat{A} \longrightarrow Q$. For an element $a \in A$, put $sa = \bigvee_{s' \in \widehat{A}} (\sigma s' \& s'a)$. It is routine to check that s is a singleton on A, the unique one such that $\sigma = [s = \cdot]$.

It remains to observe that given $a \in A$ an $s \in \widehat{A}$ the relations

$$[fa = s] = [a = a] \& sa \qquad [gs = a] = sa$$

define two inverse isomorphisms $f: A \longrightarrow \widehat{A}$, $g: \widehat{A} \longrightarrow A$ in the category of Q-sets. ∎

5 Etale maps of quantales

The second way of defining a sheaf on a quantale Q is in terms of étale maps: a sheaf on Q is a quantale étalé over Q, i.e. locally isomorphic to Q. We define very naturally:

Definition 5.1. *A morphism of quantales $f: Q \longrightarrow Q'$ is a mapping preserving arbitrary joins, the multiplication and the top element.*

We recall that a mapping between two quantales is étale when it preserves arbitrary joins and is locally an isomorphism of quantales. The generalization to quantales requires some care about what "locally" means; in fact the local character has to be compatible with the 2-sided closure operation.

Definition 5.2. *Given an arbitrary mapping $f: Q \longrightarrow Q'$ between two right quantales, an element $a \in Q$ is f-small when the restriction*

$$f: \downarrow\overline{a} \longrightarrow \downarrow f(\overline{a})$$

is an isomorphism of quantales.

Definition 5.3. *A mapping $f: Q \longrightarrow Q'$ between two quantales is étale when*

1. *f preserves arbitrary joins;*
2. *f preserves the partial order $\ll$;*
3. *every element of Q is a join of f-small elements.*

Here is a useful equivalent condition:

Proposition 5.4. *A mapping $f: Q \longrightarrow Q'$ between quantales is étale if and only if*

1. *f preserves arbitrary joins;*
2. *f preserves the partial order $\ll$, thus in particular the 2-sidedness of elements;*
3. *there is a family $(a_i)_{i\in I}$ of 2-sided f-small elements of Q such that $\bigvee_{i\in I} a_i = 1$.*

Proof: When f is étale, write $1 = \bigvee_{i\in I} u_i$ with each u_i f-small. Each $a_i = \overline{u_i}$ is still f-small, with now a_i 2-sided and again $\bigvee_{i\in I} a_i = 1$. This is the required family of elements.

Conversely, with the notations of the statement, every element $b \in Q$ can be written as

$$b = b\&1 = b\& \bigvee_{i\in I} a_i = \bigvee_{i\in I}(b\&a_i) = \bigvee_{i\in I}(b \wedge a_i)$$

with each $b \wedge a_i$ f-small. ∎

Proposition 5.5. *Let $f: Q \longrightarrow Q'$ be an étale map of quantales and $a \in Q$, $b \in Q'$. Consider the mapping $f^*: Q' \longrightarrow Q$ defined by*

$$f^*(b) = \bigvee\{a \in Q \mid f(a) \leq b\}$$

1. *$f^*: (Q', \leq) \longrightarrow (Q, \leq)$ is right adjoint to $f: (Q, \leq) \longrightarrow (Q', \leq)$, i.e. f and f^* preserve the partial orders $\leq$ and $f(a) \leq b$ iff $a \leq f^*(b)$;*
2. *when a is f-small, the inverse to $f{\downarrow}\overline{a} \longrightarrow {\downarrow}f(\overline{a})$ applies b on $f^*(b)\&a$;*
3. *f^* preserves the partial order $\ll$, thus in particular the 2-sidedness of elements;*
4. *$f(\overline{a}) = f(1)\&f(a)$;*
5. *if $b \leq f(1)$, then $b = f(a)$ for some $a \in Q$.*

Proof: The two first statements are obvious. As a left adjoint, f^* preserves arbitrary meets, thus

$$f^*(b \wedge fa) \wedge a = f^*(b) \wedge f^*(a) \wedge a = f^*(b) \wedge a$$

Let now a run through the family $(a_i)_{i\in I}$ of proposition 5.4. For each 2-sided b, $f(a_i)\&b = f(a_i) \wedge b$ is 2-sided in ${\downarrow}a_i$; by the second statement and the previous formula $f^*(b) \wedge a_i$ is 2-sided in ${\downarrow}a_i$, thus in Q; therefore the supremum, which is $f^*(b)$, is 2-sided in Q. So f^* preserves already 2-sided elements; this result, applied at the level $f^*(b)$, yields easily the third statement.

Using again the family $(a_i)_{i\in I}$ of 5.4, when $a \leq a_i$ one has $\overline{a} = a_i\&a$ and thus $f(\overline{a}) = f(a)\&f(a_i)$. For an arbitrary a, one has $a = \bigvee_{i\in I}(a\&a_i)$ and applying the previous relation to each $a\&a_i$ yields the fourth statement, since f preserves arbitrary joins.

Next observe that for a 2-sided a, the previous relations yield

$$\begin{aligned} ff^*\overline{f(a)} &\leq \overline{ff^*\overline{f(a)}} = f(1)\&ff^*\overline{f(a)} \\ &\leq f(1)\&\overline{f(a)} = f(1)\&f(a) = f(\overline{a}) = f(a). \end{aligned}$$

The converse relation holds as well since $a \leq f^*f(a) \leq f^*\overline{f(a)}$. Using again the elements a_i of 5.4, for $b \leq f(a_i)$ one has $b = f(f^*(b)\&a_i)$. For an arbitrary b, the relation $ff^*\overline{f(a_i)} = f(a_i)$ implies easily $f^*(b \wedge f(a_i)) = f^*(b\&f(a_i))$ from which the last statement by computing the supremum on $i \in I$. ∎

Corollary 5.6. *The composite of two étale maps of right quantales is still étale. This makes the right quantales and their étale maps a category.*

Proof: Consider two composable étale maps of right quantales $f: Q \longrightarrow Q'$ and $g: Q' \longrightarrow Q''$. Given $a \in Q$, write it as $a = \bigvee_{i \in I} a_i$ with each a_i f-small. Write also each $f(a_i)$ as $f(a_i) = \bigvee_{j \in J} b_{ij}$ with each b_{ij} g-small. There is a unique $a_{ij} \leq a_i$ such that $f(a_{ij}) = b_{ij}$ and since $f(\overline{a_{ij}}) = f(1) \& f(a_{ij}) \leq \overline{f(a_{ij})} = \overline{b_{ij}}$, each a_{ij} is gf-small. Moreover $\bigvee_{ij} a_{ij} = \bigvee_i a_i = a$. ■

We shall write $\mathcal{ET}$ for the category of right quantales and étale maps. Thus $\mathcal{ET}/Q$ will denote the category of right quantales étalé over Q.

6 Sheaves on a quantale

The most popular definition of a sheaf is a contravariant functor to the category of sets (= a presheaf) together with glueing requirements. This is also the definition which generalizes the less elegantly to the case of quantales. We recall (cf. section 1) that given an element v of a right quantale Q, $(\downarrow u, \preccurlyeq)$ denotes the down segment of v provided with the preorder $p \preccurlyeq q$ when $p = p \& q$.

Definition 6.1. *A presheaf on a right quantale Q is a pair (v, F) where*

- *v is an element of Q;*
- *$F: (\downarrow v, \preccurlyeq) \longrightarrow \mathcal{SET}$ is a contravariant functor to the category of sets;*
- *$v = \bigvee\{p \in \downarrow v \mid F(p) \neq\}$.*

In the previous definition, the element v is thus exactly what is generally called the "support" of the presheaf F. It is necessary to have this data as part of the definition in order to define F on just the down segment of v; indeed, defining a presheaf directly on $(Q, \preccurlyeq)$ would impose the support of a presheaf to be always a 2-sided element of Q, because of the relation $\overline{v} \preccurlyeq v$. As usual, when $p \preccurlyeq q$, we write $\rho_p^q: F(q) \longrightarrow F(p)$ for the corresponding restriction mapping.

Definition 6.2. *Let Q be a right quantale and (v, F), (w, G) two presheaves on Q. A morphism of presheaves $\alpha: F \longrightarrow G$ is defined just when $v \leq w$; it is a natural transformation from F to the restriction of G at the level v.*

Clearly, the presheaves on a right quantale and their morphisms constitute a category. The sheaves are now defined as the presheaves with the usual glueing condition:

Definition 6.3. *Let Q be a right quantale. A presheaf (v, F) on Q is a sheaf when for every covering $u = \bigvee_{i \in I} u_i \leq v$ and every family $(a_i \in Fu_i)_{i \in I}$ compatible in the sense that $\rho^{u_i}_{u_i \& u_j} a_i = \rho^{u_j}_{u_i \& u_j} a_j$ for all indices i, j, there exists a unique element $a \in Fu$ such that for every index i, $\rho^u_{u_i} a = a_i$.*

We write $\mathcal{S}h(Q)$ for the category of sheaves on the right quantale Q.

7 Q-sets and sheaves on Q

The first major theorems of this paper prove the equivalences, for a right quantale Q, of the three categories of Q-sets, étale maps over Q and sheaves on Q. First, we prove the equivalence between the categories of Q-sets and sheaves on Q.

Lemma 7.1. *Let Q be a right quantale. Every Q-set A is isomorphic to the Q-set constituted of the set A together with the Q-equality*

$$A \times A \longrightarrow Q \ ; \ (a, a') \mapsto \varepsilon(A) \& [a = a']$$

where the last bracket is the original Q-equality on A.

Proof: The two inverse isomorphisms in the category of Q-sets are defined by

$$[fa = a'] = [a = a'] \& \varepsilon(A) \qquad [ga' = a] = \varepsilon(A) \& [a' = a]$$

where $a \in A$, $a' \in A$ and the brackets $[a = a']$, $[a' = a]$ are the original Q-equality of A. ∎

Lemma 7.2. *Let Q be a right quantale. The category of Q-sets is equivalent to the following category:*

- *the objects are the pairs (v, A) where $v \in Q$ and A is a 2-Q-set with support $\overline{v}$;*
- *a morphism $(v, A) \longrightarrow (w, B)$ is defined just when $v \leq w$ and is then a morphism of 2-Q-sets $A \longrightarrow B$;*
- *the composition is that of morphisms of 2-Q-sets.*

Proof: In 7.1, observe that each element $\varepsilon(A) \& [a = a']$ is 2-sided in $\varepsilon(A)$ so that the new equality on A makes it a $2{\downarrow}\ \varepsilon(A)$-set. By 1.4 this is equivalent to giving a $\overline{\varepsilon(A)}$-set, thus a 2-$Q$-set with support $\overline{\varepsilon(A)}$. The rest is just routine. ∎

Lemma 7.3. *Let Q be a right quantale. The category of sheaves on Q is equivalent to the following category:*

- *the objects are the pairs (v, F) where $v \in Q$ and F is a sheaf on 2-Q with support $\overline{v}$;*
- *a morphism $(v, F) \longrightarrow (w, G)$ is defined just when $v \leq w$ and is then a morphism $F \longrightarrow G$ of sheaves on 2-Q;*
- *the composition is that of morphisms of sheaves on 2-Q.*

Proof: If (v, F) is a sheaf on Q, giving F is equivalent, by 1.5, to giving a sheaf on $2\text{-}{\downarrow}v$. By 1.4 this is still equivalent to giving a sheaf on $2\text{-}{\downarrow}\overline{v}$ thus finally a sheaf on 2-Q whose support is $\overline{v}$. The rest is routine. ∎

Theorem 7.4. *Let Q be a right quantale. The categories of Q-sets and sheaves on Q are equivalent.*

Proof: By 7.2, 7.3 and the corresponding result for locales (cf. [5]). ∎

8 Q-sets and étale maps

We prove now the equivalence, for a right quantale Q, of the categories of Q-sets and étale maps over Q. We need an intermediate technical notion which will make the proof more understandable.

Definition 8.1. *Let $f: \mathcal{X} \longrightarrow Q$ be an étale map of right quantales. f is stable when the f-small elements $x \in X$ are exactly those for which the restriction $f: \downarrow x \longrightarrow \downarrow fx$ is an isomorphism of quantales.*

We shall write $\mathcal{S}\text{-}\mathcal{ET}/Q$ for the full subcategory of $\mathcal{ET}/Q$ whose objects are the stable étale maps. We shall prove first that every étale map over Q is isomorphic to a stable étale maps. Next we shall prove the equivalence between the categories of Q-sets and stable étale maps over Q.

Lemma 8.2. *Let $f: \mathcal{X} \longrightarrow Q$ and $g: \mathcal{Y} \longrightarrow Q$ be stable étale maps of quantales. If $h: (\mathcal{X}, f) \longrightarrow (\mathcal{Y}, g)$ is a morphism of $\mathcal{S}\text{-}\mathcal{ET}/Q$ and $x \in \mathcal{X}$ is f-small, then $hx \in \mathcal{Y}$ is g-small.*

Proof: Let us write $1 = \bigvee_{i \in I} x_i$ with each x_i h-small and 2-sided (cf. 5.4). By definition of étale maps, both f and h restrict to isomorphisms at the levels $x \& x_i$. So g restricts to an isomorphism at each level $h(x \& x_i)$. By stability of g, each $h(x \& x_i)$ is thus g-small.

Now if $y \leq hx$, by 5.5 applied at the level x, $y = hx'$ for some $x' \leq x$. Using 5.5 together with the fact that each $h(x \& x_i)$ is g-small, one verifies first that $y = g^* gy \& hx$. From this follows easily the fact that g, restricted at the level hx, is an isomorphism of quantales. ∎

We recall that $\mathcal{S}(A)$ denotes the quantale of sub-Q-sets of a given Q-set A (cf. section 3).

Lemma 8.3. *Let Q be a right quantale and A a Q-set. The assignment*

$$\varphi(A): \mathcal{S}(A) \longrightarrow Q \;\; ; \;\; s \mapsto \bigvee_{a \in A} sa$$

is a stable étale map of quantale for which the φ_A-small elements are exactly the singletons. This assignment extends in a functor $\varphi: Q - \mathcal{SET} \longrightarrow \mathcal{S} - \mathcal{ET}/Q$.

Proof: We know by 3.2 that $\mathcal{S}(A)$ is a quantale. It is routine to verify that φ_A preserves the relation $\ll$. It preserves also arbitrary suprema since it has a right adjoint given by

$$f_A^*: Q \longrightarrow \mathcal{S}(A) \;\; ; \;\; f_A^*(v)(a) = (\varepsilon(A) \wedge v) \& [a = a]$$

where $v \in Q$ and $a \in A$.

Next consider $s \in \mathcal{S}(A)$ for which the restriction of φ_A to $\downarrow s$ is an isomorphism, with inverse $\varphi_A(\cdot)\&s$ (cf. 5.5). Applying $\overline{f_A^*\varphi_A}$ to $s\&[a = \cdot]$ yields a relation implying immediately $sa\&sa' \leq \overline{[a = a']}$, thus the fact that s is a singleton. The converse is easy.

But if s is a singleton of A, its 2-sided closure in $\mathcal{S}(A)$ is still a singleton. Therefore every singleton is φ_A-small and, moreover, the additional requirement for getting a stable étale map is already verified.

Finally let us observe that for every element $a \in A$, $s_a = \varepsilon(A)\&[a = \cdot]$ is a 2-sided singleton. It is routine to check that the family $(s_a)_{a\in A}$ satisfies the requirements of the last condition in 5.4. So $\varphi(A)$ is indeed a stable étale map of quantales.

Let us extend this construction in a functor $\varphi: Q - \mathcal{SET} \longrightarrow \mathcal{S} - \mathcal{ET}/Q$. If $f: A \longrightarrow B$ is a morphism of Q-sets, the assignment

$$\varphi(f): \mathcal{S}(A) \longrightarrow \mathcal{S}(B) \;\; ; \;\; \varphi(f)(s)(b) = \bigvee_{a\in A} sa\&[fa = b]$$

where $b \in B$ defines a mapping $\varphi(f)$ which is easily seen to preserve the partial orders $\leq$ and $\ll$. This mapping preserves arbitrary suprema since it admits the right adjoint

$$\varphi(f)^*(s)(a) = \varepsilon(A) \wedge \bigvee_{b\in B} (sb\&[fa = b])$$

where $a \in A$ and $s \in \mathcal{S}(B)$. Finally the family $(s_a)_{a\in A}$

$$s_a = \varepsilon(A)\&[a = \cdot]$$

satisfies the requirements of the last condition of 5.4. This completes the definition of the functor φ. ∎

Let us now construct a functor in the other direction.

Lemma 8.4. *Let $f: \mathcal{X} \longrightarrow Q$ be an étale map of quantales. The set $\psi(f)$ of f-small elements, provided with the Q-equality $[x_1 = x_2] = f(x_1\&x_2)$ for x_1, x_2 in $\mathcal{X}$, is a Q-set. This assignment extends to a functor $\psi: \mathcal{S} - \mathcal{ET}/Q \longrightarrow Q - \mathcal{SET}$.*

Proof: The fact for $\psi(f)$ to be a Q-set is immediate. The rest follows from 8.2. ∎

To understand the next lemma, let us make clear that an isomorphism in the category of right quantales and étale maps is of course an isomorphism for both partial orders $\leq$ and $\ll$, but just locally an isomorphism of quantales.

Lemma 8.5. *Let Q be a right quantale. Every étale map over Q is isomorphic to a stable étale map.*

Proof: Consider an étale map of quantales $f: \mathcal{X} \longrightarrow Q$. By lemmas 8.3 and 8.4 $\varphi\psi(f): \mathcal{S}(\psi f) \longrightarrow Q$ is a stable étale map. It is isomorphic to f in the category of étale maps over Q; the isomorphism is given by

$$\alpha: \mathcal{X} \longrightarrow \mathcal{S}(\psi f) \;\; ; \;\; \alpha(x) = f(x\&\cdot)$$

for $x \in \mathcal{X}$, with inverse

$$\beta: S(\psi f) \longrightarrow \mathcal{X} \;\; ; \;\; \beta(s) = \bigvee\{x \mid x \in \mathcal{X} \;\; ; \;\; x \ f\text{-small}\}$$

∎

Theorem 8.6. *Let Q be a right quantale. The category of Q-sets is equivalent to the category of étale maps over Q.*

Proof: By lemma 8.5. it suffices to prove the equivalence of the categories of Q-sets and stable étale maps over Q. We shall prove that the functors φ, ψ of lemmas 8.3 and 8.4 exhibit this equivalence. It follows immediately from the proof of 8.5 that $\varphi\psi$ is isomorphic to the identity.

Conversely, given a Q-set A, let us exhibit an isomorphism of Q-sets between A and $\psi\varphi(A)$. This is just

$$\alpha: A \times \psi\varphi(A) \longrightarrow Q \;\; ; \;\; [\alpha a = s] = [a = a] \,\&\, sa$$

where $a \in A$ and s is a singleton of A, i.e. $s \in \psi\varphi(A)$. With the same notations, the inverse isomorphism is given by

$$\beta: \psi\varphi(A) \times A \longrightarrow Q \;\; ; \;\; [\beta s = a] = sa.$$

The rest is routine computations. ∎

9 Categorical properties of sheaves

For a given right quantale Q, the two previous sections have proved the equivalence between the categories of Q-sets, étale maps over Q and sheaves on Q. We shall work with Q-sets, which is the most "flexible" of the three equivalent notions. All the properties of the category will be easily deduced from the following lemmas.

Lemma 9.1. *Let Q be a right quantale. A morphism $f: A \longrightarrow B$ of Q-sets is an epimorphism if and only if the relation*

$$[b = b] = \bigvee_{a \in A} [b = b] \,\&\, [fa = b]$$

holds for every element $b \in B$.

Proof: If f is an epimorphism, consider the disjoint union $B \coprod B$. For every $b \in B$, we write b_1, b_2 for the two corresponding elements of $B \coprod B$. We provide $B \coprod B$ with the structure of a Q-set by putting

$$[b_1 = b_1'] = [b_2 = b_2'] = [b = b']$$

$$[b_1 = b_2'] = [b_2 = b_1'] = \bigvee_{a \in A} [b = b] \,\&\, [fa = b] \,\&\, [b = b']$$

for all b, b' in B. Now we define morphisms of Q-sets $g_1, g_2 \colon B \rightrightarrows B \coprod B$ by the formulas

$$[g_1 b = c] = [b_1 = c] \qquad [g_2 b = c] = [b_2 = c]$$

for all $b \in B$, $c \in B \coprod B$. From $g_1 f = g_2 f$ one deduces $[g_1 b = b_1] = [g_2 b = b_1]$ for every $b \in B$, which is the required relation.

The converse implication is proved by arguments analogous to that developed in the case of monomorphisms (cf. 3.4). ∎

Lemma 9.2. *Let Q be a right quantale. Consider the following category $\mathcal{F}$:*

- *objects are the pairs (A, v), where $v \in Q$ and A is a Q-set such that $\varepsilon(A)$ is 2-sided in v;*
- *a morphism $(A, v) \longrightarrow (B, w)$ is defined just when $v \leq w$ and is then a morphism of Q-sets $A \longrightarrow B$;*
- *the composition is that of morphisms of Q-sets.*

The functor $\varphi \colon \mathcal{F} \longrightarrow Q$ applying (A, v) on v has the following properties:

1. *φ is both a fibration and a cofibration;*
2. *each fibre is a localic topos;*
3. *the inverse image functors between fibres are logical morphisms of toposes with both a right and a left adjoint.*
4. *every cartesian morphism is a monomorphism in Q-$\mathcal{SET}$ and every cocartesian morphism is an epimorphism in Q-$\mathcal{SET}$;*
5. *every cocartesian morphism is cartesian as well.*

Proof: The considerations of section 7 prove that the fibre at $v \in Q$ is equivalent to the topos of sheaves on the locale of 2-sided elements of $\downarrow\overline{v}$.

If $v \leq w$ in Q and A is a Q-set in the fibre over w, we get a Q-set in the fibre over v by providing A with the Q-equality $v \,\&\, [\cdot, \cdot]$. The corresponding cartesian morphism α is defined by $\alpha(a, a') = v \,\&\, [a = a']$. It is a monomorphism of Q-sets by proposition 3.4.

If $v \leq w$ in Q and A is a Q-set in the fibre over v, we get a Q-set in the fibre over w by providing A with the Q-equality $w \,\&\, [\cdot, \cdot]$. The corresponding cocartesian morphism β is defined by $\beta(a, a') = [a = a']$. It is an epimorphism of Q-sets by lemma 9.1.

If $v \leq w$ in Q and A is a Q-set in the fibre over w, transporting A in the fibre over w by the cofibration and back in the fibre over v by the fibration yields the set A with the Q-equality $v \,\&\, [\cdot, \cdot]$. Since $\varepsilon(A)$ is 2-sided in v, this Q-equality is equal to

$\varepsilon(A) \& [cdot, \cdot]$, yielding a Q-set isomorphic to the original Q-set A (cf. lemma 7.1). This implies easyly that every cocartesian morphism is also cartesian.

Finally the inverse image functor corresponding to $v \leq w$ in Q is the usual restriction functor from the category of sheaves over the locales 2-$\downarrow\overline{w}$ and 2-$\downarrow\overline{v}$, so has the required properties (cf. [6]).

■

It should be noticed that the fibration-cofibration of lemma 9.2 is generally not a bifibration, since the Beck conditions are not satisfied.

Proposition 9.3. *Let Q be a right quantale. The category of Q-sets is both complete and cocomplete.*

Proof: Given a diagram with vertices $(A_i)_{i\in I}$, we copy this diagram in the fibration-cofibration $\mathcal{F}$ of 9.2 by putting each A_i in the fibre over $\varepsilon(A_i)$. Using the fibration property, we get a diagram of the same shape in the fibre over $\bigwedge_{i\in I}\varepsilon(A_i)$. Computing the limit cone of this last diagram in the fibre and composing with the cartesian morphisms to the original objects A_i yields a cone on the original diagram in Q-$\mathcal{SET}$. If another cone with vertex M is given over the original diagram in Q-$\mathcal{SET}$, then $\varepsilon(M) \leq \varepsilon(A_i)$ for each $i \in I$. By the cofibration property we can thus transport the cone of vertex M in the fibre over $\bigwedge_{i\in I}\varepsilon(A_i)$ and get the required factorization by composing the cocartesian morphism on M with the factorization in the fibre. This proves the completeness. The cocompleteness is analogous, permuting the roles of the fibration and the cofibration. ■

Lemma 9.4. *Let Q be a right quantale. In the category of Q-sets, a monomorphism (respectively, epimorphism) $f: A \longrightarrow B$ is regular if and only if $\varepsilon(A)$ is 2-sided in $\varepsilon(B)$; in the case of an epimorphism, this implies $\varepsilon(A) = \varepsilon(B)$.*

Proof: This follows immediately from the way equalizers and coequalizers are constructed in the category of Q-sets (cf. 9.3). In the case of an epimorphism, lemma 9.1 implies the relation $\varepsilon(B) \leq \varepsilon(B)\&\varepsilon(A)$ thus $\varepsilon(B) = \varepsilon(A)$ under the conditions of the present lemma. ■

Proposition 9.5. *Let Q be a right quantale. The category of Q-sets is regular and exact.*

Proof: For the regularity, we must prove that the pullback of a regular epimorphism $f: A \longrightarrow B$ along $g: C \longrightarrow D$ in Q-$\mathcal{SET}$ is still a regular epimorphism. Using 9.4, in the fibration of 9.2 f is an epimorphism in the fibre over $\varepsilon(B)$. The result follows thus from the construction of pullbacks in Q-$\mathcal{SET}$ and the last statement of 9.2.

Next if $R \rightrightarrows A$ is an equivalence relation in Q-set, the reflexivity implies immediately $\varepsilon(R) = \varepsilon(A)$, so that the exactness of Q-$\mathcal{SET}$ follows from that of the fibres and the construction of coequalizers in Q-$\mathcal{SET}$ (cf. 9.3). ■

Applying again 9.2 and 9.3 one deduces with the same techniques a list of other properties of the category of Q-sets:

Proposition 9.6. *Let Q be a right quantale. The following are properties of the category of Q-sets:*

1. *equivalence relations are universal;*
2. *each morphism factors uniquely as an epimorphism followed by a regular monomorphism;*
3. *the pushout of a regular monomorphism is still a monomorphism and the corresponding square is a pullback;*
4. *the initial object is strict;*
5. *the canonical injections of a coproduct are monomorphisms;*
6. *the coproducts are disjoint.* ∎

10 The quantales of subobjects

By 3.2 and 3.5 it follows already that the poset of subobjects of a Q-set A, for a given right quantale Q, can be provided with the structure of a quantale. We want now to emphasize the important fact that this is not an extra structure: this structure of quantale on the subobjects of A is completely determined by the categórical properties of the category of Q-sets. In particular, for $A = 1$, the terminal object, we recapture the original quantale Q.

Lemma 10.1. *Let Q be a right quantale and A a Q-set. The regular subobjects of A constitute a locale which is both reflective and coreflective in the complete lattice of all subobjects of A.*

Proof: By 9.3, the poset of subobjects of A is complete. If $(A_i)_{i\in I}$ is a family of subobjects of A, by 9.3 again $\varepsilon(\bigcap_{i\in I} A_i) = \bigwedge_{i\in I} \varepsilon(A_i)$ and $\varepsilon(\bigcup_{i\in I} A_i) = \bigvee_{i\in I} \varepsilon(A_i)$. One concludes by 1.4 and 9.4. ∎

Theorem 10.2. *Let Q be a right quantale and A a Q-set. The complete lattice of subobjects of A becomes a quantale when provided with the multiplication*

$$S\&T = S \wedge \overline{T}$$

where S and T are subobjects of A and $\overline{T}$ is the reflection of T in the locale of regular subobjects of A (cf. 10.1).

Proof: Applying 1.4, it remains to show that the subobject $f\colon S \rightarrowtail A$ is regular precisely when the corresponding sub-Q-set

$$s\colon A \longrightarrow Q \;\; ; \;\; s(a) = \varepsilon(S)\& \bigvee_{x\in S} [fx = a]$$

described in 3.5, with $a \in A$ and $x \in S$, is 2-sided in the quantale of sub-Q-sets (cf. 3.2).

If S is regular, $\varepsilon(S)$ is 2-sided in $\varepsilon(A)$ (cf. 9.4) thus $s(a)$ is 2-sided in $\varepsilon(A)$, which is the required condition (cf. 3.3).

Conversely, one has immediately $\bigvee_{a \in A} s(a) = \varepsilon(S)$; thus if each $s(a)$ is 2-sided in $\varepsilon(A)$, so is $\varepsilon(S)$. ∎

Corollary 10.3. *Let Q be a right quantale. In the category of Q-sets, the quantale of subobjects of the terminal object* 1 *is isomorphic to the original quantale Q.*

Proof: The terminal object is the singleton provided with the Q-equality $[* = *] = 1$ (cf. 9.3). In this case, all the axioms for a sub$Q$-set vanish (cf. 3.1) so that a sub-Q-set is just an arbitrary element $s(*)$ of Q. ∎

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