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# WEAKLY LOCALLY PRESENTABLE CATEGORIES

***by J. ADÁMEK[1] and J. ROSICKÝ[2]***

Dedicated to the memory of our dear friend and excellent colleague Jan Reiterman

**Résumé.** Une catégorie faiblement localement présentable est une catégorie accessible avec des colimites faibles (ou, de maniére èquivalente, avec des produits). Les catégories faiblement localement présentables sont exactement les catégories esquissables par une limite-epi esquisse.[1]

## Introduction

Recall from [La], [MP] that a category is $\lambda$-accessible ($\lambda$ a regular cardinal) if it has $\lambda$-directed colimits, and it has a set $\mathcal{A}$ of $\lambda$-presentable objects such that each object is a $\lambda$-directed colimit of $\mathcal{A}$-objects. A category is accessible (i. e., $\lambda$-accessible for some $\lambda$) iff it is sketchable[2]. The following conditions are well-known to be equivalent for each accessible category $\mathcal{K}$:

(COLIM) $\mathcal{K}$ has colimits,

(LIM) $\mathcal{K}$ has limits,

(SKETCH) $\mathcal{K}$ is sketchable by a limit-sketch,

(FUN) $\mathcal{K}$ is equivalent to a small-orthogonality class in some functor-category $Set^{\mathcal{A}}, \mathcal{A}$ small (i. e., there exists a set $\mathcal{M}$ of morhpisms in $Set^{\mathcal{A}}$ with $\mathcal{K} \cong \mathcal{M}^{\perp}$),

(LP) $\mathcal{K}$ is locally presentable.

The concept of a locally presentable category has been fruitfully generalized in several directions. For example, Y. Diers [D] has introduced **locally multi-presentable categories** which can be characterized as accessible categories $\mathcal{K}$

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[2]"Sketchable" by a sketch $\mathcal{S}$ means: equivalent to the category of set-valued models of $\mathcal{S}$. We use terminology for sketches indicating the required properties of their models. E. g., a sketch whose all colimit-cocones are discrete (i. e., whose models are functors turning specified cones to limit-cones and specified cocones to coproduct-cocones) is called a limit-coproduct sketch, etc.

satisfying the following equivalent conditions:

(COLIM) $\mathcal{K}$ has multicolimits,

(LIM) $\mathcal{K}$ has connected limits,

(SKETCH) $\mathcal{K}$ can be sketched by a limit-coproduct sketch,

(FUN) $\mathcal{K}$ is equivalent to a small-multiorthogonality class in some functor-category $\mathrm{Set}^{\mathcal{A}}$, $\mathcal{A}$ small,

(LMP) $\mathcal{K}$ is locally multipresentable.

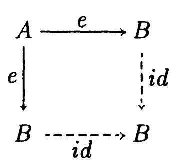
The proof can be found e. g. in [A], where also an analogous characterization of the locally polypresentable categories of F. Lamarche [L] is proved.

We are now going to introduce another very natural generalization of locally presentable categories:

(1) Concerning colimits: we generalize them to weak colimits by giving up the uniqueness of factorizations. That is, a **weak colimit** of a diagram $D$ is a compatible cocone of $D$ through which every other compatible cocone of $D$ factorizes.

(2) Instead of limits we just take products.

(3) The sketches we work with are those in which the projective cones are arbitrary (small) cones, but the inductive cones all have the following form



In other words, a model of such a sketch is a functor which (i) maps the specified projective cones to limit cones and (ii) maps specified morphisms $e\colon A \to B$ to epimorphisms. We call such a sketch **a limit-epi sketch**.

(4) Concerning orthogonality, this is generalized to injectivity by giving up the uniqueness of factorizations. For each set $\mathcal{M}$ of morphisms in a category an object $X$ is **$\mathcal{M}$-injective** provided that for each member $m\colon A \to A'$ of $\mathcal{M}$ and each morphism $f\colon A \to X$ there exists a morphism $f'\colon A' \to X$ with $f = f'.m$. The full subcategory of all $\mathcal{M}$-injective objects is denoted by $\mathcal{M}$-Inj. A full subcategory is called a **small-injectivity class** provided that it has the form $\mathcal{M}$-Inj for some set $\mathcal{M}$ of morphisms.

Thus, we are going to prove the equivalence of the following conditions on an accessible category $\mathcal{K}$:

| | |
|---|---|
| (COLIM) | $\mathcal{K}$ has weak colimits, |
| (LIM) | $\mathcal{K}$ has products, |
| (SKETCH) | $\mathcal{K}$ is sketchable by a limit-epi sketch, |
| (FUN) | $\mathcal{K}$ is equivalent to a small-injectivity class in some functor-category $\mathrm{Set}^{\mathcal{A}}$, $\mathcal{A}$ small, |
| (WLP) | $\mathcal{K}$ is weakly locally presentable. |

After the submission of our paper we have found out that C. Lair had introduced limit-epi sketches in [La], and had given a different characterization of sketchability by them as "V-qualifiables" categories.

# I. Weakly Locally Presentable Categories

**I.1 Definition.** Let $\lambda$ be a regular cardinal. A category is called **weakly locally $\lambda$-presentable** provided that it is $\lambda$-accessible, and has weak colimits.

A category is called weakly locally presentable if it is weakly locally $\lambda$-presentable for some $\lambda$.

**I.2 Examples.** (1) Every locally $\lambda$-presentable category is weakly locally $\lambda$-presentable.

(2) The category of nonempty sets and functions is weakly locally finitely presentable: every nonempty diagram has a colimit, and every object is weakly initial.

(3) The category of divisible Abelian groups is weakly locally finitely presentable.

(4) The category of CPO's with bottom (i. e., posets with joins of $\omega$-sequences and with a least element) whose morphisms are continuous functions (i. e., preserve $\omega$-joins) is weakly locally $\aleph_1$-presentable.

(5) The category of $\lambda$-complete semilattices (i. e., posets with joins of less than $\lambda$ elements) and order-preserving maps is weakly locally $\lambda$-presentable.

**I.3 Remark.** Let us call a full subcategory $\mathcal{A}$ of a category $\mathcal{K}$ **weakly reflective** provided that for each object $K$ in $\mathcal{K}$ there exists a weak reflection $r\colon K \to K^*$ in $\mathcal{A}$ (i. e., $K^* \in \mathcal{A}$ and any morphism $f\colon K \to A$ with $A \in \mathcal{A}$ factorizes as $f = f^* \cdot r$ for some, not necessarily unique, $f^*\colon K^* \to A$). It is clear that every weakly reflective subcategory of a weakly cocomplete category is weakly cocomplete.

**I.4 Proposition.** *Every small-injectivity class in a weakly locally presentable category is weakly locally presentable.*

Proof. (1) Let us observe first that every small-injectivity class $\mathcal{A}$ in a locally presentable category $\mathcal{K}$ is weakly locally presentable. It has been proved in [AR] that $\mathcal{A}$ is an accessible category (Corollary IV.4)

and

weakly reflective in $\mathcal{K}$ (Corollary III.3).

It follows that $\mathcal{A}$ is weakly locally presentable.

(2) Due to (1), it is sufficient to show that every weakly locally presentable category is equivalent to a small-injectivity class in some locally presentable category. (In fact, the relation "is a small-injectivity class of" is clearly transitive.) Let $\mathcal{K}$ be weakly locally $\lambda$-presentable. Then by [MP] the full subcategory $\mathcal{K}_\lambda$ of all $\lambda$-presentable objects of $\mathcal{K}$ is essentially small and dense in $\mathcal{K}$. Thus, the Yoneda embedding

$$E: \mathcal{K} \to \mathrm{Set}^{\mathcal{K}_\lambda^{\mathrm{op}}}, \quad K \mapsto \hom(-, K)/\mathcal{K}_\lambda^{\mathrm{op}}$$

is full and faithful. Besides, since all objects of $\mathcal{K}_\lambda$ are $\lambda$-presentable, $E$ preserves $\lambda$-directed colimits. Thus, the subcategory $E(\mathcal{K})$ of the locally presentable category $\mathrm{Set}^{\mathcal{K}_\lambda^{\mathrm{op}}}$ is full and closed under $\lambda$-directed colimits. To prove that $E(\mathcal{K})$ is a small-injectivity class, it is sufficient (by [AR]) to show that it is weakly reflective in $\mathrm{Set}^{\mathcal{K}_\lambda^{\mathrm{op}}}$. In fact, let $H$ be an object of $\mathrm{Set}^{\mathcal{K}_\lambda^{\mathrm{op}}}$, and let $el\,H$ be the category of elements of $H$: objects are pairs $(K, x)$ where $K \in \mathcal{K}_\lambda$ and $x \in HK$, morphisms $f: (K, x) \to (K', x')$ are $\mathcal{K}$-morphisms $f: K \to K'$ with $Hf(x') = x$. The natural forgetful functor $el\,H \to \mathcal{K}$ has a weak colimit, say,$((K, x) \xrightarrow{f_x} K_0)$. It follows immediately that the natural transformation $h: H \longrightarrow \hom(-, K_0)/\mathcal{K}_\lambda^{\mathrm{op}}$ given by $h_K(x) = f_x$ for all $K \in \mathcal{K}_\lambda$ is a weak reflection of $H$ in $E(\mathcal{K})$.

**I.5 Remark.** Analogously, every small-orthogonality class of a locally presentable category is locally presentable. There is, however, a substantial difference between these two situations: the choice of a concrete $\lambda$ is not analogous. Given a locally $\lambda$-presentable category and a set $\mathcal{M}$ of morphisms with $\lambda$-presentable domains and codomains, then $\mathcal{M}^\perp$ is locally $\lambda$-presentable. In contrast, the following locally finitely presentable category $\mathcal{K}$ has an injectivity-class $\{m\} - \mathrm{Inj}$ which is not finitely accessible, although the domain and codomain of $m$ is finitely presentable:

Let $\mathcal{K}$ be the category of graphs (i. e., sets with a binary relation) and graph homomorphisms. Let $m: (\{0\}, \emptyset) \longrightarrow (\{0,1\}, \{(0,1)\})$ be the inclusion morphism. Then $\{m\} - \mathrm{Inj}$ is the class of all graphs in which every vertex is the initial vertex of some edge. The last category is weakly locally $\omega_1$-presentable, but not weakly locally $\omega$-presentable. In fact, the object $(N, <)$ of natural numbers with the usual strict ordering lies in $\{m\} - \mathrm{Inj}$. This object is not a colimit of finite (= finitely presentable) objects of $\{m\} - Inj$.

**I.6 Theorem.** For each category $\mathcal{K}$ equivalent are:

(1) $\mathcal{K}$ is weakly locally presentable;

(2) $\mathcal{K}$ is accessible and has products;

(3) $\mathcal{K}$ is a small-injectivity class of a locally presentable category;

(4) $\mathcal{K}$ is equivalent to a small-injectivity class of $\mathrm{Set}^{\mathcal{A}}$ for a small category $\mathcal{A}$.

Proof. (2) $\Rightarrow$ (1). Let $\mathcal{K}$ be a $\lambda$-accessible category with products. Analogously to the proof of I.4, we can consider $\mathcal{K}$ as a full subcategory of a locally finitely presentable category $\mathcal{L}$ $(= \mathrm{Set}^{\mathcal{K}_\lambda^{op}})$ closed under $\lambda$-directed colimits. Moreover, $\mathcal{K}$ is closed under products in $\mathcal{L}$ (since the Yoneda embedding preserves all existing limits). We are going to prove that every $\lambda$-presentable object $L$ of $\mathcal{L}$ has a weak reflection in $\mathcal{K}$. It follows that $\mathcal{K}$ is weakly reflective in $\mathcal{L}$: every $L'$ of $\mathcal{L}$ is $\lambda'$-presentable for some $\lambda' \geq \lambda$, and there exists a regular cardinal $\lambda'' \geq \lambda'$ such that $\mathcal{K}$ is $\lambda''$-accessible (see [MP]). Since $\mathcal{K}$ is closed under $\lambda''$-directed colimits and products in $\mathcal{L}$, this will prove that $L'$ has a weak reflection in $\mathcal{K}$.

By [MP] for every $\lambda$-presentable object $L$ there exists a solution set $f_i : L \longrightarrow K_i$ $(i \in I)$ of the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{L}$ with the domain $L$. It then follows that $< f_i >_{i \in I} : L \longrightarrow \prod_{i \in I} K_i$ is a weak reflection of $L$ in $\mathcal{K}$.

(1) $\Rightarrow$ (4) See the proof of I.4.
(4) $\Rightarrow$ (3) Trivial.
(3) $\Rightarrow$ (2) Injectivity classes are closed under products.

**I.7 Remark.** Recall from [RTA] that under **Vopěnka's principle** (which is the large-cardinal principle stating that a locally presentable category cannot have a large, discrete, full subcategory) a category is locally presentable iff it is

(a) **bounded**, i. e., has a small, dense subcategory and

(b) complete [or cocomplete].

(In (a) it is even sufficient to take a small colimit-dense subcategory. This strengthening is, in fact, equivalent to Vopěnka's principle, see [AHR].)

Now we can ask whether, analogously, there is a set-theoretical assumption under which weakly locally presentable categories are precisely the bounded categories with products [or bounded categories with weak colimits]. This is not the case:

**I.8 Example.** Let $\mathcal{K}$ be the category whose objects are complete lattices, and whose morphisms are order-preserving functions. This category does not have $\lambda$-directed colimits for any $\lambda$, thus, it is not accessible. However $\mathcal{K}$ is weakly reflective in the category of posets (MacNeille completions are weak reflections), thus, $\mathcal{K}$ has products and weak colimits. And the two-element chain is dense in $\mathcal{K}$.

# II. Sketching a Weakly Locally Presentable Category

**II.1** By a **limit-epi sketch** is understood a triple $\mathscr{S} = (\mathcal{A}, \mathcal{L}, \mathcal{E})$ consisting of a small category $\mathcal{A}$, a set $\mathcal{L}$ of small cocones in $\mathcal{A}$, and a set $\mathcal{E}$ of morphisms in $\mathcal{A}$. A **model** of $\mathscr{S}$ is a functor $F: \mathcal{A} \to \mathrm{Set}$ which maps each cone in $\mathcal{L}$ to a limit cone in Set, and each arrow in $\mathcal{E}$ to an epimorphism in Set. We denote by

$$\mathrm{Mod}\,\mathscr{S}$$

the full subcategory of $\mathrm{Set}^{\mathcal{A}}$ of all models of $\mathscr{S}$.

**II.2 Theorem.** *A category is weakly locally presentable iff it is equivalent to* $\mathrm{Mod}\,\mathscr{S}$ *for some limit-epi sketch* $\mathscr{S}$.

Proof. (1) The category $\mathrm{Mod}\,\mathscr{S}$ is accessible for each sketch $\mathscr{S}$, see [MP], 6.2.5. If $\mathscr{S}$ is a limit-epi sketch, then $\mathrm{Mod}\,\mathscr{S}$ is closed under products in $\mathrm{Set}^{\mathcal{A}}$ because a product of epimorphisms is an epimorphism in Set. By Theorem I.4, it follows that $\mathrm{Mod}\,\mathscr{S}$ is weakly locally presentable.

(2) Let $\mathcal{K}$ be a weakly locally presentable category. Since $\mathcal{K}$ is $\lambda$-accessible, it is sketchable by the following sketch (as proved in Paragraph 4.3 of [MP]). We choose a small, full subcategory $\mathcal{B}$ representing all $\lambda$-presentable objects of $\mathcal{K}$. Since $\mathcal{B}$ is dense in $\mathcal{K}$, the following Yoneda embedding $Y: \mathcal{K}^{\mathrm{op}} \longrightarrow \mathrm{Set}^{\mathcal{B}}$, $K \mapsto \hom(K, -)/\mathcal{B}$ is full and faithful. Let $\mathcal{D}$ be a representative set of all $\lambda$-small diagrams in $Y(\mathcal{B}^{\mathrm{op}})$, where $\lambda$-small means that the underlying category has less than $\lambda$ morphisms. For each $D \in \mathcal{D}$ we choose a limit cone with a domain $A(D)$ in $\mathrm{Set}^{\mathcal{B}}$ and the codomain $D$. Then we get a sketch

$$\mathscr{S} = (\mathcal{A}, \mathcal{L}, \mathcal{C})$$

where $\mathcal{A} = Y(\mathcal{B}^{\mathrm{op}}) \cup \{A(D)\}_{D \in \mathcal{D}}$ is a small, full subcategory of $\mathrm{Set}^{\mathcal{B}}, \mathcal{L}$ are the chosen limit cones, and $\mathcal{C}$ are the canonical cocones of $A(D)$ w. r. t. $Y(\mathcal{B}^{\mathrm{op}})$. Then

$$\mathcal{K} \cong Y(\mathcal{K}^{\mathrm{op}}) = \mathrm{Mod}\,\mathscr{S}.$$

We will now present a limit-epi sketch $\mathscr{S}^*$ with $\mathrm{Mod}\,\mathscr{S} = \mathrm{Mod}\,\mathscr{S}^*$.

We first prove that any $\lambda$-small diagram $D: \mathcal{D} \longrightarrow \mathcal{B}$ has a weak colimit in $\mathcal{B}$. Let $(Dd \xrightarrow{k_d} K)_{d \in \mathcal{D}^{obj}}$ be a weak colimit in $\mathcal{K}$. Since $\mathcal{K}$ is $\lambda$-accessible, $K$ is a $\lambda$-directed colimit of $\lambda$-presentable objects, say $(D_i^* \xrightarrow{k_i^*} K)_{i \in I}$ with $D_i^* \in \mathcal{B}$. For each $d \in \mathcal{D}^{obj}$, since $Dd$ is $\lambda$-presentable, the morphism $k_d$ factorizes through $k^*_{i(d)}$ for some $i(d) \in I$. The number of objects $d$ of $\mathcal{D}$ is less than $\lambda$, and $I$ is $\lambda$-directed, thus, there is an upper bound $i \in I$ of all $i(d)$'s. Then each $k_d$ factorizes as

$$k_d = k_i^* \cdot h_d \quad \text{for some} \quad h_d: Dd \to D_i^*.$$

Given a morphism $\delta\colon d_1 \to d_2$ in $\mathcal{D}$, the equation

$$k_i^* \cdot (h_{d_2} \cdot D\delta) = k_{d_2} \cdot D\delta = k_{d_1} = k_i^* \cdot h_{d_1}$$

implies, since $Dd_1$ is $\lambda$-presentable, that there exists $i(\delta) \geq i$ such that $D^*(i \to i(\delta)) \cdot h_{d_2} \cdot D\delta = D^*(i \to i(\delta)) \cdot h_{d_1}$. Again, there is an upper bound $j \in I$ of all $i(\delta)$'s, then the morphisms

$$\bar{h}_d = D^*(i \to j) \cdot h_d \colon Dd \to D_j^*$$

form a compatible cocone of $D$: for each $\delta\colon d_1 \to d_2$ in $\mathcal{D}$ we have

$$\begin{aligned} \bar{h}_{d_2} \cdot D\delta &= D^*(i \to j) \cdot h_{d_2} \cdot D\delta \\ &= D^*(i(\delta) \to j) \cdot D^*(i \to i(\delta)) \cdot h_{d_2} \cdot D(\delta) \\ &= D^*(i(\delta) \to j) \cdot D(i \to i(\delta)) \cdot h_{d_1} \\ &= \bar{h}_{d_1}. \end{aligned}$$

The cocone $(Dd \xrightarrow{\bar{h}_d} D_j^*)$ is a weak colimit of $D$ because the given weak colimit $(Dd \xrightarrow{k_d} K)$ factorizes through it: we have $k_d = k_j^* \cdot \bar{h}_d$ for each $d$. The object $D_j^*$ is $\lambda$-presentable.

Now we define $\mathscr{S}^*$. As proved above, every $\lambda$-small diagram in $\mathcal{B}$ has a weak colimit in $\mathcal{B}$ – thus, every $\lambda$-small diagram in $Y(\mathcal{B}^{\mathrm{op}}) \cong \mathcal{B}^{\mathrm{op}}$ has a weak limit in $Y(\mathcal{B}^{\mathrm{op}})$. Let us choose a weak limit $(B(D) \xrightarrow{b_{D,d}} Dd)_{d \in \mathcal{D}^{\mathrm{obj}}}$ of $D$ with $B(D)$ in $Y(\mathcal{B}^{\mathrm{op}})$. Denote by

$$e_D \colon B(D) \to A(D)$$

the unique factorization (defined by $a_{D,d} \cdot e_D = b_{D,d}$). The sketch $\mathscr{S}^*$ is obtained from $\mathscr{S}$ by substituting each of the canonical diagrams of $A(D)$ (i. e., the diagram from $\mathcal{C}$) by the morphism $e_D$. That is, $\mathscr{S}^*$ is the limit-epi sketch

$$\mathscr{S}^* = (\mathcal{A}, \mathcal{L}, \{e_D\}_{D \in \mathcal{D}}).$$

We will prove that $\operatorname{Mod}\mathscr{S} = \operatorname{Mod}\mathscr{S}^*$. Let $G\colon \mathcal{A} \to \mathrm{Set}$ be a model of $\mathscr{S}$. For each $D \in \mathcal{L}$ we know that $G$ preserves the canonical colimit of $A(D)$ w. r. t. $Y(\mathcal{B}^{\mathrm{op}})$. Thus, to prove that $Ge_D$ is an epimorphism, it is sufficient to prove that each morphism $c\colon Yc \to A(D)$ factorizes through $e_D$. To this end, factorize the following compatible cone $(Yc \xrightarrow{a_{D,d} \cdot c} Dd)_{d \in \mathcal{D}^{\mathrm{obj}}}$ of $D$ through the above weak limit: there exists $\bar{c}\colon \mathcal{K} \to B(D)$ with

$$b_{D,d} \cdot \bar{c} = a_{D,d} \quad \text{for each} \quad d \in \mathcal{D}^{\mathrm{obj}}.$$

Then $a_{D,d} \cdot c = a_{D,d} \cdot e_d \cdot \bar{c}$ for all $d \in \mathcal{D}^{\mathrm{obj}}$, which implies $c = e_D \cdot \bar{c}$.

Conversely, let $G\colon \mathcal{A} \to \mathrm{Set}$ be a model of $\mathscr{S}^*$. By Theorem 9.2.2 of [MP], to prove that $G$ is a model of $\mathscr{S}$, it is sufficient to verify that its domain restriction

$H$ to $Y(\mathcal{B}^{op}) \cong \mathcal{B}^{op}$ is a $\lambda$-directed colimit of hom-functors. This can be proved analogously to the proof in [MP]: we make use of the fact that $Ge_d$ is an epimorphism and (in the last part of the proof) we choose, for the given $a \in H(A(D))$, an element $b \in H(B(D))$ with $He_D(b) = a$.

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