CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

B. BANASCHEWSKI

On certain localic nuclei

Cahiers de topologie et géométrie différentielle catégoriques, tome 35, nº 3 (1994), p. 227-237

<http://www.numdam.org/item?id=CTGDC_1994__35_3_227_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XXXV-3 (1994)

ON CERTAIN LOCALIC NUCLEI

by B. BANASCHEWSKI

In memoriam Jan Reiterman.

Résumé. Cette note traite d'une méthode de construction des nuclei localiques dans un quantale (bilatérale) qui ressemble à la manière dont on peut determiner ce nucleus le plus petit. En particulier, on considère des conditions qui impliquent que les quotients qui en résultent sont compacts, algébriques, on continus. En tant qu'application, on obtient des résultats qui concernent ces propriétés pour la reflection localique d'un quantale.

Introduction

This note deals with a method of constructing localic nuclei on a (two-sided) quantale that resembles the procedure by which the smallest such nucleus may be defined. Of particular interest will be conditions that ensure certain properties for the corresponding quotient, such as being compact, algebraic, or continuous. As an application, we then have results concerning these properties for the localic reflection of a quantale.

First we recall some basic concepts. A <u>quantale</u> Q is a complete lattice together with an associative multiplication such that, for all $a \in Q$ and $S \subseteq Q$,

$$a(\bigvee S) = \bigvee\{at \mid t \in S\} \text{ and } (\bigvee S)a = \bigvee\{ta \mid t \in S\},$$

called <u>two-sided</u> whenever, for all $a,b \in Q$,

$$ab \leq a \wedge b.$$

Since we shall only be dealing with two-sided quantales we let this qualification be tacitly understood. For a general introduction to quantales, see Rosenthal [7].

Principal examples of quantales are the quantales IdR and IdS of all (two-sided) ideals of a ring R or a semigroup S, with the usual operation of ideal multiplication. On the other hand, any locale (or: frame), that is, any complete lattice L in which

$$a \wedge \bigvee S = \bigvee\{a \wedge t \mid t \in S\} \qquad (a \in L,\ S \subseteq L)$$

determines a quantale by putting $ab = a \wedge b$. Moreover, one easily checks that these are exactly the idempotent quantales. For an introduction to locales see Johnstone [5] or Vickers [11].

An important process for quantales is the formation of quotients which are locales. The standard tool for this, on any quantale Q, are the localic nuclei on Q which are the closure operators k on Q such that $k(ab) = k(a) \wedge k(b)$ for all $a,b \in Q$. The corresponding quotient of Q is then

$$\mathrm{Fix}(k) = \{a \in Q \mid k(a) = a\} = k[Q],$$

with partial order induced from Q. That this is indeed a locale is seen by very simple calculation (Niefield–Rosenthal [7]).

In the following, a preclosure operator on a complete lattice L is a map $k_o : L \to L$ such that, for all $a, b \in L$, $a \le k_o(a)$ and $a \le b$ implies $k_o(a) \le k_o(b)$. Then, $\mathrm{Fix}(k_o)$ is clearly a closure system in L, that is, closed under arbitrary meet in L, and we let k be the associated closure operator, so that

$$k(a) = \bigwedge\{x \in L \mid k_o(x) = x\}$$

We call k the closure operator generated by k_o. We note that k may be viewed as the stable transfinite iterate of k_o for the usual definition of the powers k_o^{α} for all ordinals α. However, the proofs given here will make no use of this particular representation of k.

A localic prenucleus on a quantale Q will be a preclosure operator k_o on Q which satisfies the inequalities

$$k_o(a)b,\ ak_o(b),\ a \wedge b \le k_o(ab)$$

for all $a,b \in Q$. Then, Fix (k_o) is a closure system in Q, and for the closure operator k generated by k_o we have:

Lemma 1. k is a localic nucleus.

Proof. For any $a, b \in Q$, consider the set

$$W = \{x \in Q \mid a \le x \le k(a),\ xb \le k(ab)\}$$

Then $a \in W$ trivially, and $k_o(x) \in W$ for any $x \in W$ since $xb \le k(ab)$ and $k_o(x)b \le k_o(xb)$ implies that $k_o(x)b \le k(ab)$. Further, $s = \bigvee W$ obviously belongs to W by the laws of quantales, hence $s = k_o(s)$, and $a \le s \le k(a)$ then implies $s = k(a)$. This shows that $k(a)b \leq k(ab)$, and the same type of argument proves the analogous inequality $ak(b) \leq k(ab)$. Now, by these inequalities and the last inequality assumed for k_o,

$$k(a) \wedge k(b) \leq k(k(a)k(b)) \leq k(ak(b)) \leq k(ab),$$

and hence $k(ab) = k(a) \; k \wedge k(b)$, as desired.

Remark. Lemma 1 is the counterpart for quantales of a corresponding result for locales (Banaschewski [1]).

Next, we consider a particular situation that gives rise to a localic prenucleus, and thus to a localic nucleus.

A nuclear system in a quantale Q is a family $K = (K_a)_{a \in Q}$ of ideals in Q such that, for all $a,b \in Q$,

(NS1) $a \in K_a$.

(NS2) $K_a \subseteq K_b$ whenever $a \leq b$.

(NS3) $K_a \cap K_b \subseteq K_{ab}$.

Evidently, the $a \in Q$ such that $x \in K_a$ implies $x \leq a$ form a closure system in Q; we let k be the corresponding closure operator and put $K(Q) = \mathrm{Fix}(k)$.

Lemma 2. k is a localic nucleus.

Proof. Obviously, k is generated by the preclosure operator k_o defined by $k_o(a) = \bigvee K_a$, which in turn is easily seen to be a localic prenucleus, by the properties of quantales and the conditions (NS1) and (NS3). Hence the result, by Lemma 1.

Note that, for any localic nucleus k on a quantale Q, the principal ideals $\downarrow k(a) = \{x \in L \mid x \leq k(a)\}$, $a \in Q$, form a nuclear system, and hence any such k trivially arises in the manner of Lemma 2.

Our principal non–trivial example, in any quantale Q, is given by

$$R_a = \{x \in Q \mid \text{some } x^n \leq a\} \qquad (a \in Q)$$

which is easily checked to define a nuclear system. The corresponding locale R(Q) then consists of all radical (or: semiprime) elements of Q, and is known to be the localic reflection of Q (Niefield [6]). In the present context, the latter is exhibited by the fact that, for any nuclear system K in Q, $R_a \subseteq K_a$ for all $a \in Q$: since $K_x \subseteq K_{x^n}$ for any n by (NS3), $x^n \leq a$ implies $x \in K_a$ by (NS1) and (NS2). One might add that, in general, the ideals R_a are not principal, as seen for $Q = \mathrm{Id}R$ for suitable rings R, in which case they indeed do not fall under the trivial situation mentioned earlier. On the other hand, though, if all elements of Q are compact then of course $r(a) \in R_a$ for all $a \in Q$, a familiar case in point being $Q = \mathrm{IdR}$ for a commutative noetherian ring R.

Another source of nuclear systems in a given quantale Q are the m–filters in Q, that is, the filters $T \subseteq Q$ such that $ab \in T$ whenever $a,b \in T$. For any such T, one easily checks that

$$T_a = \{x \in Q \mid x \vee y \in T \text{ implies } a \vee y \in T, \text{ for all } y \in Q\}$$

defines ideals satisfying (NS1) and (NS2). For the slightly less obvious (NS3) one uses the fact that $a \vee y$, $b \vee y \in T$ implies first $(a \vee y)(b \vee y) \in T$ and then $ab \vee y. \in T$ since

$$(a \vee y)(b \vee y) = ab \vee ay \vee yb \vee y^2 \leq ab \vee y.$$

A particular case of localic nuclei thus derived was considered by Banaschewski – Erné [4].

Finally, given any nuclear system K and any m – filter T in a quantale Q,

$$\tilde{K}_a = \{x \in Q \mid uxv \in K_a \text{ for some } u, v \in T\} \qquad (a \in Q)$$

defines a nuclear system in Q, as follows readily from the inequalities

$$usxvt \leq uxv, sxt.$$

As indicated at the beginning of this note, we are interested in deriving properties of K(Q) from suitable conditions on K. The first of these is compactness.

In the following, a nuclear system K in a quantale Q will be called <u>transitive</u> whenever $K_a \subseteq K_b$ for all $a \in K_b$. Further, recall that a subset U of a complete lattice is called <u>Scott open</u> if U is an upset, that is, $x \in U$ if $x \geq y$ and $y \in U$, and for any updirected set D, $\bigvee D \in U$ implies that D meets U. Finally, for any nuclear system K in a quantale Q, let $E_K = \{a \in Q \mid e \in K_a\}$ where e is the top element of Q. Evidently, E_K is an upset by (NS1).

Now we have

Proposition 1. <u>For any transitive nuclear system K in a quantale Q, if E_K is Scott open then $k(a) = e$ iff $a \in E_K$, and K(Q) is compact.</u>

Proof. Trivially, $a \in E_K$ implies $k_o(a) = e$ and hence $k(a) = e$. For the converse, note first that $k_o(a) \in E_K$ implies $a \in E_k$: if $\bigvee K_a \in E_k$ then there exist $x \in K_a \cap E_k$ since E_K is Scott open, hence $e \in K_x$ while $K_x \subseteq K_a$ by transitivity, showing that $e \in K_a$ and thus $a \in E_K$. Next, consider the set

$W = \{x \in Q \mid a \leq x \leq k(a),\ x \in E_K$ implies $a \in E_K\}$. Here, $a \in W$ trivially. Also, $k_o(x) \in W$ for any $x \in W$ since $k_o(x) \in E_K$ implies $x \in E_K$ as just shown, and this in turn implies $a \in E_K$. Finally, for any non–void chain $C \subseteq W$, if $\bigvee C \in E_K$ then there exist $x \in C \cap E_K$, and hence $a \in E_K$, showing that $\bigvee C \in W$. Now, by Bourbaki's Fixpoint Lemma [13], W contains a fixpoint c for k_o, and since $a \leq c \leq k(a)$ we have $c = k(a)$. It follows that $k(a) \in E_K$ implies $a \in E_K$, and since $e \in E_K$ we obtain that $k(a) = e$ implies $a \in E_K$, as desired.

Regarding compactness, consider any updirected $D \subseteq K(Q)$ with join e in K(Q), that is, $k(\bigvee D) = e$. Then $\bigvee D \in E_K$ by our first result, hence there exist $a \in D \cap E_K$ and then $e = k(a) = a$, so that $e \in D$ as desired.

Remark 1. The following highlights the importance of the above hypothesis that E_k be Scott open. In any quantale Q with idempotent top in which $x,y < e$ implies $x \vee y < e$ but $e = \bigvee\{x \in Q \mid x < e\}$,

$$K_a = \{x \in Q \mid x < e\}\ (a < e),\ K_e = Q$$

defines a transitive nuclear system such that $E_K = \{e\}$ but $k(a) = e$ for all $a \in Q$.

Remark 2. For any closure operator k on a complete lattice, one easily sees that the top element of Fix(k) is compact iff $k^{-1}(e)$ is Scott open. The point of Proposition 1 is that it suffices to have the generally smaller set E_K to be Scott open.

In order to apply Proposition 1 to R(Q), note first the nuclear system R is indeed transitive: if $a^n \leq b$ and $x^m \leq a$ for any $a, b, x \in Q$ and suitable n and m then $x^{nm} \leq b$, showing that $R_a \subseteq R_b$ whenever $a \in R_b$. Further, $e \in R_a$ will imply $e = a$, and therefore $E_K = \{e\}$, provided $e = e^2$. Finally $\{e\}$ is Scott open iff e is compact, that is, Q is compact. As a result, we have:

Corollary. <u>For any compact quantale Q with idempotent top, $r(a) = e$ iff $a = e$, and R(Q) is compact</u>.

Remark 1. The following observation shows that the hypothesis $e = e^2$ is crucial in this result. If L is any locale, define a quantale Q by adding a new top u to L and putting

$$ab = a \wedge b\ (a, b \in L)$$

$$au = ua = a \ (a \in L)$$
$$u^2 = e, \text{ top of } L.$$

Then Q is compact, and $r(a) = a$ for all $a \neq e$ in L while $r(e) = u = r(u)$. Hence $R(Q) = (L - \{e\}) \cup \{u\}$ which is isomorphic to L. This proves that, for any locale L, there exists a compact quantale Q such that $L \cong R(Q)$.

Remark 2. The compactness of R(Q) does not imply that of Q even if Q is commutative and its top is the unit for the multiplication. To see this, let R be the subring without unit of the real number field consisting of all rational linear combinations of the powers t^{α}, α positive rational, of a fixed positive transcendental number t, and take Q to be the quantale of all ideals J of R such that $RJ = J$. Then $R \in Q$ since $t^{\alpha} = (t^{\alpha/2})^2$ for all α, and hence R is the multiplicative unit of Q. Further, R is not compact in Q, being the union of the chain of all Rt^{α}. Concerning the localic nucleus r on Q, it is easy to see that, for any $I \in Q$,

$r(I) = R\sqrt{I}$ where $\sqrt{I} = \{x \in R \mid \text{some } x^n \in I\}$ which readily implies that updirected joins in R(Q) are unions. On the other hand, $R = r(Rt)$ since

$$(Rt^{1/n})^{n+1} \subseteq Rt$$

for all n, and this implies that R is compact in R(Q). Of course, R being a domain, $r(0) = 0$ and hence R(Q) is non–trivial. In actual fact, R(Q) is infinite so that its compactness is a non–trivial property: since that t^{α} are linearly independent over the rationals, any positive real algebraic number c determines a homomorphism of R into the real number field, taking t^{α} to c^{α}, whose kernel is a prime ideal, which actually turns out to be maximal.

Remark 3. An argument analogous to that in the first remark shows that, for any locale L, there exists a non–compact quantale Q such that $L \cong R(Q)$. Instead of adding a new unit to L, add an interval $\{u_t \mid 0 \leq t \leq 1\}$ isomorphic to the unit interval at the top of L such that $u_o = e$, the unit of L, and define

$$ab = a \wedge b \qquad (a, b \in L)$$
$$au_t = u_t a \qquad (a \in L, \text{ all } t)$$
$$u_s u_t = u_t u_s = u_o = e \ (\text{all } t)$$

In particular, for compact L this also shows that R(Q) may be compact for non–compact Q, but here the top element is not idempotent, let alone the unit for the multiplication.

Remark 4. For any m–filter T in a quantale Q, the associated nuclear system

$$T_a = \{x \in Q \mid x \vee y \in T \text{ implies } a \vee y \in T, \text{ for all } y \in Q\}$$

mentioned earlier is also transitive. Moreover, $e \in T_a$ holds iff $a \in T$, and hence Proposition 1 yields the following result of Banaschewski – Erné [4] for the associated localic nucleus t on Q: <u>If T is a Scott open m–filter in any quantale then t(a) = e iff a ∈ T and Fix(t) is compact</u>.

Remark 5. For any m–filter T in a quantale Q, we further have the nuclear system defined by

$$K_a = \{x \in Q \mid uxv \in R_a \text{ for some } u, v \in T\}$$

which again is transitive: for a, b, x ∈ Q and u, v, s, t ∈ T, if $(uav)^n \leq b$ and $(sxt)^m \leq a$ for some n and m, then

$$(usxtv)^m \leq u(sxt)^m v \leq uav$$

and therefore

$$(usxtv)^{nm} \leq b$$

while us, tv ∈ T. In addition, $e \in K_a$ holds iff a ∈ T since $uev \in R_a$ for some u,v ∈ T iff $w \in R_a$ for some w ∈ T iff a ∈ T, each step because T is an m–filter. Hence, again by Proposition 1, for Scott open T, k(a) = e iff a ∈ T and K(Q) is compact. It is easily seen that the present localic nucleus is the smallest localic nucleus k for which k(a) = a iff a ∈ T. In general, it differs from the localic nucleus described in the preceeding remark which, in fact, is the largest localic nucleus of this kind. An obvious example is provided by any compact locale L for which the nucleus associated with the Scott open {e} is not the identity, such as any totally ordered L.

We now turn to deriving some other properties of K(Q) from suitable conditions concerning a nuclear system K on a quantale Q.

Associated with K, we have a binary relation $\ll_K$ on Q defined as follows:

$x \ll_K y$ whenever $y \in K_s$, $s = \bigvee D$ for some updirected set D, implies $x \in K_t$ for some t ∈ D.

Further, we call c ∈ Q <u>K–compact</u> if $c \ll_K c$.

Note that, for the trivial nuclear system on a locale given by the principal ideals, this relation amounts to the familiar "way below" relation $\ll$, and the corresponding notion of compactness is ordinary compactness. Now, recalling that a locale L is called

continuous if $a = \bigvee\{x \in L \mid x \ll a\}$, for all a ∈ L,

algebraic if $a = \bigvee\{x \in L \mid x \ll x \leq a\}$, for all a ∈ L, and

coherent if it is algebraic and the meet of any finite set of compact elements is compact, we shall call Q

<u>K–continuous</u> if $a = \bigvee\{x \in Q \mid x \ll_K a\}$, for all $a \in Q$,

<u>K–algebraic</u> if $a = \bigvee\{x \in Q \mid x \ll_K x \leq a\}$, for all $a \in Q$, and

<u>K–coherent</u> if Q is K–algebraic, e is K–compact, and any product of K–compact elements is K–compact. Note, incidentally, that e is K–compact iff E_K is Scott open.

Proposition 2. For any transitive nuclear system K on a quantale Q, K(Q) is

(1) <u>continuous if Q is K–continuous,</u>

(2) <u>algebraic if Q is K–algebraic, and</u>

(3) <u>coherent if Q is K–coherent</u>.

Proof. Assume, to begin with, that Q is K–continuous. Our fist step is to show k_o is algebraic, that is, preserves all updirected joins. For any updirected $D \subseteq Q$ and $a \in K_{\bigvee D}$, if $x \ll_K a$ then $x \in K_t$ for some $t \in D$, hence $x \leq k_o(t)$ and therefore $x \leq \bigvee k_o[D]$. Thus, by K–continuity, $a \leq \bigvee k_o[D]$ and consequently $k_o(\bigvee D) \leq \bigvee k_o[D]$, the nontrivial part of the desired equality.

Next, we prove that $k_o = k$. By transitivity, if $x \in K_a$ then $K_x \subseteq K_a$ so that $k_o(x) \leq k_o(a)$, and hence $\bigvee k_o [K_a] \leq k_o(a)$. On the other hand, $k_o(\bigvee K_a) = \bigvee k_o[K_a]$ by the preceding result, and this means $k_o(k_o(a)) \leq k_o(a)$, making k_o idempotent and therefore equal to k.

Finally, we obtain that $k(x) \ll k(y)$ in K(Q) whenever $x \ll_K y$ in Q. Let $D \subseteq K(Q)$ be updirected with join in K(Q) above k(y), that is $k(y) \leq k(\bigvee D)$. Now, by the results above, $k(\bigvee D) = \bigvee k[D] = \bigvee D$, hence $y \leq \bigvee D$ so that $y \in K_{\bigvee D}$ and therefore $x \in K_t$ for some $t \in D$, showing that $x \leq k_o(t) = t$ and finally $k(x) \leq t$ since $t \in K(Q)$.

It now follows easily that K(Q) is continuous: for any $a \in K(Q)$, the fact that $a = \bigvee\{x \in Q \mid x \ll_K a\}$ in Q implies that $a = k(\bigvee\{k(x) \mid x \ll_K a\})$ while $x \ll_K a$ implies $k(x) \ll a$ in K(Q). The case of K–algebraic Q follows analogously, but for K–coherent Q some additional considerations are needed. To begin with, $e \ll_K e$ in Q implies $e \ll e$ in K(Q) by the earlier parts of this proof. Further, one readily sees that finite joins of K–compact elements in Q are K–compact, and hence for any compact $a \in K(Q)$ there exists a K–compact $b \in Q$ such that $a = k(b)$. Thus, if also $c = k(d)$ with K–compact

$d \in Q$ then $a \wedge c = k(b) \wedge k(d) = k(bd)$ is compact in Q since bd is K–compact by hypothesis.

Concerning the special case $K = R$, note that $x \ll_R y$ means that, for any updirected set D, $y^n \leq \bigvee D$ implies that $x^m \leq t$ for some m and some $t \in D$. The obvious explicit formulation of the corresponding results for the localic reflection $R(Q)$ of Q given by Proposition 2 is left to the reader. There is, however, a particular situation that might merit detailed mention.

Given that the difference between quantales and locales lies exactly in non–idempotency, a natural definition of the "way below" relation in a quantalic sense would be $x \ll_q y$ whenever $x^n \ll y^n$, for all n.

Based on this, call a quantale continuous (in the quantalic sense) if $a = \bigvee\{x \in Q \mid x \ll_q a\}$ for each $a \in Q$ and use the quantalic terms algebraic and coherent analogously. Note in particular that $c \ll_q c$ means that all powers c^n are compact.

Now, $x \ll_q y$ clearly implies $x \ll_R y$ in any quantale, and hence we have:

Corollary 1. R(Q) is continuous, algebraic, or coherent whenever Q has the corresponding quantalic property.

For the following, recall that a subset M of a complete lattice is called join dense if every element is a join of elements in M. Then we have, concerning coherence:

Corollary 2. R(Q) is coherent for any compact quantale Q which contains a join dense multiplicatively closed set of compact elements.

Proof. Let M be the set of compact elements in question. Then, for any compact $c \in Q$, $c = a_1 \vee \ldots \vee a_n$ with suitable $a_i \in M$, and if $d = b_1 \vee \ldots \vee b_m$ is another such element then $cd = \bigvee\{a_i b_j \mid i=1,\ldots,n;\ j=1,\ldots,m\}$ is again compact by hypothesis on M. It follows from this that Q is coherent in the quantalic sense, and hence $R(Q)$ is coherent by the preceding corollary.

The last result has a nice application to ring theory. For this, recall that, in a ring R, an element $a \in R$ is called quasicentral if $Ra = aR$, and that R is said to have dense quasicentre if every ideal of R is generated by quasicentral elements (van der Walt [10], Banaschewski– Harting [2]). Now, for quasicentral a in a ring R with unit, the principal two–sided ideal (a) generated by a is $Ra = aR$, and hence $(a)(b) = (ab)$ for any quasicentral $a,b \in R$ where ab is of course also quasicentral. This means that, for a ring R with unit and with dense quasicentre, the principal ideals generated by the quasi–central elements form a join dense multiplicatively closed set of compact elements in IdR. Hence we have:

Corollary 3. R(IdR) is coherent for any ring R with unit which has dense quasicentre.

Remark 1. Of course this corollary covers the familiar commutative case. The result as such is essentially due to Banaschewski–Harting [2] who proved that the locale L(IdR) of Levitzki radical ideals is coherent for these rings, but an easy argument shows that L(IdR) = R(IdR) in this situation. A direct proof was given by Sun [10].

Remark 2. It is known that, in general, R(IdR) need not be coherent, as discussed by Banaschewski–Harting [2]. On the other hand, it seems to be an open question whether it must always be algebraic.

Remark 3. A result similar to Proposition 2(1) for K = R, in the case of a commutative quantale Q whose top element is its multiplicative unit was obtained by Rosický [9].

Remark 4. Sun [10] calls a ring R (i) an m^*–ring if all powers of principal ideals are finitely generated, and (ii) an m–ring if the product of any two finitely generated ideal is finitely generated. Obviously, then, the quantale IdR is aglebraic, or coherent, in the quantalic sense whenever R is an m^*–ring, or an m–ring, respectively. Hence Corollary 1 implies the result of [10] that R(IdR) is algebraic, or coherent, if R is an m^*–ring, or an m–ring, respectively. For the more general situation of a compact quantale Q for which the top is the multiplicative unit, analogous conditions, with the corresponding results for R(Q), were also presented by Banaschewski–Harting [3].

Acknowledgements

This note was written during a visit to the Department of Mathematics, Applied Mathematics, and Astronomy at the University of South Africa in December 1992. The hospitality of my colleagues there, and in particular some stimulating conversations with Johannes Heidema, are gratefully acknowledged. Further thanks go to the Natural Science and Engineering Research Council for continuing research grant support.

References

1. B. Banaschewski, Another look at the localic Tychonoff Theorem. Comm. Math. Univ. Carolinae 29(1988), 647–656.

2. B. Banaschewski and R. Harting, Lattice aspects of radical ideals and choice principles. Proc. London Math. Soc. 50(1985), 385–404.

3. B. Banaschewski and R. Harting, V–rings. Unpublished manuscript, presented at the Category Theory conference, Murten, Switzerland, 1984.

4. B. Banaschewski and M. Erné, On Krull's separation lemma. Preprint 1993.

5. P. T. Johnstone, Stone spaces. Cambridge Studies in Advanced Mathematics no. 3, Cambridge University Press 1982.

6. S. B. Niefield, Localic and Boolean quotients of a quantale. Preprint 1992.

7. S. B. Niefield and K. Rosenthal, Constructing locales from quantales. Math. Proc. Cambridge Phil. Soc. 104 (1988), 215–234.

8. K. Rosenthal, Quantales and their applications. Pitman Research Notes in Mathematics 234, Longman Scientific and Technical Essex, 1990.

9. J. Rosický, Multiplicative lattices and frames. Acta Math. Hung. 49 (1987), 391–395.

10. S. H. Sun, On the least multiplicative nucleus on a ring. J. Pure Appl. Alg. 78 (1992), 221–236.

11. S. Vickers, Topology via logic. Cambridge University Press 1988.

12. A. P. J. van der Walt, Rings with dense quasicentre. Math. Z. 97(1967), 38–44.

13. E. Witt, Beweisstudien zum Satz von M. Zorn, Math. Nachr. 4(1951), 434–438.

RW/BB4/BB25/PG
February 15, 1994