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## A NEW PROOF OF REITERMAN'S THEOREM

by Grzegorz JARZEMBSKI

In [3] J.Reiterman introduced a concept of an implicit operation in order to get a very elegant finite analog of the Birkhoff variety theorem. In this note we analyse implicit operations from a categorical point of view. As a result we obtain a simple, purely categorical proof of Reiterman theorem.

### 1 Basic concepts

For all unexplained notions of category theory we refer the reader to [2].

We shall identify classes of objects of a category considered with full subcategories they generate. For every natural number  $n$ , by  $\tilde{n}$  we denote the set  $\{1, 2, \dots, n\}$ .

Let  $(\mathcal{A}, U : \mathcal{A} \rightarrow \text{Set})$  be an arbitrary concrete category over sets. For any natural number  $n$ , by an  $n$ -ary implicit operation in  $\mathcal{A}$  we mean any natural transformation  $\phi : U^n \rightarrow U$  [3].

Roughly speaking, an implicit operation is a family of functions  $(\phi_A : U^n A \rightarrow UA : A \in \text{Ob}\mathcal{A})$  "compatible" with all  $\mathcal{A}$ -morphisms.

By  $IO(\mathcal{A})_n$  we shall denote the class of all  $n$ -ary implicit operations in  $\mathcal{A}$ .

**Definition 1** Any pair  $(\phi, \psi)$  of  $n$ -ary implicit operations in  $\mathcal{A}$  is called here an "equation"; (notation  $(\phi = \psi)$ ).

We say that an equation  $(\phi = \psi)$  is satisfied in  $B \in \text{Ob}\mathcal{A}$  iff  $\phi_B = \psi_B$ .

For a given set of equations  $E$ , by  $\text{Mod}E$  we denote the full subcategory of  $\mathcal{A}$  consisting of  $\mathcal{A}$ -objects satisfying all equations in  $E$ .

We call a full subcategory  $\mathcal{D} \subset \mathcal{A}$  equationally definable iff  $\mathcal{D} = \text{Mod}E$  for some set of equations  $E$ .

The following observation needs only a routine verification:

**Lemma 1** Every equationally definable full subcategory  $\mathcal{D} \subset \mathcal{A}$  satisfies the following:

1.  $\mathcal{D}$  is closed under formation of all existing concrete limits,
2. whenever  $A \in \mathcal{D}$ ,  $h : A \rightarrow B$  and  $Uh$  is a surjection, then  $B \in \mathcal{D}$  ( $\mathcal{D}$  is closed under formation of homomorphic images),
3. whenever  $A \in \mathcal{D}$ ,  $m : B \rightarrow A$  and  $Um$  is a monomorphism, then  $B \in \mathcal{D}$  ( $\mathcal{D}$  is closed under formation of subobjects).

J. Reiterman has proved the converse of Lemma 1 for equationally definable classes of finite algebras of an arbitrary finite type  $\Omega$  ([3]). His result has been next generalized by Banaschewski ([1]) for arbitrary finitary types. In this paper we present a new proof of Reiterman theorem based on a categorical analysis of the concept of an implicit operation.

## 2 Two restrictions

**Definition 2** A concrete category  $(\mathcal{A}, U)$  is said to be small-based provided for every natural number  $n$   $IO(\mathcal{A})_n$  is a set.

Let  $(\tilde{n} \downarrow U)$  denote the category of all  $U$ -morphisms with a domain  $\tilde{n}$ . Let

$$U_n : (\tilde{n} \downarrow U) \rightarrow \text{Set}$$

be the "forgetful functor", i.e.  $U_n(h : \tilde{n} \rightarrow UA, A) = UA$ , for every  $(h, A)$  in  $(\tilde{n} \downarrow U)$ .

**Lemma 2** A concrete category  $(\mathcal{A}, U)$  is small-based if and only if for every natural number  $n$  there exists a limit of the functor  $U_n$  and then

$$IO(\mathcal{A})_n \cong \lim U_n$$

**Proof.** A limit of  $U_n$  is formed by the set  $IO(\mathcal{A})_n$  together with limit projections  $\pi_{(h,A)} : IO(\mathcal{A})_n \rightarrow U_n(h, A) = UA$  such that

$$\pi_{(h,A)}(\phi) = \phi_A(h)$$

for every  $(h, A) \in (\tilde{n} \downarrow U)$  and  $\phi \in IO(\mathcal{A})_n$ .

**Definition 3** We say that a concrete category  $(\mathcal{A}, U)$  has enough implicit operations iff for every  $A, B \in \mathcal{A}$  and  $h : UA \rightarrow UB$ ,  $h = Ug$  for some  $\mathcal{A}$ -morphism  $g : A \rightarrow B$  provided  $h$  preserves all implicit operations ( $h \cdot \phi_A = \phi_B \cdot h^n$  for every  $n \in N$  and  $\phi \in IO(\mathcal{A})_n$ ).

**Lemma 3** Assume that a  $(\mathcal{A}, U)$  is a concrete, small-based with enough implicit operations. Then there exists a (finitary) monad  $T$  over  $Set$  such that the following conditions are satisfied:

1.  $(\mathcal{A}, U)$  is concretely isomorphic to a full subcategory of the Eilenberg-Moore category  $Set^T$ ,
2. for every  $n \in N$ ,  $IO(\mathcal{A})_n \cong \lim U_n$  is a carrier of a finitely generated monadic  $T$ -algebra such that every limit projection is a  $T$ -morphism.

Proof. Since  $(\mathcal{A}, U)$  is small-based, the family of all implicit operations form a finitary type:  $IO = (IO(\mathcal{A})_n : n \in N)$ .

Let  $T$  be a monad over  $Set$  such that its Eilenberg-Moore category  $Set^T$  is the category of all  $IO$ -algebras. Since the concrete category considered has enough implicit operations, it is concretely isomorphic to a full subcategory of  $T$ -algebras.

In the considered case the functor  $U_n$  factorizes through the forgetful functor  $U^T : Set^T \rightarrow Set$  in an obvious way.  $U^T$  creates limits, hence, by Lemma 2, every set  $IO(\mathcal{A})_n$  carries a structure of a monadic  $T$ -algebra such that every limit-projection  $\pi_{(h, \mathcal{A})} : IO(\mathcal{A})_n \rightarrow A$  is a homomorphism.

Obviously, every  $IO(\mathcal{A})_n$  is then a finitely generated algebra of the type  $IO$  - it is generated by the set of natural projections  $\{\pi_i^n : U^n \rightarrow U : i = 1, 2, \dots, n\}$ .

It must be stressed however that a monad  $T$  satisfying the conditions of Lemma 3 is not uniquely determined. It need not to be finitary too. An embedding  $\mathcal{A} \rightarrow Set^T$  will be used only in order to support an investigation on equationally definable subcategories of  $\mathcal{A}$ . As we will show later, in particular cases a "good" choice of a monad  $T$  is crucial.

### 3 Characterization of equationally definable subcategories

Assume that a concrete category  $(\mathcal{A}, U)$  is a small-based category with enough implicit operations. Let  $T = (T, \mu, \eta)$  be an arbitrary but fixed monad over  $Set$  such that the conditions stated in Lemma 3 are satisfied. We shall assume that  $\mathcal{A}$  is a full subcategory of the Eilenberg-Moore category  $Set^T$ . The  $T$ -algebra of  $n$ -ary implicit operations will be denoted by  $IO(\mathcal{A})_n$ .

Let  $\pi^n : \tilde{n} \rightarrow IO(\mathcal{A})_n$  be a function such that for every  $i \in \tilde{n}$ ,  $\pi^n(i) = \pi_i^n : U^n \rightarrow U =$  the  $i$ -th natural projection.

We assume also that the extension of  $\pi^n$  to the  $T$ -morphism  $(T\tilde{n}, \mu_{\tilde{n}}) \rightarrow IO(\mathcal{A})_n$  is surjective for every natural number  $n$ .

**Definition 4** We call a  $T$ -algebra  $D$  a  $T_{\mathcal{A}}$ -algebra iff for every  $n \in N$  and a function  $f : \tilde{n} \rightarrow U^T D$  there exists a (unique)  $T$ -morphism  $f^* : IO(\mathcal{A})_n \rightarrow D$  such that

$$f^* \cdot \pi^n = f$$

We shall call  $f^*$  an extension of  $f$ .

Clearly,  $\mathcal{A}$  is a subclass of  $T_{\mathcal{A}}$ -algebras: for every  $B \in Ob\mathcal{A}$  and  $h : \tilde{n} \rightarrow UB$  its extension is the limit-projection  $\pi_{(h, \mathcal{A})} : IO(\mathcal{A})_n \rightarrow B$  (Lemma 2).

Notice also that every  $IO(\mathcal{A})_n$  is a  $T_{\mathcal{A}}$ -algebra. This is a consequence of the fact that the class of  $T_{\mathcal{A}}$ -algebras is closed under formation of limits. It is easily checked that the class of  $T_{\mathcal{A}}$ -algebras is closed within  $Set^T$  under formation of homomorphic images and subalgebras, i.e. the class of  $T_{\mathcal{A}}$ -algebras is the "Birkhoff subcategory generated by  $\mathcal{A}$ " within  $Set^T$ .

**Definition 5** For each  $\phi \in IO(\mathcal{A})_n$  and a  $T_{\mathcal{A}}$ -algebra  $D$  we define an  $n$ -ary operation in  $D$ ,  $\phi_D : (U^T D)^n \rightarrow U^T D$  as follows:

$$\phi_D(h) = h^*(\phi)$$

for every  $h : \tilde{n} \rightarrow U^T D$ .

Thus  $h^*$  gives "values of all  $n$ -ary implicit operations in  $D$  at the valuation  $h$ ".

It is easily checked that  $T$ -morphisms between  $T_{\mathcal{A}}$ -algebras preserve these operations.

Having all implicit operations extended to all  $T_{\mathcal{A}}$ -algebras we extend also a notion of satisfaction for equations of implicit operations:

**Definition 6** *Let  $\phi, \psi \in IO(\mathcal{A})_n$ . A  $T_{\mathcal{A}}$ -algebra  $A$  satisfies the equation  $(\phi = \psi)$  iff for every  $h : \tilde{n} \rightarrow U^T A$ ,*

$$h^*(\phi) = h^*(\psi)$$

If  $A \in Ob\mathcal{A}$  and  $h : \tilde{n} \rightarrow UA$  then  $h^*(\phi) = \pi_{(h,A)}(\phi) = \phi_A(h)$ , i.e. the newly defined concept of satisfaction and that of Definition 1 coincide for  $\mathcal{A}$ -objects.

For any class  $\mathcal{B}$  of  $T_{\mathcal{A}}$ -algebras we write  $H(\mathcal{B})$ ,  $S(\mathcal{B})$ ,  $P(\mathcal{B})$  in order to denote the class of all homomorphic images, subalgebras and products of the algebras from  $\mathcal{B}$ , resp.

We call a class  $\mathcal{B}$  an *HSP-class* iff  $\mathcal{B} = HSP(\mathcal{B})$ .

Since we deal with finitary implicit operations only, we shall use one more closure operator. In what follows by  $C(\mathcal{B})$  we shall denote the class of all  $T_{\mathcal{A}}$ -algebras with all finitely generated subalgebras in  $\mathcal{B}$  (a monadic  $T$ -algebra  $A$  is said to be *finitely generated* provided it is isomorphic to a quotient of a free monadic  $T$ -algebra  $(T\tilde{n}, \mu_{\tilde{n}})$  for some natural number  $n$ ).

**Theorem 4** *For an arbitrary class  $\mathcal{B}$  of  $T_{\mathcal{A}}$ -algebras the following conditions are equivalent:*

1.  $\mathcal{B}$  is an HSP-class and  $\mathcal{B} = C(\mathcal{B})$
2. There exists a set  $E$  of equations (of implicit operations) such that  $\mathcal{B}$  consists of all  $T_{\mathcal{A}}$ -algebras satisfying all equations in  $E$ .

Proof. 2.  $\Rightarrow$  1. The equations  $\mathcal{B} = P(\mathcal{B}) = H(\mathcal{B}) = S(\mathcal{B})$  need only a routine verification.

For every  $T_{\mathcal{A}}$ -algebra  $B$  and a function  $h : \tilde{n} \rightarrow U^T B$  its extension  $h^*$  factorizes as a surjective  $T$ -morphism followed by a monomorphism:

$$h^* = m \cdot e : IO(\mathcal{A}_n) \rightarrow D_h \rightarrow B$$

Clearly,  $D_h$  is a finitely generated  $T_{\mathcal{A}}$ -algebra. Hence a  $T_{\mathcal{A}}$ -algebra  $B$  satisfies an equation  $(\phi = \psi)$  iff every finitely generated subalgebra of  $B$  satisfies it.

This proves  $C(\mathcal{B}) \subset \mathcal{B}$ .

1.  $\Rightarrow$  2. Since  $\mathcal{B}$  is closed under products and subalgebras, for every natural number  $n$  there exists a "reflection map" - a surjective  $T$ -morphism  $e_n : IO(\mathcal{A})_n \rightarrow B_n$  such that  $B_n \in \mathcal{B}$  and every morphism of  $T$ -algebras  $h : IO(\mathcal{A})_n \rightarrow A$  with  $A \in \mathcal{B}$  factorizes through  $e_n$ . Let

$$E_n = \ker(e_n) = \{(\phi, \psi) : \phi, \psi \in IO(\mathcal{A})_n, e_n(\phi) = e_n(\psi)\}$$

and

$$E = \bigcup (E_n : n \in N)$$

We prove that  $E$  is a set of equations we are looking for.

Clearly, each  $T_{\mathcal{A}}$ -algebra in  $\mathcal{B}$  satisfies all equations in  $E$ .

Assume that  $A$  is a finitely generated  $T_{\mathcal{A}}$ -algebra satisfying all equations in  $E$ .  $A$  is finitely generated hence there exists a surjective  $T$ -morphism  $f : IO(\mathcal{A})_n \rightarrow A$  for some  $n \in N$ . Since  $A$  satisfies all equations in  $E$ , we obtain  $E_n \subset \ker f$ . Hence  $f = f^\circ \cdot e_n$  for some  $T$ -morphism  $f^\circ$ .  $f$  is surjective, hence  $f^\circ$  is a surjective morphism, too.

Thus  $A \in H(\mathcal{B}) = \mathcal{B}$ .

An arbitrary  $T_{\mathcal{A}}$ -algebra  $B$  satisfies all equations from  $E$ , iff all its finitely generated subalgebras satisfy all these equations, i.e. if all those subalgebras are in  $\mathcal{B}$ . This means  $B \in C(\mathcal{B}) = \mathcal{B}$ .

The proof is complete.

**Corollary 5** . *For any class  $\mathcal{B}$  of  $T_{\mathcal{A}}$ -algebras, the class  $CHSP(\mathcal{B})$  is the smallest equationally definable class of  $T_{\mathcal{A}}$ -algebras containing  $\mathcal{B}$ .*

Proof. Observe that  $HC(\mathcal{B}) \subset CH(\mathcal{B})$  and  $SPC(\mathcal{B}) \subset CSP(\mathcal{B})$  for any class of  $T_{\mathcal{A}}$ -algebras  $\mathcal{D}$ . We omit a routine calculation.

The next Corollary summarizes investigation of this section.

**Corollary 6** *For any class  $\mathcal{D} \subset \mathcal{A}$ , the following conditions are equivalent*

1.  $\mathcal{D}$  is an equationally definable class,
2.  $\mathcal{D} = CHSP(\mathcal{D}) \cap \mathcal{A}$

## 4 A new proof of Reiterman theorem

Consider now equationally definable classes in the concrete category  $Alg_{fin}\Omega$  of all finite algebras of a given finitary type  $\Omega$ . We are going to give a new proof of Reiterman characterization theorem based on the results of the previous section. Our proof also covers the generalization of Reiterman theorem given by Banaschewski ([1]).

Since  $Alg_{fin}\Omega$  is a small category, it is small-based. It follows from Lemma 2 that in this case the set of  $n$ -ary implicit operations is represented as a limit of the poset of all finite quotients of a free  $\Omega$ -algebra generated by the set  $\tilde{n} = \{1, 2, \dots, n\}$ . From this it easily follows that every implicit operation  $\phi$  is "locally explicit" - i.e., for every finite algebra  $A$  there is a term  $t$  with  $t_A = \phi_A$  ([3]).

Obviously, the concrete category  $Alg_{fin}\Omega$  has enough implicit operations.

For the category  $Alg_{fin}\Omega$  we may consider three (at least) concrete full embeddings of it into the following Eilenberg-Moore categories over sets:

- the category  $Alg\Omega$  of all  $\Omega$ -algebras,
- the category  $AlgIO$ , where  $IO$  is the type built of all implicit operations in  $Alg_{fin}\Omega$ ,
- the category of all compact Hausdorff  $IO$ -algebras and continuous homomorphisms (by endowing each finite algebra with the discrete topology).

In our proof of Reiterman theorem we shall use the third embedding.

Recall that by a *pseudovariety* of finite  $\Omega$ -algebras we mean each class of finite algebras closed under formation of finite products, subalgebras and homomorphic images.

### **Theorem 7** (*Reiterman theorem*)

*For any class  $\mathcal{D} \subset Alg_{fin}\Omega$  the following conditions are equivalent:*

1.  $\mathcal{D}$  is a pseudovariety,
2.  $\mathcal{D} = ModE$  for some set of equations of implicit operations of finite algebras.

*Proof.* Each  $ModE$  is clearly a pseudovariety.

Conversely, let  $D$  be a pseudovariety. To find  $E$  with  $D = ModE$  consider the monad  $T$  over  $Set$  whose Eilenberg-Moore category is the category of all compact Hausdorff  $IO$ -algebras (and continuous homomorphisms).

Clearly,  $Alg_{fin}\Omega$  is a full concrete subcategory of  $Set^T$  and for every natural  $n$ , the set of  $n$ -ary implicit operations in  $Alg_{fin}\Omega$  is a carrier of a finitely generated monadic  $T$ -algebra (Lemma 3).

By Corollary 6, it is enough to prove that  $\mathcal{D}$  contains every finite algebra  $C$  lying in  $HSP(\mathcal{D}) \subset Set^T$ . That is, there is a continuous and surjective homomorphism  $e : B \rightarrow C$ , where  $B$  is a compact Hausdorff  $IO$ -algebra, and  $B$  is a closed subalgebra of  $\prod_{i \in I} A_i$  for a collection  $(A_i : i \in I) \subset \mathcal{D}$ .

We may assume that every  $A_i$  has the form  $A_i = B/\rho_i$  for some congruence  $\rho_i$  on  $B$ . Then

$$\bigcap (\rho_i : i \in I) = \nabla_B$$

where  $\nabla_B$  denotes the diagonal in  $B \times B$ .

Every congruence  $\rho_i$  has a finite index and it is a kernel of a continuous projection  $pr_i : B \rightarrow A_i$ , hence it is a closed subset of the compact space  $B \times B$ .

The epimorphism  $e : B \rightarrow C$  is continuous, the diagonal  $\nabla_C$  is open in the compact space  $C \times C$ , hence the inverse image  $ker(e) = (e \times e)^{-1}(\nabla_C)$  is an open subset of  $B \times B$  containig  $\nabla_B$ . Hence

$$\bigcap (\rho_i : i \in I) = \nabla_B \subset ker(e)$$

Since  $B \times B$  is compact, there exists a finite subset  $I_1 \subset I$  such that

$$\bigcap (\rho_i : i \in I_1) \subset ker(e)$$

Clearly, the congruence  $\bigcap (\rho_i : i \in I_1)$  has a finite index. Thus  $B_0 = B/\bigcap (\rho_i : i \in I_1)$  is a finite algebra and  $C$  is a homomorphic image of  $B_0$ .

Moreover,  $B_0$  is a subalgebra of the product of the family of finite algebras  $\{A_i : i \in I_1\}$ .

Hence  $C \in \mathcal{D}$ .

The proof is complete.

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