CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 35, nº 3 (1994), p. 249-261

<http://www.numdam.org/item?id=CTGDC_1994__35_3_249_0>

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Volume XXXV-3 (1994)

RELATIVELY BOOLEAN AND DE MORGAN TOPOSES AND LOCALES

by Anders KOCK and Gonzalo E. REYES

Résumé Nous présentons une notion de topos booléen rélative a un topos de base. Si le topos de base est booléen dans le sens standard, la notion se reduit a la notion booléen standard. La notion rélative nous permet de retenir le théorème de structure pour les topos booléens de prefaisceaux, sans hypothèse sur le topos de base. De même pour une notion de topos deMorgan. Nous considerons de plus ces notions pour les topos localiques.

Introduction

The motivation for the present work was partly the topos theoretic analysis of modal operators - notably the question of naturality of the possibility operator, this question being related to de Morgan's law [10] -; and partly the feeling that the two well known results

- a presheaf category $\widehat{\mathbf{C}}$ is Boolean iff $\mathbf{C}$ is a groupoid;
- a presheaf category $\widehat{\mathbf{C}}$ is de Morgan iff $\mathbf{C}$ satisfies the Ore condition

are too good to be abandoned if one abandons the hypothesis that the base topos is Boolean. We introduce relative notions of 'Boolean' and 'de Morgan' which make these two results valid over any base topos (see Section 3). The notions are, of course, lattice theoretic, so deal with *frames* in toposes; this leads us to consider the question when the topos of sheaves on a locale (or frame) is relatively Boolean over the base topos, see Section 4.

In so far as the frame theory is concerned, we have benefited from correpondence between the first author and M. Jibladze and P. T. Johnstone, arising partly as comments on [8]. In particular, Jibladze gives (March 1990) a list of frame theoretic conditions, all of which reduce to the condition of being a Boolean algebra if the base topos is boolean. Thus, the beautiful generalisation of the law of excluded middle $1 = a \vee \neg a$, namely

$$1 = \bigvee_{\lambda} (a \leftrightarrow \lambda)$$

(λ ranging over the set Ω of truth values), we learned from him.

For *open* geometric morphisms, our notion of relatively Boolean, respectively relatively de Morgan, geometric morphism $\gamma : \underline{E} \to \underline{S}$ can be stated: the canonical

$$\gamma^* \Omega_{\underline{S}} \to \Omega_{\underline{E}},$$

(monic, by openness) is an isomorphism, respectively has a left adjoint. The notion of 'relatively Boolean' in [8] is weaker than the one introduced in the present paper; in fact it is precisely the de Morgan condition which upgrades the former to the latter, see remarks after Definition 1.1.

1 Clopen and regular elements in a frame

Let A be a frame. Given a map $\tau : D \rightarrow A$ to A from an arbitrary set D, we consider the following two subsets of A (omitting τ from notation; in the applications, τ will often be the inclusion of a subset),

$$Clp(A) := \{a \in A \mid 1_A = \bigvee_{\lambda \in D} (a \leftrightarrow \lambda)\}$$

$$Reg(A) := \{a \in A \mid a = \bigwedge_{\lambda \in D} (a \rightarrow \lambda) \rightarrow \lambda\}.$$

In case where A is the frame of open sets of a topological space, and D is the subset $\{0_A, 1_A\}$, $Clp(A)$ and $Reg(A)$ consist of, respectively, the *clopen* and the *regular open* subsets, respectively.

We shall apply these notions in two situations: the first, as in [1], where τ is the unique frame map $\Omega \rightarrow A$; the second where $\gamma : \underline{E} \rightarrow \underline{S}$ is a geometric morphism, and τ is the canonical comparison map $\gamma^*\Omega_{\underline{S}} \rightarrow \Omega_{\underline{E}}$which classifies the monic $\gamma^*(true) : \gamma^*1 \rightarrow \gamma^*\Omega_{\underline{S}})$.

As in topology, clopen implies regular open:

Proposition 1.1 *For any $\tau : D \rightarrow A$, $Clp(A) \subseteq Reg(A)$.*

Proof. Let $a \in Clp(A)$, so $1_A = \bigvee_\lambda (a \leftrightarrow \lambda)$ where λ ranges over D. We should prove $\bigwedge_\lambda (a \rightarrow \lambda) \rightarrow \lambda \leq a$ (the other inequality always holds). It suffices to see that for any b with $b \leq (a \rightarrow \lambda) \rightarrow \lambda$ for all $\lambda \in D$ we have $b \leq a$. The assumption on b may be reformulated

$$b \wedge (a \rightarrow \lambda) \leq \lambda \ \forall \lambda. \tag{1}$$

Since $1_A = \bigvee_\lambda (a \leftrightarrow \lambda)$, b is covered by the family $\{b \wedge (a \leftrightarrow \lambda) \mid \lambda \in D\}$, so it suffices to prove that $b \wedge (a \leftrightarrow \lambda) \leq a$. But

$$\begin{aligned} b \wedge (a \leftrightarrow \lambda) &= b \wedge (a \rightarrow \lambda) \wedge (a \leftrightarrow \lambda) \\ &\leq \lambda \wedge (a \leftrightarrow \lambda) \\ &\leq \lambda \wedge (\lambda \rightarrow a) \leq a, \end{aligned}$$

using (1) for the first inequality. This proves the Proposition.

It is well known that for fixed $d \in A$, $(- \rightarrow d) \rightarrow d$ is a nucleus on the frame A. Hence the (pointwise) meet

$$\bigwedge_{\lambda \in D} (- \rightarrow \lambda) \rightarrow \lambda \tag{2}$$

is a nucleus L_D. If $D \subseteq A$, it may be characterized as the largest nucleus (under pointwise order) which fixes D; see [5]. Note that $Reg(A)$ is by definition the set $Fix(L_D)$ of fixpoints for the nucleus L_D, and that the inclusion $Reg(A) \subseteq A$ therefore has a left exact left adjoint.

Proposition 1.2 *For any $\tau : D \to A$, the following conditions are equivalent:*

1. *$Clp(A) \subseteq A$ has a left exact left adjoint (which is then necessarily given by (2))*
2. *$Clp(A) = Reg(A)$*
3. *For all $a \in A$,*

$$1_A = \bigvee_\lambda [\bigwedge_\mu (a \to \mu) \to \mu] \leftrightarrow \lambda$$

(λ and μ ranging over D,)

Proof. Assume 1. Combining the assumed left adjoint with the inclusion $Clp \subseteq A$, we get a nucleus P on A with Clp as its fixpoint set. (We omit A from notation.) Since $Clp \subseteq Reg$, by the previous Proposition, we have the opposite inequality for the corresponding nuclei, so $L_D \leq P$. To see $L_D = P$ (which implies 2.), it thus suffices to see that $P \leq L_D$. Since L_D is the largest nucleus fixing (the image under τ of) D, it suffices to see that P fixes that image, that is, to see that $\tau(\mu) \in Clp$ for every $\mu \in D$. But (omitting τ from notation),

$$\bigvee_\lambda (\mu \leftrightarrow \lambda) \geq \mu \leftrightarrow \mu = 1_A,$$

whence μ belongs to Clp. Conversely, since $Reg \subseteq A$ has a left exact left adjoint (given by the nucleus L_D), 2. implies 1.

Now assume 2. Since $\bigwedge_\mu (a \to \mu) \to \mu \in Reg$, it belongs also to Clp, by assumption, so 3. holds; conversely if 3. holds, every element of form $\bigwedge_\mu (a \to \mu) \to \mu$ is in Clp; but every element of Reg is of this form, so $Reg \subseteq Clp$, hence by Proposition 1.1, $Reg = Clp$.

It is well known that a Heyting algebra where every element a is regular (in the classical sense, $a = \neg\neg a$), has the property that every element is complemented, $1 = a \vee \neg a$, i.e. is a Boolean algebra. A *Stone* algebra (cf. e.g. [2]), or *de Morgan* algebra, is a Heyting algebra where every $\neg a$ is complemented, $1 = \neg a \vee \neg\neg a$. This is equivalent to saying that $\neg\neg$ commutes with finite joins, cf loc.cit. These notions and results, being purely finitary-algebraic, make sense for Heyting algebras in an arbitrary topos $\underline{S}$. We may refer to them as the ***absolute*** notions. However, for the reasons explained in the introduction, we want relative notions that take the full set $\Omega_{\underline{S}}$ of truth values into account. Since $\Omega_{\underline{S}}$ is not in general finite, the relative notions are best formulated for *complete* Heyting algebras, i.e. for frames.

Let A be a frame in a topos $\underline{S}$, and $\tau : \Omega_{\underline{S}} \to A$ the unique frame map. We now let the the notions Clp and Reg refer to this map. We then pose

Definition 1.1 *Let A be a frame in* $\underline{S}$. *We say that*

1. *A is* relatively Boolean *if* $A = Clp$;
2. *A is* pre-boolean *if* $A = Reg$;
3. *A is* relatively de Morgan *if* $Reg = Clp$.

Essentially, 1) was considered by Jibladze and Johnstone, and 2) was considered in [8] under the name 'relatively Boolean'. Since by Proposition 1, $Clp \subseteq Reg$, it is clear that 1. is at least as strong as 2. Jibladze has given a simple example (with $\underline{S}$ = Sierpinski topos) showing that 1. is strictly stronger than 2. Finally it is clear that 2. and 3. together are equivalent to 1. If $\Omega_{\underline{S}} = 1 + 1$ (i.e. $\underline{S}$ is a Boolean topos), 2. is equivalent to 1., and the frames satisfying this condition are exactly the Boolean algebras in the absolute sense; and the frames satisfying 3. are the de Morgan algebras in the absolute sense.

2 Notions relative to a geometric morphism

We consider a geometric morphism $\gamma : \underline{E} \to \underline{S}$. Recall that it gives rise to a comparison map $\tau : \gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$. If γ is open, τ is monic (cf e.g. [4]); in any case, it may be safely omitted from notation. The object $\Omega_{\underline{E}}$ is a frame in $\underline{E}$; $\gamma^*\Omega_{\underline{S}}$ will in general only be a Heyting algebra; and τ will be a lattice homomorphism, in particular, it will preserve complements.

We now apply our general considerations to the frame $\Omega_{\underline{E}}$, and with $\gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$ as our $\tau : D \to A$. As long as we reason entirely in in $\underline{E}$, we may reason as if $\underline{E}$ were Sets (constructively). But because $\Omega_{\underline{E}}$ now is the frame of truth values, some principles hold which do not hold for frames in general; such principles are crucial in the proofs of Propositions 2.1 and 2.2 below. In analogy with Definition 1.1, we pose

Definition 2.1 *Let* $\gamma : \underline{E} \to \underline{S}$ *be a geometric morphism. We let* $Clp \subseteq \Omega_{\underline{E}}$ *and* $Reg \subseteq \Omega_{\underline{E}}$ *refer to the comparison map* $\tau : \gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$. *Then we say*

1. $\underline{E}$ *is* relatively Boolean *over* $\underline{S}$ *if* $\Omega_{\underline{E}} = Clp$;
2. $\underline{E}$ *is* pre-Boolean *over* $\underline{S}$ *if* $\Omega_{\underline{E}} = Reg$;
3. $\underline{E}$ *is* relatively de Morgan *over* $\underline{S}$ *if* $Reg = Clp$.

Just as in the remarks after Definition 1.1, we see that 1. holds iff 2. and 3. hold.

We shall relate these notions to the absolute notions in Propositions 2.4 and 2.5 below. Let us first note that *Reg* and *Clp* are defined as subobjects of $\Omega_{\underline{E}}$ by interpretation of certain logical formulae, involving $\bigwedge_\lambda$ and $\bigvee_\lambda$ with λ ranging over $\gamma^*\Omega_{\underline{S}}$, thus they are not finitary, and thus not easy to work with. But at least for *Clp*, we can avoid logic, since we have

Proposition 2.1 *The subobject $Clp \subseteq \Omega_{\underline{E}}$ equals the image of $\gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$. More generally, for any $\tau : D \to \Omega_{\underline{E}}$ in a topos $\underline{E}$, the extension of the formula (with free variable a ranging over $\Omega_{\underline{E}}$),*

$$true = \bigvee_{\lambda \in D} (a \leftrightarrow \tau(\lambda)) \tag{3}$$

equals the image of τ.

Proof. Let us write Ω for $\Omega_{\underline{E}}$. The element $true \in \Omega$ is 'inaccessible by sup' in the sense that $true = \sup U$ (for $U \subseteq \Omega$) implies $true \in U$; for, for any subset $U \subseteq \Omega$, $\sup U$ equals the truth value of the statement $true \in U$, see e.g. [9] Corollary 2.5. Thus the formula (3) is equivalent to the formula

$$\exists \lambda \in D \; : \; true = (a \leftrightarrow \tau(\lambda))$$

and then again to

$$\exists \lambda \in D \; : \; a = \tau(\lambda),$$

whose extension clearly is just the image of τ.

Another example of the special properties of the frame Ω appears in the (probably well known)

Proposition 2.2 *Any closure operator P on Ω is left exact, i.e. is a nucleus.*

Proof. We first prove that

$$a \wedge P(b) \leq P(a \wedge b) \tag{4}$$

for all $a, b \in \Omega$. Since we are dealing with truth values, it suffices to see that if the left hand side is *true*, then so is the right hand side. So assume $a \wedge P(b)$ is true. Then a is true, and $P(b)$ is true. Since a is true, $b = a \wedge b$, so $P(a \wedge b) = P(b) = true$. This proves the inequality (4). To see $P(c) \wedge P(b) \leq P(c \wedge b)$, use instances of (4) twice,

$$P(c) \wedge P(b) \leq P(P(c) \wedge b) \leq P(P(c \wedge b)) = P(c \wedge b).$$

(Alternatively, the inequality (4) furnishes the monad P with a tensorial strength; and strong monads carry monoidal structure, by [7]). The other inequality is clear, and likewise $P(1) = 1$ (where $1 = true$). (Note that we need not assume $a \leq P(a)$; this is automatic from $P(1) = 1$.)

The notions of Definition 2.1 should perhaps be combined with the assumption that we are dealing with an *open* geometric morphism γ. (Recall [4] that if the codomain of γ is an (absolutely) Boolean topos, then γ is automatically open.) Recall from [3] the comparison map $\gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$ for an open geometric morphism is monic (this latter property of γ is called "sub-open" in [3]). Thus we get from Proposition 2.1 (using also Proposition 1.2, for the de Morgan case):

Proposition 2.3 *Let $\gamma : \underline{E} \to \underline{S}$ be an open geometric morphism (actually sub-open suffices). Then $\underline{E}$ is relatively Boolean (respectively relatively de Morgan) over $\underline{S}$ iff the canonical comparison $\gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$ is an isomorphism (respectively: has a left adjoint).*

Also, we have easily

Proposition 2.4 *Let $\gamma : \underline{E} \to \underline{S}$ be a geometric morphism with $\underline{S}$ an (absolutely) Boolean topos. Then $\underline{E}$ is relatively Boolean over $\underline{S}$ iff $\underline{E}$ is absolutely Boolean. And $\underline{E}$ is relatively de Morgan over $\underline{S}$ iff $\underline{E}$ is absolutely de Morgan.*

Proof. By assumption, $\Omega_{\underline{S}} = 1 + 1$, so also $\gamma^*\Omega_{\underline{S}} = 1 + 1$. The formulae defining, respectively, Clp and Reg as subobjects of $\Omega_{\underline{E}}$ involve therefore only a binary join, respectively meet,

$$Clp = \{a \in \Omega_{\underline{E}} \mid true = a \vee \neg a\}$$

$$Reg = \{a \in \Omega_{\underline{E}} \mid a = (true \wedge \neg\neg a)\} = \{a \in \Omega_{\underline{E}} \mid a = \neg\neg a\}.$$

So $\underline{E}$ is relatively Boolean iff $Clp = \Omega_{\underline{E}}$ iff $true = a \vee \neg a$ holds as an identity for $\Omega_{\underline{E}}$, which is the case iff $\underline{E}$ is a Boolean topos in the absolute sense. And $\underline{E}$ is relatively de Morgan iff $Reg \subseteq Clp$ iff $true = \neg\neg a \vee \neg\neg\neg a (= \neg a)$ holds as an identity for $\Omega_{\underline{E}}$, which is the case iff $\Omega_{\underline{E}}$ is a de Morgan algebra, thus $\underline{E}$ is a de Morgan topos in the absolute sense.

We finally prove

Proposition 2.5 *Let $\gamma : \underline{E} \to \underline{S}$ be a geometric morphism, and assume that $\underline{S}$ is absolutely de Morgan. If $\underline{E}$ is relatively de Morgan over $\underline{S}$, it is absolutely de Morgan.*

Proof. Since Clp by Proposition 2.1 is the image of a lattice homomorphism, it is a sublattice of $\Omega_{\underline{E}}$. And since $\Omega_{\underline{S}}$ is a Stone algebra, then so is $\gamma^*\Omega_{\underline{S}}$ and hence also its quotient Clp. On the other hand, since $Clp = Reg$, Clp is the fixpoint set for a nucleus L on $\Omega_{\underline{E}}$, and since it is a sublattice of $\Omega_{\underline{E}}$, this nucleus preserves finite joins. The result then follows from the following purely equational

Lemma 1 *Let L be a nucleus on a Heyting algebra, preserving finite joins. Then if its fixpoint set A_L is a Stone algebra, then so is A itself.*

Proof. Since $L(0_A) = 0_A$, $0_A \in A_L$. Since $\neg\neg$ is the largest nucleus on A that fixes 0_A, $L \leq \neg\neg$, so $A_{\neg\neg} \subseteq A_L$. Since A_L is a sublattice of A, the inclusion preserves complements, so the double negation nucleus N for the Heyting algebra A_L is the restriction of the nucleus $\neg\neg$ on A. So, first, $(A_L)_N \subseteq A_{\neg\neg}$; but also $A_{\neg\neg} \subseteq (A_L)_N$ since the values of $\neg\neg$ are in A_L and $\neg\neg = \neg\neg\neg\neg$. The assumption that A_L is a Stone algebra gives that the inclusion $(A_L)_N \subseteq A_L$, and hence $A_{\neg\neg} \subseteq A_L$, preserves finite joins; and $A_L \subseteq A$ preserves finite joins by assumption on the nucleus L. Composing, we get that the inclusion $A_{\neg\neg} \subseteq A$ preserves finite joins, and so A is a Stone algebra.

3 Presheaf toposes

Let $\mathbf{C}$ be a category object in a topos $\underline{S}$. It is easy to see (and follows also from Proposition 2.6 in [3]) that the geometric morphism $\gamma : \widehat{\mathbf{C}} \to \underline{S}$ is open (where $\widehat{\mathbf{C}}$ is the topos of $\underline{S}$-valued presheaves on $\mathbf{C}$). The introduction of the notions of relatively Boolean and relatively de Morgan geometric morphism is now partially justified by the following relativization of two classical results for $\underline{S} = \underline{Set}$ (due to Freyd (?) and Johnstone [2], respectively; the part concerning groupoids was also known to Squire [11], for general base topos $\underline{S}$):

Theorem 3.1 *The geometric morphism $\gamma : \widehat{\mathbf{C}} \to \underline{S}$ is relatively Boolean iff $\mathbf{C}$ is a groupoid; and it is relatively de Morgan iff $\mathbf{C}$ satisfies the Ore condition: every diagram of two arrows with common codomain embeds into a commutative square.*

Proof. We may argue as if $\underline{S}$ were $\underline{Sets}$, provided the argument is positive and constructive. Let $\mathbf{C}$ be a category in $\underline{S}$. We describe the canonical $\tau : \gamma^*\Omega_{\underline{S}} \to \Omega_{\underline{E}}$ where $\underline{E} = \widehat{\mathbf{C}}$. For $C \in \mathbf{C}$ an object, τ_C is the unique frame map

$$\Omega_{\underline{S}} = \gamma^*(\Omega_{\underline{S}})(C) \to P(y(C)),$$

where $P(y(C))$ denotes the set (frame) of subfunctors of the representable functor $y(C) = hom_{\mathbf{C}}(-, C)$. We describe a left adjoint left inverse σ for τ_C, namely given by

$$\sigma(R) = \| R \text{ is inhabited } \| \tag{5}$$

for any subfunctor (sieve) $R \subseteq y(C)$, where $\| \ldots \|$ denotes 'truth value of ...'. Clearly $\sigma(\tau_C(\lambda)) = \lambda$; and if $f \in R(D)$, R is inhabited, so $f \in \tau_C(\| R$ is inhabited $\|$), so $R \subseteq \tau_C(\sigma(R))$. To say that $\widehat{\mathbf{C}} \to \underline{S}$ is relatively Boolean is, in view of its openness and by Proposition 2.1, equivalent to saying that τ_C is iso for all C. Now let $\mathbf{C}$ be a groupoid. To prove τ_C iso, it suffices to prove $\tau_C(\sigma(R)) \subseteq R$ for any sieve $R \subseteq y(C)$. Let $f : D \to C$, and assume $f \in \tau_C(\sigma(R))$, so $\sigma(R)$ is true, so R is inhabited, say $g : C' \to C$ is in R for some g. But then $g \circ (g^{-1} \circ f)$ is also in R, since R is a sieve. So $f \in R$.

Conversely, assume $\widehat{\mathbf{C}} \to \underline{S}$ is relatively Boolean, so τ_C is iso for all C, with σ as inverse. Let $f : D \to C$ be arbitrary, and let $R \subseteq y(C)$ be the sieve generated by f. It is inhabited (witness: f), so $\sigma(R) = true$, so $id_C \in \tau_C(\sigma(R)) = R$. But to say that id_C belongs to the sieve generated by f is to say that f has a right inverse. So every arrow of $\mathbf{C}$ has a right inverse, and this implies that $\mathbf{C}$ is a groupoid.

Next, assume that $\mathbf{C}$ satisfies the Ore condition. To prove that $\widehat{\mathbf{C}} \to \underline{S}$ is de Morgan is equivalent to proving that $\tau : \gamma^*\Omega_{\underline{S}} \to \Omega_{\widehat{\mathbf{C}}}$ has a left adjoint. We have already *pointwise* a left adjoint, given by the description (5) above; now we write σ_C for σ. It suffices to see that σ_C is natural in C, that is, to prove that for each $f : D \to C$ the diagram

$$\begin{array}{ccc} P(y(C)) & \xrightarrow{\pi} & P(y(D)) \\ \sigma_C \downarrow & & \downarrow \sigma_D \\ \Omega_{\underline{S}} & \xrightarrow[id]{} & \Omega_{\underline{S}} \end{array}$$

commutes, where the top map π to a sieve $R \subseteq y(C)$ associates the set of arrows g with codomain D and with $f \circ g \in R$. Now let $R \in P(y(C))$ be a sieve on C. Assume $\sigma_C(R)$ is true, so R is inhabited, say witnessed by $(h : C' \to C) \in R$. Completing the square

$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ h' \uparrow & & \uparrow h \\ D' & \longrightarrow & C' \end{array}$$

we get that $\pi(R)$ is inhabited (witnessed by h'), so $\sigma_D(\pi(R))$ is true. This implies that $\sigma_C(R) \leq \sigma_D(\pi(R))$. The other inequality $\sigma_D(\pi(R)) \leq \sigma_C(R)$ is trivial: if $\sigma_D(\pi(R))$ is true, $\pi(R)$ is inhabited which implies that R is inhabited, so $\sigma_C(R)$ is true.

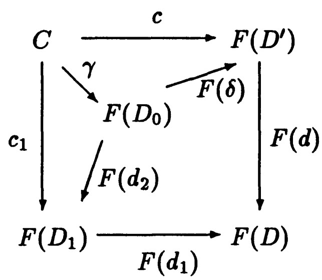
Conversely, if $\widehat{\mathbf{C}} \to \underline{S}$ is relatively de Morgan, $\tau : \gamma^*\Omega_{\underline{S}} \to \Omega_{\widehat{\mathbf{C}}}$ has a left adjoint, hence the pointwise left adjoint σ_C of τ_C is natural in C. Contemplating the naturality square above for $f : D \to C$ and applying it to the principal sieve generated by h, we get that the sieve of those h', which fit in the Ore square above, is inhabited. Thus $\mathbf{C}$ satisfies the Ore condition.

We have a generalization of part of Theorem 3.1. Given a functor $F : \mathbf{D} \to \mathbf{C}$ between category objects in a topos $\underline{S}$. It induces an (essential) geometric morphism $\gamma = \widehat{F} : \widehat{\mathbf{D}} \to \widehat{\mathbf{C}}$ (whose inverse image functor is just "composing with F"). For simplicity, we formulate the result as if $\underline{S}$ were Sets, but the character of the proof makes it clear that it works over any base topos. We first prove

Proposition 3.1 *Let $F : \mathbf{D} \to \mathbf{C}$ be a functor between small categories, inducing the (essential) geometric morphism $\gamma : \widehat{\mathbf{D}} \to \widehat{\mathbf{C}}$. Then the canonical $\tau : \gamma^*\Omega_{\widehat{\mathbf{C}}} \to \Omega_{\widehat{\mathbf{D}}}$ has a left adjoint if and only if F satisfies the condition (*): to every commutative square*

$$F(d) \circ c = F(d_1) \circ c_1 \qquad (6)$$

in $\mathbf{C}$*, there exists a commutative square* $d \circ \delta = d_1 \circ d_2$ *in* $\mathbf{D}$ *and an arrow* γ *in* $\mathbf{C}$ *with* $c = F(\delta) \circ \gamma$*, as displayed in*



Proof. This is much in spirit and notation as the proof of Proposition 2.6 in [3], and it generalizes the proof of Theorem 3.1. For $D \in \mathbf{D}$, $\tau_D : (\gamma^* \Omega_{\widehat{\mathbf{C}}})(D) \to \Omega_{\widehat{\mathbf{D}}}$ takes a sieve R on $F(D)$ to the sieve $\{\beta : D_1 \to D \mid F(\beta) \in R\}$. For each D, τ_D has a left adjoint σ_D given by $\sigma_D(S) =$

$$= \{c : C \to F(D) \mid c \text{ factors through some } F(\delta) : F(D_0) \to F(D) \text{ with } \delta \in S\}.$$

Clearly τ has a left adjoint (as a map in $\widehat{\mathbf{D}}$) precisely when the σ_D's are natural in D, which amounts to the equality, for each $d : D' \to D$ in $\mathbf{D}$ and each sieve S on D,

$$F(d)^*(\sigma_D(S)) = \sigma_{D'}(d^*(S))$$

(where $d^*(S) = \{\delta \mid d \circ \delta \in S\}$, and similarly for $F(d)^*$). Unravelling the description of σ and d^* etc, the two sides of this equation are, respectively,

$$\{c \mid \exists d_1 \in S \exists c_1 : F(d) \circ c = F(d_1) \circ c_1\} \tag{7}$$

and

$$\{c \mid \exists \delta, \gamma : d \circ \delta \in S \wedge c = F(\delta) \circ \gamma\}. \tag{8}$$

The inclusion (8) $\subseteq$ (7) always holds.

Now assume that the condition (*) holds, and let $c \in$(7), for some sieve S. By condition (*), we find γ, δ, d_2 as displayed in the diagram, with $d \circ \delta = d_1 \circ d_2$ and $c = F(\delta) \circ \gamma$. By the commutativity $d \circ \delta = d_1 \circ d_2$ and $d_1 \in S$, it follows that $d \circ \delta \in S$, and thus $c \in$(8).

Conversely, assume (7) $\subseteq$ (8), for all sieves S on D, and consider a commutative square (6). Apply the condition (7) $\subseteq$ (8) for the sieve generated by d_1. Since $c_1 \in$(7), $c_1 \in$(8), and this provides us with the δ, γ required by the condition (*). This proves the Proposition.

For the case where the functor $F : \mathbf{D} \to \mathbf{C}$ makes the corresponding geometric morphism γ sub-open, i.e. the canonical $\tau : \gamma^* \Omega_{\widehat{\mathbf{C}}} \to \Omega_{\widehat{\mathbf{D}}}$ monic (and an elementary condition in terms of $F : \mathbf{D} \to \mathbf{C}$ for this is given in [3] Proposition 2.6), the existence of a left adjoint for τ is equivalent to γ being relatively de Morgan, by Proposition 2.3. We therefore get as an immediate corollary that, assuming sub-openness of γ, the geometric morphism $\gamma = \widehat{u}$ makes $\widehat{\mathbf{D}}$ a relatively de Morgan topos over $\widehat{\mathbf{C}}$ iff the functor u satisfies the condition * above; and this result is valid over any base topos.

4 Toposes defined by frames

We remind the reader of some results of intuitionistic frame theory from [4]. Let A be a frame in a topos $\underline{S}$. An element $a \in A$ is called ***positive*** if every cover $\bigvee_I a_i$ of a has I inhabited. The frame is called ***open*** if every element of A is a supremum of positive elements. This is equivalent to $sh(A) \to \underline{S}$ being an open geometric morphism. The map $pos : A \to \Omega_{\underline{S}}$ which to $a \in A$ associates the truth value of the statement that a is positive, is left adjoint to the unique frame map $\Omega_{\underline{S}} \to A$. Finally, A is called ***surjective*** if this frame map is injective.

Consider the geometric morphism $\gamma : sh(A) \to \underline{S}$, where $sh(A)$ is the topos of ($\underline{S}$-valued) sheaves on A. We need to recall some standard sheaf theory. If D is a presheaf on A, a global section of the associated sheaf $\tilde{D}$ is given by an ***atlas***, meaning a family

$$\{(a_i, \lambda_i) \mid i \in I\}, \tag{9}$$

where $\bigvee a_i = 1_A$, and $\lambda_i \in D(a_i)$ is a compatible family, meaning that for each $i, j \in I$, there is a cover $\{a'_k \mid k \in K\}$ of $a_i \wedge a_j$ such that $\lambda_i \mid a'_k = \lambda_j \mid a'_k$ for each $k \in K$. If D is a ***constant*** presheaf, $D(a) = D$ for all a, then this condition reads: $\lambda_i = \lambda_j$ to the extent that K is inhabited. If the family (9) is compatible,

$$pos(a_i \wedge a_j) \text{ implies } \lambda_i = \lambda_j;$$

and if the frame is open, this condition implies conversely compatibility of the family. We apply this for the case where $D = \Omega_{\underline{S}}$ and A is a surjective open frame. A global section of $\gamma^*\Omega_{\underline{S}}$ given by such an atlas $\{(a_i, \lambda_i)\}$ ($\lambda_i \in \Omega_{\underline{S}}$) maps by the canonical map $\gamma_*\gamma^*\Omega_{\underline{S}} \to \gamma_*\Omega_{sh(A)} = A$ to $\bigvee \lambda_i \wedge a_i$ (by surjectivity, the map $\Omega_{\underline{S}} \to A$ is monic, so we identify elements λ of $\Omega_{\underline{S}}$ with their image in A. Also, by openness, $\gamma^*\Omega_{\underline{S}} \to \Omega_{sh(A)}$ is monic and hence $\gamma_*\gamma^*\Omega_{\underline{S}} \to \gamma_*\Omega_{sh(A)} = A$ is monic.)

Proposition 4.1 *Let A be a surjective open frame. The canonical map $\gamma_*\gamma^*\Omega_{\underline{S}} \to \gamma_*\Omega_{sh(A)} = A$ identifies $\gamma_*\gamma^*\Omega_{\underline{S}}$ with $Clp(A)$.*

Proof. Let $a \in Clp(A)$. Then we have an atlas for a global section of $\gamma^*\Omega_{\underline{S}}$, ie. for an element of $\gamma_*\gamma^*\Omega_{\underline{S}}$, given by

$$\{(a \leftrightarrow \lambda, \lambda) \mid \lambda \in \Omega_{\underline{S}}\}; \tag{10}$$

for, the $a \leftrightarrow \lambda$'s cover 1_A since $a \in Clp(A)$; the compatibility is seen as follows: if

$$pos((a \leftrightarrow \lambda) \wedge (a \leftrightarrow \mu)),$$

then $pos(\lambda\leftrightarrow\mu)$; now, the bi-implication here is formed in A, but the map $i : \Omega_{\underline{S}} \to A$ preserves (bi-)implication, because of the left adjoint and [6], Propositions V.1.1 and V.3.1; thus $pos(i(\lambda\leftrightarrow\mu))$ is true. But pos is a left inverse for i, so $\lambda\leftrightarrow\mu$ is true, thus $\lambda = \mu$. Also, the element given by the atlas (10) maps to a since

$$\begin{aligned}\bigvee_\lambda \lambda \wedge (a\leftrightarrow\lambda) &= \bigvee_\lambda a \wedge (a\leftrightarrow\lambda)\\ &= a \wedge \bigvee_\lambda (a\leftrightarrow\lambda) = a \wedge 1 = a.\end{aligned}$$

Conversely, let $a \in \gamma_*\gamma^*\Omega_{\underline{S}}$ be given by an atlas $\{(a_i, \lambda_i) \mid i \in I\}$; we have

$$pos(a_i \wedge a_j) \text{ implies } \lambda_i = \lambda_j. \tag{11}$$

and $a = \bigvee \lambda_i \wedge a_i$. To prove $1_A = \bigvee_\lambda (a\leftrightarrow\lambda)$, it suffices to prove that $a_i \leq \bigvee_\lambda (a\leftrightarrow\lambda)$ for each $i \in I$. We shall prove that we even have $a_i \leq (a\leftrightarrow\lambda_i)$. This amounts to proving $a_i \wedge \lambda_i \leq a$ and $a_i \wedge a \leq \lambda_i$. The first is obvious from the construction of a. For the second, we have

$$\begin{aligned}a_i \wedge a &= a_i \wedge \bigvee_j \lambda_j \wedge a_j\\ &= \bigvee_j (a_i \wedge a_j) \wedge \lambda_j,\end{aligned}$$

so it suffices to prove $(a_i \wedge a_j) \wedge \lambda_j \leq \lambda_i$ for each i, j, or $a_i \wedge a_j \leq (\lambda_j \to \lambda_i)$. Since $pos : A \to \Omega_{\underline{S}}$ is left adjoint to the inclusion $\Omega_{\underline{S}} \subseteq A$, this is equivalent to $pos(a_i \wedge a_j) \leq (\lambda_j \to \lambda_i)$. This is an equality in $\Omega_{\underline{S}}$, so assume $pos(a_i \wedge a_j)$ is true. Then by (11), $\lambda_i = \lambda_j$ is true, hence so is $\lambda_j \leq \lambda_i$. This proves the Proposition.

We now have as a Corollary:

Theorem 4.1 *Let A be a surjective open frame. Then $\gamma : sh(A) \to \underline{S}$ is a relatively Boolean geometric morphism iff A is a relatively Boolean frame.*

Proof. By openness, the canonical $\gamma^*\Omega_{\underline{S}} \to \Omega_{sh(A)}$ is monic; if γ is further relatively Boolean, it is an isomorphism, and hence so is $\gamma_*\gamma^*\Omega_{\underline{S}} \to \gamma_*\Omega_{sh(A)} = A$. But by Proposition 4.1, the domain of this map is $Clp(A)$. Thus $A = Clp(A)$. Conversely, assume A is relatively Boolean. We have to prove that for each $a \in A$, $(\gamma^*\Omega_{\underline{S}})(a) \to \Omega_{sh(A)}(a) =\downarrow a$ is an isomorphism. It suffices actually to prove this for *positive* a, since every element in A is covered by positive elements. But if $a \in A$ is positive, $\downarrow a$ is a surjective open frame, and since the map $- \wedge a : A \to \downarrow a$ preserves everything involved in the definition of relative Booleanness, $\downarrow a$ is a relatively Boolean frame. Thus by applying Proposition 4.1 to $\downarrow a$ gives that the canonical

$$\gamma'_*\gamma'^*\Omega_{\underline{S}} \to \gamma'_*\Omega_{sh(\downarrow a)} \tag{12}$$

is an isomorphism (where $\gamma' : sh(\downarrow a) \to \underline{S}$). But the map (12) identifies with

$$\gamma^* \Omega_{\underline{S}}(a) \to \downarrow a.$$

This proves the Theorem.

In a similar vein, we prove

Theorem 4.2 *Let A be a surjective open frame. Then $\gamma : sh(A) \to \underline{S}$ is a relatively de Morgan geometric morphism iff A is a relatively de Morgan frame.*

Proof. By openness, the canonical $\gamma^*\Omega_{\underline{S}} \to \Omega_{sh(A)}$ is monic; if γ is further relatively de Morgan, $\gamma^*\Omega_{\underline{S}} \hookrightarrow \Omega_{sh(A)}$ has a left adjoint (Proposition 2.1), which is necessarily left exact (Proposition 2.2). Thus $\gamma_*\gamma^*\Omega_{\underline{S}} \hookrightarrow \gamma_*\Omega_{sh(A)} = A$ has a left exact left adjoint, but this inclusion identifies by Proposition 4.1 with $Clp \hookrightarrow A$. Thus A is relatively de Morgan.

Conversely, assume that A is a relatively de Morgan frame (and open, surjective). In analogy with the proof of the previous Theorem, to see that $\gamma^*\Omega_{\underline{S}} \hookrightarrow \Omega_{sh(A)}$ has a left adjoint, it suffices to see that for each positive $a \in A$,

$$\gamma^*\Omega_{\underline{S}}(a) \hookrightarrow \Omega_{sh(A)}(a) = \downarrow a \tag{13}$$

has a left adjoint which is natural in a. In analogy with the previous proof, $\downarrow a$ is an open surjective relatively de Morgan frame. By Proposition 4.1, the domain of (12) gets identified with $Clp(\downarrow a)$, and thus the map (12) has a left adjoint. But (12) gets identified with the map (13). So we just have to prove that these left adjoints are natural. Since for de Morgan frames $Clp = Reg$, the left adjoint for $Clp(\downarrow a) \hookrightarrow$ is given by the nucleus $\bigwedge_\lambda(- \to \lambda) \to \lambda$ with the implication signs on the right being the one of the Heyting algebra $\downarrow a$. But for $b \leq a$, $- \wedge b : \downarrow a \to \downarrow b$ is a Heyting algebra homomorphism. This proves naturality, and hence we have a left adjoint for $\gamma^*\Omega_{\underline{S}} \to \Omega_{sh(A)}$. This proves the Theorem.

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