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A NOTE ON r-EMBEDDINGS
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Résumé. Dans cet article on étudie la classe des espaces r-compacts [10,12] et la notion connexe de sous-espace r-plongé, où r est un épireflecteur topologique. Cette notion permet d'obtenir des résultats dans le contexte du problème de la conservation des produits topologiques par r. De plus, elle donne une caractérisation des réflecteurs totaux, définis en [2,3].

Introduction.
Let r : Top → R be a topological epireflector. In [10,12] we have introduced the category r – Comp of r-compact spaces, and the related r-closure operator.
In this note we go on studing such classes of spaces. Since it turns out that r-compactness is not an absolute property, then we are concerned, in particular, with a concept that overcomes such a difficulty, namely that of r-embedded subset. A subset A of a space X is r-embedded in X whenever r(S) is a subspace of r(X).
After presenting the general properties of r-embeddings, we concentrate on two arguments.
First, we describe the close relation existing between the classes of r-embeddings in Top and that of embeddings in R with respect to which r is total, in the sense of [2]. Actually, r is total with respect to the class of open embeddings if and only if every r-open subset of a given space is r-embedded there.
Furthermore, we give a characterization of the maximal class Tr of embeddings in R, such that r is total with respect to Tr. We obtain a somewhat surprising result (Th.2.3) in this direction, by making use

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of the notion of pullback complement [3].
Secondly, the concept of $r$-embedding turns out to be useful in connection with the question whether $r$ preserves products. Given two $r$-compact spaces $X$ and $Y$, and two $r$-embedded subspaces $A \subseteq X$, $B \subseteq Y$, then $r(A \times B) = r(A) \times r(B)$ holds. Such kind of result applies to a great variety of cases and, in particular, when $r = \tau$ is the Tychonoff modification functor.

1. $r$-Compact Spaces and $r$-Embeddings.

In this paper we shall be concerned with an epireflective subcategory $\mathbf{R}$ of the category $\mathbf{Top}$ of all topological spaces and maps, with reflector $r : \mathbf{Top} \to \mathbf{R}$.

For every space $X \in \mathbf{Top}$, let $r_X : X \to r(X)$ be its onto reflection map, and let us decompose it as follows

\[
\begin{array}{ccc}
X & \xrightarrow{r_X} & r(X) \\
\downarrow{j_X} & & \downarrow{r'_X} \\
X_r & & 
\end{array}
\]

where $X_r$ is the space having the same underlying set as $X$, with the initial topology induced by $r_X$. Then $j_X$ acts as the identity, and $r_X = r'_X$ as set maps.

Such a procedure is functorial and gives a decomposition $r = r' \cdot j$ of the reflector $r$ by means of the two functors $j : \mathbf{Top} \to \mathbf{Top}_r$, and $r' : \mathbf{Top}_r \to \mathbf{R}$. Here $\mathbf{Top}_r = \{X_r | X \in \mathbf{Top}\}$ is the bireflective hull of $\mathbf{R}$ in $\mathbf{Top}$ with bireflector $j = \{j_X\}$, while $r'$ is the restriction of $r$ to $\mathbf{Top}_r$, in fact $r'_X = r'_{X_r} = r'_X$, for every space $X$. The topology on $X_r$ is generated by what we called in [12] the $r$–closure operator, defined as follows

\[cl_r(M) = r^{-1}_X(cl(r_X(M))),\]

where $M \subseteq X$ and $cl$ denotes ordinary closure. $M$ is $r$–closed in $X$ whenever $M = cl_r(M)$. A subset $U$ of $X$ will be called $r$–open if
EXAMPLES 1.1. (a) When \( r : \text{Top} \to \text{Top}_o \) is the \( T_o \)-identification functor, then \( U \subseteq X \) is \( r \)-open if and only if \( U \) is open and \( x \in U \) implies \( \text{cl}\{x\} \subseteq U \).

In general, for a quotient reflector \( r \), the \( r \)-open subsets of \( X \in \text{Top} \) coincide with those subsets \( S \) that are open and saturated with respect to the reflection map \( r_X \), that is \( S = r_X^{-1}(r_X(S)) \).

(b) Let \( \tau : \text{Top} \to \text{Tych} \) be the Tychonoff modification functor \([5,6]\). The \( \tau \)-open subsets of a space \( X \) are those which are union of cozero sets of \( X \).

(c) Let \( r : \text{Top} \to 0 - \text{dim} \) be the reflector functor on the category of zero-dimensional spaces; in such a case the \( r \)-open subsets of \( X \in \text{Top} \) are those subsets that are union of clopen subsets of \( X \).

(d) Let \( R \subseteq \text{Top} \) be such that the Salbany closure operator \([9]\) (called \( R \)-closure) induced by it, coincides with the ordinary closure on each \( X \in R \). This happens, e.g., for \( R = \text{Haus} \), the category of Hausdorff spaces, \( R = \text{Reg} \), the category of regular spaces, \( R = \text{Tych} \), \( R = \text{O - dim} \), etc. In such a case the \( r \)-open subsets of \( X \) are the same as the \( R \)-open subsets. For the general situation see \([11]\).

A topological space \( X \) is said to be \( r - \text{compact} \) whenever its reflection \( r(X) \) in \( R \) is a compact space (no separation axiom is assumed); this amounts to say that every cover of \( X \) by \( r \)-open subsets has a finite subcover.

The category \( r - \text{Comp} \subseteq \text{Top} \) of \( r \)-compact spaces has been studied in \([10,12]\), where we are concerned with the question whether \( r \) preserves topological products.

Recently, \( r \)-compact spaces have been studied further in \([8]\).

\( r \)-compactness is not an absolute property, as shown by the following example concerning the Tychonoff reflector \( \tau \).

EXAMPLE 1.2. Let \( X \) be the space having the closed unit interval \( I \) as underlying set and a topology which differs from the usual one in that the neighbourhoods of the point 0 do not contain points of the
set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

$X$ is a non-compact, $\tau$-compact space, in fact one has $\tau(X) = X_\tau = I$.

If we consider the subset $S = A \cup \{0\}$ of $X$, then $S$ is relatively $\tau$-compact with respect to $X$, but it is not $\tau$-compact considered as a topological space, with the topology inherited from $X$.

In order to avoid this kind of situation, it is useful to introduce the notion of $r$–embedding, which also turns out to be of interest in connection with other concepts, as we shall see below.

**DEFINITION 1.3.** A subset $S$ of a space $X$ is said to be $r$–embedded in $X$ provided that every $r$-open subset of $S$ is the intersection of an $r$-open subset of $X$ with $S$.

**THEOREM 1.4.** For a subset $S$ of a space $X$, the following are equivalent:

1. $S$ is $r$-embedded in $X$.
2. $S_r$ is a subspace of $X_r$.
3. $r(S)$ is a subspace of $r(X)$.
4. $cl_{r,S}(M) = cl_{r,X}(M) \cap S$, for every $M \subset S$.

**PROOF:** The equivalence of (1) and (2) follows directly from the definitions.

(2) $\rightarrow$ (3). Let $s$ be the embedding of $S$ in $X$. We have the following situation: $r(s_r) \cdot r_s = r_{X_r} \cdot s_r$, where $s_r$ is the embedding of $S_r$ in $X_r$ and $r_{X_r} \cdot s_r$, $r(s_r)$ are initial maps. Our first task is to show that $r(s_r)$ is also initial. To this end, let $V \subset r(S)$ be an open subset; then there exists an open subset $T \subset r(X)$ such that

$$r_{S_r}^{-1}(V) = (r_{S_r} \cdot s_r)^{-1}(T) = (r(s_r) \cdot r_{S_r})^{-1}(T) = r_{S_r}^{-1}(r(s_r)^{-1}(T)).$$

Since $r_{S_r}$ is onto, we obtain that $V = r(s_r)^{-1}(T)$. For $r(s_r)$ to be an embedding it remains to show that it is injective. In order to do this let us recall [7] that $R$ is either bireflective in $\text{Top}$ or it is contained in the category $\text{Top}_0$ of $T_0$-spaces; in the former case $r(s)$ must be injective, hence an embedding. In the latter case $r(S)$ is a $T_0$-space. If $x$ and $y$ are two distinct points in $r(S)$ then there is an open subset $U$ of $r(S)$ such that $x \in U$ and $y \notin U$. By the first part of the proof there
exists an open set \( V \subset r(X) \) such that \( U = r(s)^{-1}(V) \). It follows that \( r(s)(x) \) and \( r(s)(y) \) must be different.

(3) \( \rightarrow \) (4). Let us recall that \( cl_{r,S}(M) \) is given by the intersection of all \( r \)-closed subsets \( F \) of \( S \) that contain \( M \). Every such \( F \) is of the form \( F = r_S^{-1}(F') \), being \( F' \) a closed subset of \( r(S) \). Note also that, by assumption, \( F' \) is of the form \( F' = r(s)^{-1}(F'') \) for some closed subset \( F'' \) of \( r(X) \). Then one has \( r_S^{-1}(r(s)^{-1}(F'')) = s^{-1}(r_X^{-1}(F'')) = r_X^{-1}(F'') \cap S \). It follows

\[
cl_{r,S}(M) = \cap \{ r_S^{-1}(F') : M \subset F \} = \cap \{ s^{-1}(r_X^{-1}(F'')) : M \subset F \} = \cap \{ r_X^{-1}(F'') \cap S : M \subset F \} = \cap \{ r_X^{-1}(F'') : M \subset F \} \cap S = cl_{r,S}(M) \cap S.
\]

(4) \( \rightarrow \) (1). This is immediate.

**COROLLARY 1.5.** Let \( S \) be an \( r \)-embedded subspace of \( X \). Then:

1. \( r(S) = r(X)(S) \) and \( r_S = r_X|_S \).
2. \( S \) is \( r \)-compact iff it is \( r \)-compact relatively to \( X \).

**COROLLARY 1.6.** Every retract of \( X \) is \( r \)-embedded in \( X \).

**EXAMPLES 1.7.** (a) Let \( r \) be a quotient reflector with \( r_X : X \rightarrow r(X) = X/\sim \), for every \( X \in \text{Top} \). A subset \( M \subset X \) is then \( r \)-embedded exactly when \( M/\sim \) is a subset of \( X/\sim \). It follows that a sufficient condition in order that \( M \) be \( r \)-embedded in \( X \) is that \( M \) is either open or closed in \( X \) and saturated with respect to the quotient map \( r_X \).

If \( r \) is the \( T_0 \)-identification functor, then every closed subset of \( X \) is \( r \)-embedded in \( X \), as one can easily verify.

(b) Recall that a subset \( S \) of a space \( X \) is \( z \)–embedded [1] whenever every cozero set in \( S \) is the intersection with \( S \) of a cozero set in \( X \). Hence it is clear that every \( z \)-embedded subset is also \( \tau \)-embedded. Moreover, since every cozero set is \( z \)-embedded, it follows that every \( \tau \)-open subset of \( X \) is \( \tau \)-embedded.

**REMARK 1.8.** In the non-surjective case of the Hewitt realcompactification functor \( \nu : \text{Top} \rightarrow r-\text{Comp} \), the concept of \( \nu \)-embedding coincides with that defined in [1].
The situation described in 1.7. (b) above is not typical of $\tau$, but it is fairly general, since it is shared by all those reflectors that are total (in the sense of [2]), with respect to the class of open embeddings. Besides $\tau$, other examples are the reflectors onto the categories $\textbf{Haus}$, $\textbf{Reg}$, $0$–$\text{dim}$.

Let us recall that the epireflector $r$ is said to be total [2,3] with respect to a class $S$ of topological embeddings, whenever, given an embedding $s : S \to r(X)$, with $s \in S$, then the restriction of $r_X$ to $r_X^{-1}(S)$, that is the map

$$r_X|_{r_X^{-1}(S)} : r_X^{-1}(S) \to S$$

is uniquely $R$-extendable, hence a reflection map.

**Proposition 1.9.** The epireflector $r$ is total with respect to the class of open embeddings iff, for every $X \in \textbf{Top}$, every $r$-open subset of $X$ is $r$-embedded in $X$.

**Proof:** Let $r$ be total with respect to the class of open embeddings and let $A$ be an $r$-open subset of $X$. Then $r_X(A)$ is open in $r(X)$ and $A = r_X^{-1}(r_X(A))$. It follows that $r_X|_{A} : A = r_X^{-1}(r_X(A)) \to r_X(A)$ is the reflection map for $A$, hence $r(A) = r_X(A)$ and $r_A = r_X|_{A}$, that is $A$ is $r$-embedded in $X$.

Conversely, assume that, for any space $X$, every $r$-open subset of $X$ is $r$-embedded. Let $s : S \to r(X)$ be an open embedding; then $r_X^{-1}(S)$ is $r$-open in $X$. It follows that $r_X|_{r_X^{-1}(S)} : r_X^{-1}(S) = r_X^{-1}(r_X(r_X^{-1}(S))) \to S$ is a reflection map. This shows that $r$ is total with respect to the class of open embeddings.
2. Further Properties of r-Embeddings.

Let the epireflector \( r : \text{Top} \rightarrow \text{R} \) be given and let us denote by \( T_r \) the maximal class of topological embeddings such that \( r \) is total with respect to it.

A natural question that arises is then to characterize the class \( T_r \).

The next results deal with this problem.

**PROPOSITION 2.1.** Let \( X \in \text{Top} \). An embedding \( s : S \rightarrow r(X) \) is an element of \( T_r \) if and only if there exists a subset \( A \) of \( X \) such that:

1. \( S = r_X(A) \).
2. \( A \) is saturated with respect to \( r_X \).
3. \( A \) is \( r \)-embedded in \( X \).

**PROOF:** The proof is straightforward and depends essentially on the definition of \( r \)-embedding.

The following theorem indicates how one must choose the subset \( A \) of \( X \), involved in the previous proposition. It make use of the notion of pullback complement [3] which we recall for sake of completeness.

**DEFINITION 2.2.** Let \( f : A \rightarrow U \) and \( s : U \rightarrow Y \) be morphisms in a category \( \mathbf{C} \). A pullback complement of the pair \((f, s)\) is a pair \((s', f')\) in such a way that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & U \\
\downarrow{s'} & & \downarrow{s} \\
P & \xrightarrow{f'} & Y
\end{array}
\]

is a pullback square with the property that for every other pullback square
and any morphism \( h : V \to A \) with \( f \cdot h = g \), there is a unique \( h' : T \to P \) such that \( f' \cdot h' = g' \) and \( s' \cdot h = h' \cdot t \).

**THEOREM 2.3.** Let \( a : A \to X \) be an embedding that is saturated with respect to \( r_X \). Then \( a \) is an \( r \)-embedding if and only if the pair \( (r_A, r(a)) \) is a pullback complement of \( (a, r_X) \).

**PROOF:** Assume that \( a \) is a saturated \( r \)-embedding with respect to \( r_X \); then the following diagram is a pullback:

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow{r_A} & & \downarrow{r_X} \\
r(A) & \longrightarrow & r(X)
\end{array}
\]

Suppose there is another pullback square

\[
\begin{array}{ccc}
V & \longrightarrow & X \\
\downarrow{t} & & \downarrow{r_X} \\
T & \longrightarrow & r(X)
\end{array}
\]

and a map \( h : V \to A \) with \( a \cdot h = g \).

By commutativity it follows \( g^* \cdot t = r_X \cdot g = r_X \cdot a \cdot h = r(a) \cdot r_A \cdot h \). Because of Theorem 1.4(3) we know that \( r(a) \) is an embedding. Moreover, \( t \) as a pullback of the onto map \( r_X \) is onto as well. Now the fact that onto maps and embeddings form a factorization system implies the existence of the required \( h^* : T \to r(A) \); it arises as a "diagonal fill-in". This concludes the first half of the proof.
Conversely, assume that $a : A \to X$ is a saturated embedding such that the first square is a pullback complement. Consider the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow & & \downarrow \\
\r_X|A & \xrightarrow{r_A} & \r_X \\
\downarrow & & \downarrow \\
r_X(A) & \xrightarrow{t} & \r(a) \\
\end{array}
\]

where $a^*$ is the inclusion.

By the universal property of the reflection there is a unique $s : \r(A) \to \r_X(A)$ such that $s \cdot r_A = r_X|A$; by the universal property of the pullback complement, there is a unique $t : \r_X(A) \to \r(A)$ such that $\r(a) \cdot t = a^*$. Then $\r(a) \cdot t \cdot r_X|A = a^* \cdot r_X|A = \r_X \cdot a$; by the universal property of the right outer pullback square, it follows that $1_A$ must be the unique map which renders the left outer square commutative, that is, $r_A = t \cdot r_X|A$. Finally, the composition $t \cdot s$ must be the identity, hence $s$ is an onto map having a left inverse; $s$ is a homeomorphism, and this concludes the proof.

Let now $X$ and $Y$ be two topological spaces. One says that the epireflector $\r$ preserves their product and writes $\r(X \times Y) = \r(X) \times \r(Y)$, whenever the unique map $t_{X \times Y} : \r(X \times Y) \to \r(X) \times \r(Y)$ such that $t_{X \times Y} \cdot \r_{X \times Y} = \r_X \times \r_Y$, which is induced by the universal property of the reflection, is a homeomorphism. This happens, e.g., whenever $X$ and $Y$ are $\r$-compact spaces.

In [12] it was shown that $\r$ preserves the product of $X$ and $Y$ if and only if every $\r$-open subset of $X \times Y$ is union of subsets of the form
THEOREM 2.4. Let $X$ and $Y$ be topological spaces $r$-embedded in $r$-compact spaces $X^*$ and $Y^*$, respectively. Then $r(X \times Y) = r(X) \times r(Y)$ iff $X \times Y$ is $r$-embedded in the product space $X^* \times Y^*$.

PROOF: Let us assume that $X \times Y$ is $r$-embedded in the product $X^* \times Y^*$. Let $P \subset X \times Y$ be an $r$-open subset, then there is a $r$-open subset $Q \subset X^* \times Y^*$ such that $P = (X \times Y) \cap Q$. For every $q \in Q$, there exists a rectangular $r$-open set $A_q \times B_q$ in $X^* \times Y^*$ such that $q \in A_q \times B_q$. It follows that $Q = \bigcup_{q \in Q} A_q \times B_q$, therefore $P = \bigcup_{q \in Q}((X \times A_q) \times (Y \times B_q))$, where $(X \times A_q)$ and $(Y \times B_q)$ are $r$-open subsets in $X, Y$, respectively. Hence $P$ can be written as a union of rectangular $r$-open sets of $X \times Y$.

Conversely, assume that $r(X \times Y) = r(X) \times r(Y)$, then every $r$-open subset of $X \times Y$ is a union of rectangular $r$-open subsets. Let $P \subset X \times Y$ be an $r$-open subset; there are $r$-open subsets $A_i \subset X$ and $B_i \subset Y$, $i \in I$, such that $P = \bigcup_{i \in I} A_i \times B_i$. By assumption there are $r$-open subsets $A_i^* \subset X^*$ and $B_i^* \subset Y^*$ with $A_i = X \cap A_i^*$, $B_i = Y \cap B_i^*$. Now $P^* = \bigcup_{i \in I} A_i^* \times B_i^*$ is an $r$-open subset of $X^* \times Y^*$ with the property that $P = (X \times Y) \cap P^*$.

COROLLARY 2.5. Assume that $r$ is total with respect to the class of open embeddings and let $X, Y$ be $r$-compact spaces. For any pair of $r$-open subsets $A \subset X$ and $B \subset Y$, we have $r(A \times B) = r(A) \times r(B)$.

REMARKS 2.6. (a) The category of $w$-compact spaces, defined and studied in [5,6], is contained in $\tau$-Comp. Since $\tau$ is total with respect to the open embeddings, because of Proposition 1.9, Corollary 2.5 applies to any two (unions of) cozero sets $A \subset X$ and $B \subset Y$, whenever $X$ and $Y$ are $w$-compact. The same is true, e.g., when $A, B$ are Lindeloff subspaces of $X, Y$, respectively, since then they are $z$-embedded [1], hence $\tau$-embedded.

(b) The corollary above applies to most of the usual situations; see, in fact, [2] for a list of topological epireflectors that are total with respect to the class of open embeddings.
References.


