

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

J. G. CABELLO

Simplicial groups as models for n -types

Cahiers de topologie et géométrie différentielle catégoriques, tome 38, n° 1 (1997), p. 67-92

http://www.numdam.org/item?id=CTGDC_1997__38_1_67_0

© Andrée C. Ehresmann et les auteurs, 1997, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

SIMPLICIAL GROUPS AS MODELS FOR n -TYPES

by J. G. CABELLO

RESUME. On sait que la catégorie des groupes simpliciaux $\mathbf{Simp}(\mathbf{Gp})$ fournit des modèles pour tous les types d'espaces connexes. C'est aussi un exemple de catégorie-modèle pour la théorie de l'homotopie. Dans cet article, on définit une nouvelle structure de catégorie-modèle sur $\mathbf{Simp}(\mathbf{Gp})$ pour chaque $n > 0$ (n -structure), redonnant la version classique lorsque n tend vers l'infini. On démontre que $\mathbf{Simp}(\mathbf{Gp})$ muni de cette structure est un modèle pour les $(n+1)$ -types d'espaces connexes, non seulement au niveau des catégories d'homotopie mais aussi au niveau des théories d'homotopie.

1 Introduction

As proved by Kan, [15], the category of all simplicial groups, $\mathbf{Simp}(\mathbf{Gp})$, provides algebraic models for all connected homotopy types. $\mathbf{Simp}(\mathbf{Gp})$ is also an outstanding example of what a closed model category is (in the sense of Quillen), [18], that is, the weak equivalences -those morphisms which are formally inverted to produce the homotopy category- are accompanied in this case by other two types of relevant morphisms, fibrations and cofibrations, subordinated all of them to a set of axioms. The consequence is that one can reproduce in $\mathbf{Simp}(\mathbf{Gp})$ -in any closed model category in general- analogues to the classical constructions of homotopy theory of spaces since the homotopy category is extended by the notion of *homotopy theory associated to a closed model category*. This includes the extra structure of loop and suspension functors and fibrations and cofibration sequences.

On the other hand, one has the notion of n -type, first studied by J.H.C. Whitehead, [20], (his "homotopy systems", now generalized by crossed complexes, were a partial solution to the problem of finding an algebraic model for this notion). Since then, several proposals have been given in this direction. One of them is the category $\mathbf{n} - \mathbf{HXC}(\mathbf{Gp})$ of n -hypercrossed complexes of groups, a model of n -types of simplicial groups and, consequently, of $(n + 1)$ -types of connected spaces, [5]. This category consists of certain complexes of non-abelian groups and reduces in the low dimensional levels to the well-known cases of 2-types (crossed modules) and 3-types (2-crossed modules).

The results of this paper are part of the author's Ph.D. thesis, [2], where a standard procedure is introduced to supply any category, suitably related to $\mathbf{Simp}(\mathbf{Gp})$, with a closed model structure. It is proved that this procedure particularly gives a closed model structure for the category $\mathbf{n} - \mathbf{HXC}(\mathbf{Gp})$. It is possible, then, to regard $\mathbf{Simp}(\mathbf{Gp})$ as a candidate for modelling $(n + 1)$ -types of connected spaces as well (or n -types of simplicial groups). In this paper, this is proved by showing that the Quillen's closed model structure on $\mathbf{Simp}(\mathbf{Gp})$ can be generalized by notions of n -weak equivalence, n -fibration and n -cofibration, producing several model structures (one for each n , called the n -structure). First, one observes that when n runs to infinite, the original Quillen's structure on $\mathbf{Simp}(\mathbf{Gp})$ is recovered, satisfying our prime purpose. Note that our definition of n -weak equivalences as a truncated version of classical weak equivalences (see 3.6) is the most natural when dealing with n -types. This fact, together with the expression of Kan fibrations in terms of the RLP, suggested the possibility of truncating them as well (level by level) in order to reach the n -fibrations (compare 2.1 with 3.2). This is also the spirit of [14], where something similar was proposed for simplicial sets.

The comparison between both structures (the n -structure and the classical one), is made by means of the pair of adjoint functors $(n + 1) - skeleton$ and $(n + 1) - coskeleton$, denoted by Sk^{n+1} and $Cosk^{n+1}$. This gives rise to an equivalence of categories

$$Ho_n(\mathbf{Simp}(\mathbf{Gp})) \simeq Ho(\mathbf{Simp}(\mathbf{Gp})|n - cc)$$

between the localization of $\mathbf{Simp}(\mathbf{Gp})$ with respect to the n -weak equiv-

alences (on the left) and the full subcategory of the homotopy category $Ho(\mathbf{Simp}(\mathbf{Gp}))$ whose objects are those simplicial groups n -coconnected, that is, with vanishing homotopy groups in dimensions greater than or equal to $n + 1$, $\Pi_i = 0$ for $i \geq n + 1$ (on the right). This shows $\mathbf{Simp}(\mathbf{Gp})$ as a category of n -types of simplicial groups.

As the category $\mathbf{n} - \mathbf{HXC}(\mathbf{Gp})$ has been converted into a closed model category ([2], [3]) it is also reasonable to think of an equivalence between the corresponding homotopy theories. This is achieved by considering in $\mathbf{Simp}(\mathbf{Gp})$ this n -model structure.

Let us mention that many other categories which, in principle, supported a Quillen's closed model structure (topological spaces, simplicial sets, [18], crossed complexes, [1]) have also been equipped with such an n -structure, generalizing the original one, [9], [10], [14]. In addition, the category of sheaves has been endowed recently with a closed model structure, [6], by a procedure similar to that proposed here.

The plan of this paper is as follows. Next section is devoted to the revision of the set of axioms most frequently used to define a closed model category, together with some other known results of interest for this paper. Section 3 introduces n -weak equivalences, n -fibrations and n -cofibrations for simplicial groups and gives some characterizations of these notions, relevant for the sequel. The detailed proof of the main result, showing that the above definitions lead to the n -closed model structure, is given in section 4. Finally, the last section is left to the conclusions: the comparison of both model structures in $\mathbf{Simp}(\mathbf{Gp})$, the classical and the n -structure, is made, obtaining the desired equivalence of homotopy theories.

Acknowledgements: This work was partially supported by DGI-CYT: PB94-0823.

2 Preliminaires

Through all this paper $\mathbf{Simp}(\mathbf{Gp})$ will denote the category of simplicial groups, i.e., the category of functors $Gp^{\Delta^{op}}$ where Δ is the category whose objects are the ordered sets $[0] = \{0\}$, $[1] = \{0, 1\}$, $[2] = \{0, 1, 2\}$, \dots , and whose morphisms are the order-preserving functions between

them.

Recall that the n -th simplicial kernel of an $(n - 1)$ -truncated simplicial group $G_{\bullet, tr}$, written $\Delta^n(G_{\bullet, tr})$, is the subgroup of G_{n-1}^{n+1} whose elements are those (x_0, x_1, \dots, x_n) such that $d_i x_j = d_{j-1} x_i$ for $i < j$. If $d_i : \Delta^n(G_{\bullet, tr}) \rightarrow G_{n-1}$ denotes the restriction of the canonical projections, there are unique morphisms $s_j : G_{n-1} \rightarrow \Delta^n(G_{\bullet, tr})$, for $0 \leq j \leq n - 1$ such that

$$\Delta^n(G_{\bullet, tr}) \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{\dots} \\ \xrightarrow{d_n} \end{array} G_{n-1} \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{\dots} \\ \xrightarrow{d_{n-1}} \end{array} \dots G_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} G_0$$

is an n -truncated simplicial group. By iterating the simplicial kernel construction one has the functor $cosk^{n-1}$ from the category of $(n - 1)$ -truncated simplicial groups to $\mathbf{Simp}(\mathbf{Gp})$. The resulting functor $cosk^{n-1}$ is right adjoint to the $(n - 1)$ -truncating functor, tr^{n-1} , that truncates any simplicial group at level $n - 1$. By convention, $Cosk^{n-1} = cosk^{n-1}tr^{n-1}$ and $\Delta^n(G_{\bullet}) = \Delta^n(tr^{n-1}(G_{\bullet}))$.

On the other hand, given any truncated simplicial group,

$$G_{\bullet, tr} : \quad G_{n-1} \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{\dots} \\ \xrightarrow{d_{n-1}} \end{array} \dots G_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} G_0$$

its n th-simplicial cokernel $\nabla(G_{\bullet, tr})$ is defined as the quotient of the coproduct $\sqcup_{i=0}^{n-1} G_{n-1}$, i.e., free product in \mathbf{Gp} , under the congruence generated by the elements $(s_i s_j x, s_{j-1} s_i x)$, $x \in G_{n-2}$ for $0 \leq i < j \leq n - 2$, where $s_j : G_{n-1} \rightarrow \nabla(G_{\bullet, tr})$ are the morphisms induced by the canonical inclusions. The $(n - 1)$ -skeleton of the $(n - 1)$ -truncated simplicial group $G_{\bullet, tr}$, denoted by $sk^{n-1}(G_{\bullet, tr})$ may be defined by iteration of successive simplicial cokernels. Writing $Sk^{n-1} = sk^{n-1}tr^{n-1}$, one has the following adjunction

$$Sk^{n-1} : \mathbf{Simp}(\mathbf{Gp}) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{Simp}(\mathbf{Gp}) : Cosk^{n-1}, \quad Sk^{n-1} \dashv Cosk^{n-1}$$

The same adjunction is known to exist for simplicial sets, which allows us to identify, for any simplicial set X_{\bullet} , simplicial maps $\hat{\Delta}[n] = Sk^{n-1}\Delta[n] \rightarrow X_{\bullet}$ with simplicial maps $\Delta[n] \rightarrow Cosk^{n-1}(X_{\bullet})$ and so with elements of $\Delta^n(X_{\bullet})$ (see [7], prop. 1.5).

Given any $(G_\bullet \in \mathbf{Simp}(\mathbf{Gp}))$, $\Lambda_i^n(G_\bullet)$ will stand for the universal group having morphisms $d_j : \Lambda_i^n(G_\bullet) \rightarrow G_{n-1}$, for $0 \leq j \leq n, j \neq i$ satisfying $d_j d_k = d_{k-1} d_j, j < k, k \neq i$, which is called the object of open i -horns at dimension n . If $\Delta[n, k], 0 \leq k \leq n > 0$, denotes the simplicial subset of $\Delta[n]$ generated by $d_i(i_n), 0 \leq i \leq n, i \neq k$, (where $i_n = (0, 1, \dots, n) \in \Delta[n]_n$, see [7]), note the equivalence between giving an element of $\Lambda_k^n(G_\bullet)$ and giving a simplicial map $\Delta[n, k] \rightarrow G_\bullet$. A morphism in $\mathbf{Simp}(\mathbf{Gp})$, $f_\bullet : E_\bullet \rightarrow B_\bullet$, is said to be a Kan fibration iff the canonical morphism

$$E_s \rightarrow B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$$

is surjective, for $0 \leq k \leq s, s > 0$.

Let us recall as well that the Moore complex $N(G_\bullet)$ associated to any simplicial group, G_\bullet , [18], is the following

$$N(G_\bullet) : \dots N_q(G_\bullet) \xrightarrow{\delta_q} N_{q-1}(G_\bullet) \longrightarrow \dots \longrightarrow N_1(G_\bullet) \xrightarrow{\delta_1} N_0(G_\bullet)$$

where $N_0(G_\bullet) = G_0$ and $N_q(G_\bullet) = \bigcap_{i=0}^{q-1} \text{Ker} d_i \subseteq G_q$ and δ_q is the restriction of $d_q : G_q \rightarrow G_{q-1}$ to $N_q(G_\bullet)$. For later use, let us recall that the Moore complex of the simplicial group $\text{Cosk}^{n-1}(G_\bullet)$ is the following:

$$N_q(\text{Cosk}^{n-1}(G_\bullet)) = \begin{cases} N_q(G_\bullet) & \text{if } 0 \leq q \leq n-1 \\ \text{Ker}(\delta_{n-1} : N_{n-1}(G_\bullet) \rightarrow N_{n-2}(G_\bullet)) & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

The (Moore) homotopy groups of any simplicial group G_\bullet are defined then as the corresponding homology groups of its Moore complex, that is, for $n \geq 0$

$$\Pi_n(G_\bullet) = \frac{\bigcap_{i=0}^n \text{Ker}(d_i : G_n \rightarrow G_{n-1})}{d_{n+1}(\bigcap_{i=0}^n \text{Ker}(d_i : G_{n+1} \rightarrow G_n))}$$

A closed model category in the sense of Quillen, [19], is a category \mathcal{C} with three distinguished classes of morphisms called fibrations, cofibrations and weak equivalences, satisfying the usual axioms,

CM1. The category \mathcal{C} has finite limits and colimits.

CM2. For any composable pair of arrows f, g in \mathcal{C} , if any two of f, g, fg are weak equivalences so is the third.

CM3. If f is a retract of g and g is a fibration, cofibration or weak equivalence, then f is also such, where one says that a morphism $f : X \rightarrow Y$ is a retract of $g : W \rightarrow Z$ if there are morphisms $i : X \rightarrow W, r : W \rightarrow X, j : Y \rightarrow Z$ and $s : Z \rightarrow Y$ such that $ri = Id_X, sj = Id_Y, gi = jf$ and $fr = sg$.

CM4. Lifting axiom. Given a solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array} \quad (0)$$

where i is a cofibration, p is a fibration and either i or p is also a weak equivalence, then the dotted arrow exists.

CM5. Factorization axiom. Any morphism f in \mathcal{C} can be factored both as $f = pi$ and $f = qj$, where p and q are fibrations, i, j are cofibrations and p, j are weak equivalences.

This set of axioms is, of course, equivalent to the original (see [18]). A morphism in \mathcal{C} , $i : A \rightarrow B$, is said to have the *left lifting property*, LLP, with respect to a morphism $p : X \rightarrow Y$ (and p is said to have the *right lifting property*, RLP, with respect to i) if the dotted arrow exists in any diagram of the form (0).

Finally, let us recall that the Quillen's closed model structure over $\mathbf{Simp}(\mathbf{Gp})$ described in [18], has the Kan fibrations as fibrations, those morphisms which induce isomorphisms between the homotopy groups as weak equivalences and the cofibrations are defined by the LLP with respect to trivial fibrations (fibrations and weak equivalences). By means of the free group functor $F : \mathbf{Simp}(\mathbf{Sets}) \rightarrow \mathbf{Simp}(\mathbf{Gp})$ from the category of all simplicial sets, (trivial) fibrations in $\mathbf{Simp}(\mathbf{Gp})$ can be characterized as follows:

Proposition 2.1 ([18], Prop. 1.2). *Let $f_\bullet : X_\bullet \rightarrow Y_\bullet$ be a morphism of simplicial groups. Then,*

1. *f_\bullet is a fibration iff it has the RLP with respect to the morphisms $F\Delta[n, k] \rightarrow F\Delta[n]$, induced by the inclusions $\Delta[n, k] \hookrightarrow \Delta[n]$, for $0 \leq k \leq n, n > 0$.*
2. *f_\bullet is a trivial fibration iff it has the RLP with respect to the family of morphisms $F\dot{\Delta}[n] \rightarrow F\Delta[n]$, induced by the inclusions $\dot{\Delta}[n] \hookrightarrow \Delta[n]$, for all $n \geq 0$.*

3 Truncating the closed model structure in $\mathbf{Simp}(\mathbf{Gp})$

Definition 3.1. *A morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ is said to be a truncated Kan fibration at dimension n if it satisfies the Kan condition up to dimension n , that is, the canonical morphism*

$$E_s \rightarrow B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$$

is surjective for $0 \leq k \leq s$ and $0 < s \leq n$. A morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ is said to be an n -fibration, $n \geq 1$, if p_\bullet is a truncated Kan fibration at dimension $n+2$ and the induced morphism on the $(n+1)$ -th homotopy groups, $\bar{p}_{n+1} : \Pi_{n+1}(E_\bullet) \rightarrow \Pi_{n+1}(B_\bullet)$, is surjective.

Hence, one observes that an ∞ -fibration in $\mathbf{Simp}(\mathbf{Gp})$ is exactly a Kan fibration, retrieving so the classical definition. This can also be observed from the following characterization of n -fibrations in terms of the RLP:

Proposition 3.2. *For any morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ the following are equivalent:*

1. *p_\bullet is an n -fibration.*
2. *p_\bullet has the RLP with respect to the following two families of morphisms in $\mathbf{Simp}(\mathbf{Sets})$:*

$$* \rightarrow S^{n+1} = \frac{\Delta[n+1]}{\dot{\Delta}[n+1]}$$

and

$$\Delta[s, k] \hookrightarrow \Delta[s], 0 \leq k \leq s, 0 < s \leq n + 2,$$

or equivalently with respect to the induced families of morphisms of simplicial groups:

$$F* \rightarrow F, S^{n+1} \text{ and } F\Delta[s, k] \rightarrow F\Delta[s], 0 \leq k \leq s, 0 < s \leq n + 2,$$

Proof. In order to show the existence of lifting D_\bullet in any given commutative square

$$\begin{array}{ccc} \Delta[s, k] & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \nearrow D_\bullet & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b_\bullet} & B_\bullet \end{array} \quad (1)$$

with $0 \leq k \leq s, 0 < s \leq n + 2$, first one applies the condition of Kan fibration at dimension s to the element $(y, (x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_s)) \in B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$, where $(x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_s) \in \Lambda_k^s(E_\bullet)$ is determined by the simplicial map a_\bullet , and $y \in B_s$ is the s -simplex represented by b_\bullet . The analogy in properties between simplicial maps coming from $\Delta[s]$ and s -simplices of the target -as remarked in section 2- is again present since the element $(y, (x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_s))$ belongs to the fiber product $B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$, because of the commutativity of (1). As p_\bullet is a truncated Kan fibration at dimension $n + 2$ there exists $x \in E_s$ such that $d_i x = x_i$, for $i \neq k$ and $p_s(x) = y$. The required lifting for (1), D_\bullet , is then the representing map of $x \in E_s$. As for the RLP with respect to $* \rightarrow S^{n+1}$, consider any commutative square of type

$$\begin{array}{ccc} * & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \nearrow d_\bullet & \downarrow p_\bullet \\ S^{n+1} & \xrightarrow{b_\bullet} & B_\bullet \end{array} \quad (2)$$

and take $y \in B_{n+1}$ represented by the simplicial map b_\bullet . As it happened above, $d_i y = e$, $0 \leq i \leq n + 1$ because of the commutativity of the diagram (2). For the surjectivity of \bar{p}_{n+1} there is then $\bar{x} \in \Pi_{n+1}(E_\bullet)$ such that $\bar{p}_{n+1}(\bar{x}) = \bar{y}$. Let $z \in B_{n+2}$ the homotopy

between $p_{n+1}(x)$ and y , i.e., z holds $d_{n+2}z = y, d_{n+1}z = p_{n+1}(x)$ and $d_i z = s_n d_i p_{n+1}(x) = s_n d_i y$, for $i = 1, \dots, n$. Using then the condition of p_\bullet being a Kan fibration at dimension $n + 2$ for the element $(z, (x_0, \dots, x_k, x_{k+1}, \dots, x_{n+1})) \in B_{n+2} \times_{\Lambda_{n+2}^{n+2}(B_\bullet)} \Lambda_{n+2}^{n+2}(E_\bullet)$ where $x_j = s_n d_j x, j = 0, \dots, n, x_{n+1} = x$, and taking into consideration that z verifies that $d_i z = p_{n+1}(x_i)$ for $i \neq n + 2$, there exists $x' \in E_{n+2}$ such that $d_i x' = x_i, i \neq n + 2$ and $p_{n+2}(x') = z$.

Note that $d_{n+2}x' \in E_{n+1}$ verifies $d_i d_{n+2}x' = e, i = 0, \dots, n + 1$ and $p_{n+1}(d_{n+2}x') = y$; hence, the required lifting for (2), d_\bullet , is defined as the representing map of $d_{n+2}x'$. That proves the first half.

As for the opposite direction, it should be noted that if p_\bullet has the RLP with respect to $\Delta[s, k] \hookrightarrow \Delta[s]$ for $0 \leq k \leq s, 0 < s \leq n + 2$, then it is a truncated Kan fibration at dimension $n + 2$ just by reversing the same argument given for the first implication. Then, it only remains to show that $\bar{p}_{n+1} : \Pi_{n+1}(E_\bullet) \rightarrow \Pi_{n+1}(B_\bullet)$ is surjective. To this end, take $y \in B_{n+1}$ with $d_i y = e, i = 0, \dots, n + 1$ a representative of any $\bar{y} \in \Pi_{n+1}(B_\bullet)$ and consider the following square:

$$\begin{array}{ccc}
 * & \xrightarrow{a_\bullet} & E_\bullet \\
 \downarrow & \nearrow D_\bullet & \downarrow p_\bullet \\
 S^{n+1} & \xrightarrow{b_\bullet} & B_\bullet
 \end{array}$$

where a_\bullet is the unique simplicial map from $*$ and b_\bullet is the representing map of y . This diagram commutes owing to $d_i y = e, i = 0, \dots, n + 1$ and so there is a lifting $D_\bullet : S^{n+1} \rightarrow E_\bullet$. If d_\bullet is the composition $\Delta[n + 1] \longrightarrow S^{n+1} \xrightarrow{D_\bullet} E_\bullet$, it represents a simplex $x \in E_{n+1}$ verifying $d_i x = e, i = 0, \dots, n + 1$ and $p_{n+1}(x) = y$. Then, $\bar{x} \in \Pi_{n+1}(E_\bullet)$ is the desired preimage which shows \bar{p}_{n+1} to be surjective. \square

The following lemma summarizes truncated versions of some results due to Quillen ([18]):

Lemma 3.3. *Let $f_\bullet : G_\bullet \rightarrow H_\bullet$ be a morphism in $\mathbf{Simp}(\mathbf{Gp})$. Then,*

1. $f_q : G_q \rightarrow H_q$ is surjective (injective) for $0 \leq q \leq n$ iff $N_q(f) : N_q(G_\bullet) \rightarrow N_q(H_\bullet)$ is surjective (injective) for $0 \leq q \leq n$.

2. f_\bullet is a truncated Kan fibration at dimension n iff $N_q(f) : N_q(G_\bullet) \rightarrow N_q(H_\bullet)$ is surjective for $0 < q \leq n$.
3. $f_q : G_q \rightarrow H_q$ is surjective for $0 \leq q \leq n$ iff f_\bullet is a truncated Kan fibration at dimension n and the induced morphism on the 0-th homotopy group, $\bar{f}_0 : \Pi_0(G_\bullet) \rightarrow \Pi_0(H_\bullet)$ is surjective.

After the previous result, the n -fibrations can be also characterized as follows:

Corollary 3.4. *For any morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ the following are equivalent:*

1. p_\bullet is an n -fibration.
2. $N_q(p_\bullet)$ is surjective, $0 < q \leq n + 2$ and $\bar{p}_{n+1} : \Pi_{n+1}(E_\bullet) \rightarrow \Pi_{n+1}(B_\bullet)$ is surjective.

As a direct consequence of this, the n -fibrant simplicial groups are identified:

Corollary 3.5. *Every object G_\bullet in $\mathbf{Simp}(\mathbf{Gp})$ is n -fibrant, $n > 0$, i.e., the morphism $G_\bullet \rightarrow *$ is an n -fibration.*

Next, other kind of relevant morphisms for the future closed model n -structure.

Definition 3.6. *A morphism $f_\bullet : X_\bullet \rightarrow Y_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ is said to be an n -weak equivalence, $n > 0$, if the induced morphism on the homotopy groups, $\bar{f}_q : \Pi_q(X_\bullet) \rightarrow \Pi_q(Y_\bullet)$, $0 \leq q \leq n$, is an isomorphism.*

The morphism f_\bullet will be called an n -trivial fibration if it is an n -fibration and an n -weak equivalence.

Formulation and proof of the corresponding characterization in terms of the RLP can be given now for n -trivial fibrations.

Proposition 3.7. *For any morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ the following are equivalent:*

1. p_\bullet is an n -trivial fibration.

2. p_\bullet has the RLP with respect to the following two families of maps in $\mathbf{Simp}(\mathbf{Sets})$:

$$\dot{\Delta}[s] \hookrightarrow \Delta[s], \quad 0 \leq s \leq n + 1$$

and

$$\Delta[n + 2, k] \hookrightarrow \Delta[n + 2], \quad 0 \leq k \leq n + 2,$$

or equivalently with respect to the induced families of morphisms of simplicial groups:

$$F\dot{\Delta}[s] \rightarrow F\Delta[s], \quad 0 \leq s \leq n + 1$$

and

$$F\Delta[n + 2, k] \rightarrow F\Delta[n + 2], \quad 0 \leq k \leq n + 2,$$

Proof. Let us prove the first direction. Note that, by 3.2, p_\bullet has the RLP with respect to $\Delta[n + 2, k] \hookrightarrow \Delta[n + 2]$ particularly. Hence, it remains to show the existence of lifting in commutative squares such as

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \nearrow & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b_\bullet} & B_\bullet \end{array} \quad (3)$$

For $s = 0$, it can be observed that the existence of lifting in (3) is equivalent to p_0 being surjective. To this end, take any $y \in B_0$ and consider its homotopy class into $\Pi_0(B_\bullet)$, \bar{y} . As $\bar{p}_0 : \Pi_0(E_\bullet) \rightarrow \Pi_0(B_\bullet)$ is surjective (even more, it is an isomorphism), there exists $\bar{x} \in \Pi_0(E_\bullet)$ such that $\bar{p}_0(\bar{x}) = \bar{y}$. Let z be the corresponding homotopy between them, i.e., $d_0z = p_0(x)$ and $d_1z = y$ and then, since p_\bullet is an 1-truncated Kan fibration, there is $x' \in E_1$ such that $d_0x' = x$ and $p_1(x') = z$. The element d_1x' is then the required preimage for y .

In order to prove that there exists a lifting in (3), $0 < s \leq n + 1$, let $y \in B_s$ be the simplex with representing morphism b_\bullet and $(x_0, \dots, x_{s-1}, x_s)$ the element of $\Delta^n(E_\bullet)$ determined by a_\bullet .

As usual, the commutativity of (3) establishes $(y, (x_0, \dots, x_{s-1}))$ as an element of $B_s \times_{\Lambda_s^s(B_\bullet)} \Lambda_s^s(E_\bullet)$. Now, p_\bullet is a truncated Kan fibration

at dimension $n + 2$ so there exists $\alpha \in E_s$ such that $d_i\alpha = x_i, i \neq s$ and $p_s(\alpha) = y$. It is not difficult to confirm that $x_s - d_s\alpha \in Ker\bar{p}_{s-1}$, which is particularly injective, so $x_s - d_s\alpha = \bar{e}$. Let us call α' to the homotopy between $x_s - d_s\alpha$ and e , i.e., $d_s\alpha' = x_s - d_s\alpha$ and $d_i\alpha' = e, i = 0, \dots, s-1$, and consider the s -simplex of E_\bullet , $\alpha'' = \alpha' + \alpha$, which verifies $d_i\alpha'' = x_i$ and $d_i(p_s(\alpha'')) = y, i = 0, \dots, s$. That allows one to consider $p_s(\alpha'') - y \in \Pi_s(B_\bullet)$ and, since \bar{p}_s is an epimorphism for $0 < s \leq n + 1$ (an isomorphism up to dimension n , indeed), there exists $\bar{z} \in \Pi_s(E_\bullet)$ such that $\bar{p}_s(\bar{z}) = p_s(\alpha'') - y$.

Let $w \in B_{s+1}$ be the homotopy between $p_s(\alpha'') - y$ and $p_s(\bar{z})$ and apply again the condition of Kan fibration at dimension $s + 1$ to the element $(w, (e, \dots, e, z)) \in B_s \times_{\Lambda_{s+1}^{s+1}(B_\bullet)} \Lambda_{s+1}^{s+1}(E_\bullet)$ to find $\lambda \in E_{s+1}$ which holds $d_i\lambda = e, i = 0, \dots, s-1, d_s\lambda = z$ and $p_{s+1}(\lambda) = w$.

Finally, the required lifting for (3) is defined as the representing morphism of $x = -d_{s+1}\lambda + \alpha''$.

The converse will be proved in two steps:

First, it will be showed that if p_\bullet has the RLP with respect to $\dot{\Delta}[s] \hookrightarrow \Delta[s]$, for $0 \leq s \leq n + 1$ then \bar{p}_s is surjective and \bar{p}_{s-1} is injective. That leads to \bar{p}_{n+1} surjective and \bar{p}_i isomorphism for $0 \leq i \leq n$ (particularly, p_\bullet will be an n -weak equivalence).

In order to prove that \bar{p}_s is surjective, take any $\bar{y} \in \Pi_s(B_\bullet)$ and consider the following square

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \nearrow D_\bullet & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b_\bullet} & B_\bullet \end{array}$$

constructed by the representing map b_\bullet of any representative element y of \bar{y} , and the simplicial map a_\bullet defined by the element of the simplicial kernel (e, \dots, e) . It is clear that this square commutes so there exists a lifting $D_\bullet : \Delta[s] \rightarrow E_\bullet$. The homotopy class of the s -simplex of E_\bullet represented by D_\bullet proves then that \bar{p}_s is surjective, $0 \leq s \leq n + 1$.

To prove now that \bar{p}_{s-1} is injective, take any $\bar{x} \in Ker\bar{p}_{s-1}$ and let $w \in B_s$ be the homotopy between $p_{s-1}(x)$ and e (so $d_s w = p_{s-1}(x)$ and $d_i w = e$ for $i = 0, \dots, s-1$). With the representing map of w , say b'_\bullet ,

and that of the element of the simplicial kernel (e, \cdot^s, e, x) , written a'_s , the following commutative square can be constructed for $0 \leq s \leq n + 1$,

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{a'_s} & E_\bullet \\ \downarrow & \nearrow D'_s & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b'_s} & B_\bullet \end{array}$$

where the lifting D'_s , that there exists by applying the hypothesis, determines the homotopy which shows that $\bar{x} = \bar{e}$. Hence, \bar{p}_{s-1} is injective, $0 \leq s \leq n + 1$.

The second step consists of showing that to have the RPL with respect to $\dot{\Delta}[s] \hookrightarrow \Delta[s]$, for $0 \leq s \leq n + 1$ implies that p_\bullet is a truncated Kan fibration at dimension $n + 1$. To this end, take any element $(y, (x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_s)) \in B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$, $0 \leq k \leq s$, $0 \leq s \leq n + 1$, and define the following collection of s $(s - 2)$ -simplices of E_\bullet :

$$z_i = \begin{cases} d_{k-1}x_i & \text{if } i \leq k - 1 \\ d_k x_{i+1} & \text{if } i > k - 1 \end{cases}$$

It appears clear that $(z_0, \dots, z_{k-1}, z_{k+1}, \dots, z_s) \in \Delta^{s-1}(E_\bullet)$. Then, considering the simplicial map $c_\bullet : \dot{\Delta}[s - 1] \rightarrow E_\bullet$ associated to this, and the representing map of $d_k y \in B_{s-1}$, $d_\bullet : \Delta[s - 1] \rightarrow B_\bullet$, one may form the following commutative square

$$\begin{array}{ccc} \dot{\Delta}[s - 1] & \xrightarrow{c_\bullet} & E_\bullet \\ \downarrow & \nearrow H_\bullet & \downarrow p_\bullet \\ \Delta[s - 1] & \xrightarrow{d_\bullet} & B_\bullet \end{array}$$

for which there is a lifting $H_\bullet : \Delta[s - 1] \rightarrow E_\bullet$. Let x_k be the $(s - 1)$ -simplex of E_\bullet with representing map H_\bullet and note that the element $(x_0, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_s) \in \Delta^s(E_\bullet)$. Since $d_i y = p_{s-1}(x_i)$ for $i = 0, \dots, k, \dots, s$, one may find a lifting G_\bullet on the square

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{c'_s} & E_\bullet \\ \downarrow & \nearrow G_\bullet & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{d'_s} & B_\bullet \end{array}$$

where c'_\bullet is associated to $(x_0, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_s)$ and d'_\bullet is the representing map of $y \in B_s$. The s -simplex x' of E_\bullet with representing map G_\bullet , verifies that $d_i x' = x_i, i \neq k$ and $p_s(x') = y$. Consequently, p_\bullet is a truncated Kan fibration at dimension $n + 1$. This brings the proof to an end since the condition of being a Kan fibration at dimension $n + 2$ is equivalent (see proof of 3.2) to the RLP with respect to $\Delta[n + 2, k] \hookrightarrow \Delta[n + 2]$, for $0 \leq k \leq n + 2$. \square

Another characterization of n -trivial fibrations can be given as a consequence of the following lemma, which is a truncated version of that given by Quillen (see [18], chapter II, §3, Prop. 2).

Lemma 3.8. *For any morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ the following are equivalent:*

1. p_\bullet is a truncated Kan fibration at dimension n , $\bar{p}_q : \Pi_q(E_\bullet) \rightarrow \Pi_q(B_\bullet)$ is an isomorphism for $0 \leq q \leq n$ and \bar{p}_{n+1} is surjective.
2. p_q is surjective and $\Pi_q(Ker p_\bullet) = 0$, for $0 \leq q \leq n$.

Then, the announced result is the following:

Corollary 3.9. *For any morphism $p_\bullet : E_\bullet \rightarrow B_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$, p_\bullet is an n -trivial fibration iff p_q is surjective, $0 \leq q \leq n + 2$ and $\Pi_q(Ker p_\bullet) = 0$ for $0 \leq q \leq n$.*

The third distinguished class of morphisms in a closed model category is defined as follows:

Definition 3.10. *A morphism $f_\bullet : X_\bullet \rightarrow Y_\bullet$ in $\mathbf{Simp}(\mathbf{Gp})$ is said to be an n -cofibration, $n > 0$, if it has the LLP with respect to the n -trivial fibrations.*

At this point, it is interesting to remark the relationship between all these truncated definitions. Note that one has the following sequence of inclusions for n -trivial fibrations:

$$(\infty - \text{triv.fib.}) \subseteq \dots \subseteq ((n + 1) - \text{triv.fib.}) \subseteq (n - \text{triv.fib.}) \subseteq \dots$$

Consequently, the definition of n -cofibration sets the corresponding sequence for them:

$$\dots \subseteq (n - \text{cofib.}) \subseteq ((n + 1) - \text{cofib.}) \subseteq \dots \subseteq (\infty - \text{cofib.})$$

4 The n -structure in $\mathbf{Simp}(\mathbf{Gp})$

The above definitions of n fibration, n -cofibration and n -weak equivalence will be used now to determine the n -structure in $\mathbf{Simp}(\mathbf{Gp})$, i.e., a closed model structure in the sense of Quillen for each n , that generalizes the classical one.

The following lemma will be needed in the proof of the main theorem of this section.

- Lemma 4.1.** 1. If $\{f_{\bullet}^i : A_{\bullet}^i \rightarrow B_{\bullet}^i\}_{i \in I}$ is a family of n -cofibrations in $\mathbf{Simp}(\mathbf{Gp})$, then the induced morphism on the coproducts, $\sqcup_I A_{\bullet}^i \rightarrow \sqcup_I B_{\bullet}^i$ is an n -cofibration.
2. If $f_{\bullet} : A_{\bullet} \rightarrow B_{\bullet}$ is an n -cofibration in $\mathbf{Simp}(\mathbf{Gp})$ and $g_{\bullet} : A_{\bullet} \rightarrow C_{\bullet}$ is arbitrary, the induced morphism into the pushout, $C_{\bullet} \rightarrow B_{\bullet} \sqcup_{A_{\bullet}} C_{\bullet}$ is an n -cofibration.
3. If $A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_m \rightarrow \cdots$ is a sequence of n -cofibrations (resp. n -weak equivalences) in $\mathbf{Simp}(\mathbf{Gp})$, the canonical morphism $A_0 \rightarrow A_{\infty} = \varinjlim A_m$ is an n -cofibration (resp. n -weak equivalence).

Proof. The first two properties, together with the half of the third one concerning n -cofibrations, can be easily proved by the universal property of coproducts, pushouts and direct limits, in each case. As for the n -weak equivalences, just recall that in $\mathbf{Simp}(\mathbf{Gp})$, one has that $\Pi_i(\varinjlim X_m) = \varinjlim \Pi_i(X_m)$. \square

The main theorem can be enounced and showed now:

Theorem 4.2. *With the given definitions of n -fibration, n -cofibration and n -weak equivalence, $n > 0$, the category $\mathbf{Simp}(\mathbf{Gp})$ is, in each case, a closed model category.*

Proof. The first axiom is well-known and the second can be proved without any difficulty. As for the third related to n -fibrations, let $p_{\bullet} : E_{\bullet} \rightarrow B_{\bullet}$ be an n -fibration and $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ a retract of p_{\bullet} with

$r_\bullet : E_\bullet \rightarrow X_\bullet$ as retraction. The required liftings in the commutative diagrams

$$\begin{array}{ccc} \Delta[s, k] & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow f_\bullet \\ \Delta[s] & \longrightarrow & Y_\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} * & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow f_\bullet \\ S^{n+1} & \longrightarrow & Y_\bullet \end{array}$$

are given by the composition with r_\bullet of the corresponding liftings for p_\bullet . The cases involving n -cofibrations and n -weak equivalences are analogous.

We prove now **CM5**, concerning the factorization of any morphism $f_\bullet : X_\bullet \rightarrow Y_\bullet$ into n -cofibration followed by n -trivial fibration, by means of the so-called "small object argument". Consider the following diagram

$$\begin{array}{ccccccc} X_\bullet & \xrightarrow{i_{\bullet 0}} & X_{\bullet 0} & \xrightarrow{i_{\bullet 1}} & X_{\bullet 1} & \xrightarrow{i_{\bullet 2}} & \dots \\ f_\bullet \downarrow & q_{\bullet 0} \swarrow & & q_{\bullet 1} \swarrow & & & \\ Y_\bullet & & & & & & \end{array}$$

where, having obtained $X_{\bullet m}$ and considering all commutative diagrams of the form

$$\begin{array}{ccc} F\hat{\Delta}[s] & \longrightarrow & X_{\bullet m} \\ \downarrow & \nearrow \lambda & \downarrow q_{\bullet m} \\ F\Delta[s] & \longrightarrow & Y_\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} F\Delta[n+2, k] & \longrightarrow & X_{\bullet m} \\ \downarrow & \nearrow \mu & \downarrow q_{\bullet m} \\ F\Delta[n+2] & \longrightarrow & Y_\bullet \end{array}$$

$0 \leq s \leq n+1$ and $0 \leq k \leq n+2$, distinguished by indexes λ and μ , then the morphism $i_{\bullet, m+1} : X_{\bullet m} \rightarrow X_{\bullet, m+1}$ is defined by the pushout diagram

$$\begin{array}{ccc} (\sqcup_\lambda F\hat{\Delta}[s]) \sqcup (\sqcup_\mu F\Delta[n+2, k]) & \longrightarrow & X_{\bullet m} \\ \downarrow & & \downarrow i_{\bullet, m+1} \\ (\sqcup_\lambda F\Delta[s]) \sqcup (\sqcup_\mu F\Delta[n+2]) & \longrightarrow & X_{\bullet, m+1} \end{array}$$

Let us define now $X_{\bullet\infty} = \varinjlim X_m$ and $q_{\bullet} = \varinjlim q_m$ to have a factorization

$$\begin{array}{ccc} X_{\bullet} & \xrightarrow{f_{\bullet}} & Y_{\bullet} \\ & \searrow i_{\bullet} & \nearrow q_{\bullet} \\ & X_{\bullet\infty} & \end{array}$$

where q_{\bullet} is an n -trivial fibration by 3.7. In order to show now that i_{\bullet} is an n -cofibration, note that $F\Delta[s] \rightarrow F\Delta[s], 0 \leq s \leq n+1$ and $F\Delta[n+2, k] \rightarrow F\Delta[n+2], 0 \leq k \leq n+2$ are n -cofibrations and so, by 4.1, the coproducts of them are also such, so every $i_{\bullet m}$ is. Finally i_{\bullet} is an n -cofibration, again by 4.1.

We shall be concerned now about the factorization of any morphism f_{\bullet} into an n -trivial cofibration followed by an n -fibration. To this end, let us repeat the same argument used in the factorization already proved, taking in this case all diagrams of the form

$$\begin{array}{ccc} F_{\bullet} \longrightarrow X_{\bullet m} & & F\Delta[s, k] \longrightarrow X_{\bullet m} \\ \downarrow & \searrow \lambda & \downarrow p_{\bullet m} \\ F S^{n+1} \longrightarrow Y_{\bullet} & & F\Delta[s] \longrightarrow Y_{\bullet} \end{array} \text{ and } \begin{array}{ccc} F\Delta[s, k] \longrightarrow X_{\bullet m} & & \\ \downarrow & \searrow \mu & \downarrow p_{\bullet m} \\ F\Delta[s] \longrightarrow Y_{\bullet} & & \end{array}$$

for $0 \leq k \leq s, 0 < s \leq n+2$. Let us define then $j_{\bullet m} : X_{\bullet m} \rightarrow X_{\bullet m+1}$ by the pushout diagram

$$\begin{array}{ccc} (\sqcup_{\lambda} F_{\bullet}) \sqcup (\sqcup_{\mu} F\Delta[s, k]) & \longrightarrow & X_{\bullet m} \\ \downarrow & & \downarrow j_{\bullet m+1} \\ (\sqcup_{\lambda} F S^{n+1}) \sqcup (\sqcup_{\mu} F\Delta[s]) & \longrightarrow & X_{\bullet m+1} \end{array}$$

Let $X_{\bullet\infty}$ be $\varinjlim X_m$ and $p_{\bullet} = \varinjlim p_m$ one obtains, as above, a factorization $f_{\bullet} = p_{\bullet} j_{\bullet}$, where p_{\bullet} is an n -fibration by 3.2 and j_{\bullet} is an n -cofibration by 4.1, since $F_{\bullet} \rightarrow F S^{n+1}$ and $F\Delta[s, k] \rightarrow F\Delta[s], 0 \leq k \leq s, 0 < s \leq n+2$ are n -cofibrations.

It only remains to prove that j_{\bullet} is an n -weak equivalence, which will be achieved by showing that every $j_{\bullet m}$ is, applying 4.1 again. For this, it should be noted that each $j_{\bullet m} : X_{\bullet m-1} \rightarrow X_{\bullet m}$ is the composition

$X_{\bullet, m-1} \xrightarrow{\alpha_{\bullet}} T_{\bullet} \xrightarrow{\beta_{\bullet}} X_{\bullet, m}$ of simplicial maps α_{\bullet} and β_{\bullet} , both obtained from the following pushouts diagrams:

$$\begin{array}{ccc} \sqcup_{\lambda} F\Delta[s, k] & \longrightarrow & X_{\bullet, m-1} \\ \downarrow & & \downarrow \alpha_{\bullet} \\ \sqcup_{\lambda} F\Delta[s] & \longrightarrow & T_{\bullet} \end{array} \quad \text{and} \quad \begin{array}{ccc} \sqcup_{\mu} F* & \longrightarrow & T_{\bullet} \\ \downarrow & & \downarrow \beta_{\bullet} \\ \sqcup_{\mu} FS^{n+1} & \longrightarrow & X_{\bullet, m} \end{array}$$

It is clear that α_{\bullet} is a weak equivalence -with the classical model structure in $\mathbf{Simp}(\mathbf{Gp})$ - so it is an n -weak equivalence, for all $n > 0$. In order to prove that β_{\bullet} is an n -weak equivalence as well, note that the free simplicial groups $F*$ and FS^{n+1} are equal to \mathbb{Z} up to dimension n ($F*$ is the constant simplicial group having \mathbb{Z} at each dimension whereas FS^{n+1} has \mathbb{Z} up to dimension n and the free product $\mathbb{Z} \sqcup \mathbb{Z}$ at level $n+1$, indeed). Hence, $\beta_{\bullet} : T_{\bullet} \rightarrow X_{\bullet, m}$ is an isomorphism up to dimension n , inducing then isomorphisms on the corresponding homotopy groups up (and including) dimension $n-1$.

Let us analyze now the case of the n th-homotopy groups. First, note the existence of a morphism $v_{\bullet} : X_{\bullet, m} \rightarrow T_{\bullet}$ such that $v_{\bullet}\beta_{\bullet} = Id_{T_{\bullet}}$, so β_{\bullet} is a section. This comes from the fact that $*$ is the terminal object in $\mathbf{Simp}(\mathbf{Sets})$ and hence, there is a morphism $S^{n+1} \rightarrow *$ such that, for the uniqueness of such maps, verifies that the composition $* \rightarrow S^{n+1} \rightarrow *$ is equal to the identity on $*$. As for the rest, it comes from the properties of the free group functor, F , coproducts and pushouts diagrams, respectively.

In order to compute the n th-homotopy groups, let us consider them as the n th-homology groups of the corresponding Moore complex. This situation can be pictured in the following diagram,

$$\begin{array}{ccccccc} N_{n+1}(T_{\bullet}) & \longrightarrow & N_n(T_{\bullet}) & \longrightarrow & N_{n-1}(T_{\bullet}) & \longrightarrow & \dots \\ \uparrow \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ N_{n+1}(X_{\bullet, m}) & \longrightarrow & N_n(X_{\bullet, m}) & \longrightarrow & N_{n-1}(X_{\bullet, m}) & \longrightarrow & \dots \end{array}$$

which clearly shows that $\Pi_n(T_{\bullet}) \cong \Pi_n(X_{\bullet, m})$. Then, j_{\bullet} is an n -trivial cofibration. In addition, j_{\bullet} has by construction the LLP with respect to any n -fibration.

It only remains to prove the lifting axiom **CM4**. In fact, the only half to be shown is that dealing with the existence of lifting in any commutative square of the form

$$\begin{array}{ccc}
 A_{\bullet} & \xrightarrow{a_{\bullet}} & X_{\bullet} \\
 \downarrow k_{\bullet} & & \downarrow q_{\bullet} \\
 B_{\bullet} & \xrightarrow{b_{\bullet}} & Y_{\bullet}
 \end{array} \quad (4)$$

with q_{\bullet} n -fibration and k_{\bullet} n -trivial cofibration, because the other half comes straightforward from 3.10. To this end, factor the morphism k_{\bullet} by **CM5** into an n -fibration p_{\bullet} followed by an n -trivial cofibration i_{\bullet} and note that, since both k_{\bullet} and i_{\bullet} are n -weak equivalences so it is p_{\bullet} , by **CM2**. On the other hand, there exists a lifting h_{\bullet} in the following commutative square

$$\begin{array}{ccc}
 A_{\bullet} & \xrightarrow{a_{\bullet}} & X_{\bullet} \\
 \downarrow i_{\bullet} & \nearrow h_{\bullet} & \downarrow q_{\bullet} \\
 A_{\infty} & \xrightarrow{b_{\bullet} p_{\bullet}} & Y_{\bullet}
 \end{array}$$

since i_{\bullet} has the LLP with respect to all n -fibrations. Finally, the required lifting for (4) is given by composing h_{\bullet} with the lifting s_{\bullet} obtained from the diagram

$$\begin{array}{ccc}
 A_{\bullet} & \xrightarrow{i_{\bullet}} & A_{\infty} \\
 \downarrow k_{\bullet} & \nearrow s_{\bullet} & \downarrow p_{\bullet} \\
 B_{\bullet} & \xlongequal{\quad} & B_{\bullet}
 \end{array}$$

which exists because k_{\bullet} is n -cofibration (n -trivial cofibration really) and p_{\bullet} is an n -trivial fibration, as remarked before. \square

Throughout the above proof the key to characterize the n -cofibrations has also been given and so, the n -cofibrant simplicial groups are identified.

Proposition 4.3. *A morphism $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ in $\mathbf{Simp}(\mathbf{Gp})$ is an n -cofibration iff it is a retract of the morphism $i_{\bullet} : X_{\bullet} \rightarrow X_{\infty}$ obtained*

from the factorization of f_\bullet into a n -cofibration followed by an n -trivial fibration.

Proof. Supposing f_\bullet is an n -cofibration, let us factor it by the first half of **CM5** into n -cofibration, i_\bullet , and n -trivial fibration, q_\bullet . The following diagram shows, then, that f_\bullet is a retract of i_\bullet :

$$\begin{array}{ccc} X_\bullet & \xlongequal{\quad} & X_\bullet \\ \downarrow f_\bullet & & \downarrow i_\bullet \\ Y_\bullet & \xrightleftharpoons[q_\bullet]{d_\bullet} & X_\infty \end{array}$$

where d_\bullet is the lifting found by **CM4** in the commutative square

$$\begin{array}{ccc} X_\bullet & \xrightarrow{i_\bullet} & X_\infty \\ \downarrow f_\bullet & \nearrow d_\bullet & \downarrow q_\bullet \\ Y_\bullet & \xlongequal{\quad} & Y_\bullet \end{array}$$

Conversely, if f_\bullet is a retract of the n -cofibration i_\bullet , f_\bullet is, by **CM3**, an n -cofibration. □

5 Comparing model structures in $\mathbf{Simp}(\mathbf{Gp})$

In previous sections, the n -structure in $\mathbf{Simp}(\mathbf{Gp})$ has been showed to be a generalization of the classical Quillen's closed model structure. In this one, it will be appear as the tool to present the category $\mathbf{Simp}(\mathbf{Gp})$ as a model for connected $(n + 1)$ -types of spaces.

To this end, the machinery used will be the skeletal-coskeletal adjunction. Let us point out then that $\Pi_i(\mathit{Cosk}^{n+1}G_\bullet) = 0$ for $i > n$ and so, the image of Cosk^{n+1} could be identified with the full subcategory of $\mathbf{Simp}(\mathbf{Gp})$ formed by the n -coconnected simplicial groups, that is, those simplicial groups H_\bullet such that $\Pi_i(H_\bullet) = 0$ for $i > n$.

Theorem 5.1. *For the adjunction*

$$Sk^{n+1} : \mathbf{Simp}(\mathbf{Gp}) \rightleftarrows \mathbf{Simp}(\mathbf{Gp}) : \mathit{Cosk}^{n+1} \quad (5)$$

it holds:

1. If $p_\bullet : E_\bullet \rightarrow B_\bullet$ is an n -fibration (resp. n -weak equivalence), then $Cosk^{n+1}p_\bullet : Cosk^{n+1}E_\bullet \rightarrow Cosk^{n+1}B_\bullet$ is a Kan fibration (resp. weak equivalence).
2. If $i_\bullet : A_\bullet \rightarrow C_\bullet$ is a cofibration (resp. weak equivalence) in $\mathbf{Simp}(\mathbf{Gp})$, then $Sk^{n+1}i_\bullet$ is an n -cofibration (resp. n -weak equivalence).

Proof. Suppose $p_\bullet : E_\bullet \rightarrow B_\bullet$ is a n -fibration. By 3.4, this implies that $N_q(p_\bullet) : N_q(E_\bullet) \rightarrow N_q(B_\bullet)$ is surjective for $0 < q \leq n + 2$. In order to prove that $Cosk^{n+1}p_\bullet$ is a Kan fibration, $N_q(Cosk^{n+1}p_\bullet)$ to be surjective, for $q > 0$, will be proved: for the special feature of $N_q(Cosk^{n+1}G_\bullet)$ for any simplicial group G_\bullet , the surjectivity is clear for $0 < q \leq n + 1$. In the next level, the situation is as pictured:

$$\begin{array}{ccccc}
 N_{n+2}(E_\bullet) & \xrightarrow{d_{n+2}} & N_{n+1}(E_\bullet) & & \\
 \downarrow & \searrow & \nearrow & & \downarrow \\
 & & Imd_{n+2} & & \\
 N_{n+2}(B_\bullet) & \xrightarrow{d'_{n+2}} & N_{n+1}(B_\bullet) & & \\
 \downarrow & \searrow & \nearrow & & \downarrow \\
 & & Imd'_{n+2} & &
 \end{array}$$

From this, $Imd_{n+2} \rightarrow Imd'_{n+2}$ is surjective and then, since $\bar{p}_{n+1} : \Pi_{n+1}(E_\bullet) \rightarrow \Pi_{n+1}(B_\bullet)$ is onto for being p_\bullet an n -fibration, the following diagram of short exact sequences gives the surjectivity of $N_{n+2}(Cosk^{n+1}p_\bullet) : Kerd_{n+1} \rightarrow Kerd'_{n+1}$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Imd_{n+2} & \longrightarrow & Kerd_{n+1} & \longrightarrow & \Pi_{n+1}(E_\bullet) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Imd'_{n+2} & \longrightarrow & Kerd'_{n+1} & \longrightarrow & \Pi_{n+1}(B_\bullet) \longrightarrow 0
 \end{array}$$

The case involving n -weak equivalences follows straightforward: for $i > n$ because $\Pi_i(\text{Cosk}^{n+1}E_\bullet) = 0$ and for $0 \leq i \leq n$, because p_\bullet is an n -weak equivalence. This ends the first part of the theorem.

As for the second half, let $i_\bullet : A_\bullet \rightarrow C_\bullet$ be any cofibration in $\mathbf{Simp}(\mathbf{Gp})$. Consequently, to prove that $Sk^{n+1}i_\bullet$ is an n -cofibration consists of finding lifting in any commutative square of the form

$$\begin{array}{ccc} Sk^{n+1}A_\bullet & \longrightarrow & E_\bullet \\ Sk^{n+1}i_\bullet \downarrow & & \downarrow p_\bullet \\ Sk^{n+1}C_\bullet & \longrightarrow & B_\bullet \end{array}$$

with p_\bullet any n -trivial fibration. This is equivalent, by the adjunction (5), to doing it through the diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & \text{Cosk}^{n+1}E_\bullet \\ i_\bullet \downarrow & & \downarrow \text{Cosk}^{n+1}p_\bullet \\ C_\bullet & \longrightarrow & \text{Cosk}^{n+1}B_\bullet \end{array}$$

where the desired lifting exists for being $\text{Cosk}^{n+1}p_\bullet$ a trivial fibration (by the first half, already proved). Hence, $Sk^{n+1}i_\bullet$ is an n -cofibration.

As for the weak equivalences, let $i_\bullet : A_\bullet \rightarrow C_\bullet$ be a weak equivalence, note that, for any simplicial group G_\bullet , one has $\Pi_i(Sk^{n+1}G_\bullet) = \Pi_i(G_\bullet), 0 \leq i \leq n$. Then, $Sk^{n+1}i_\bullet$ is an n -weak equivalence. \square

If $Ho_n(\mathbf{Simp}(\mathbf{Gp}))$ stands for the homotopy category of $\mathbf{Simp}(\mathbf{Gp})$ associated to the n -structure and $Ho(\mathbf{Simp}(\mathbf{Gp})|n - \text{coconnected})$ denotes the full subcategory of $Ho(\mathbf{Simp}(\mathbf{Gp}))$ consisting of n -coconnected simplicial groups (i.e., with trivial homotopy groups for $i \geq n + 1$), one has

Corollary 5.2. *The functors Sk^{n+1} and Cosk^{n+1} induce an equivalence of categories*

$$Ho_n(\mathbf{Simp}(\mathbf{Gp})) \simeq Ho(\mathbf{Simp}(\mathbf{Gp})|n - \text{coconnected})$$

Theorem 5.1 also produces a new approach to the n -cofibrant simplicial groups. To this end, a simplicial group G_\bullet will be said to be

n - *skeletal* if $Sk^n G_\bullet = G_\bullet$. Then, a large number of n -cofibrant simplicial groups are determined by the following result.

Corollary 5.3. *Any retract of a free and $(n + 1)$ -skeletal simplicial group is n -cofibrant.*

Proof. Let G_\bullet be a retract of a free and $(n + 1)$ -skeletal simplicial group Z_\bullet , i.e., there are simplicial morphisms

$$G_\bullet \begin{array}{c} \xrightarrow{i_\bullet} \\ \xleftarrow{j_\bullet} \end{array} Z_\bullet$$

such that $j_\bullet i_\bullet = Id_{G_\bullet}$. Note then that the unique morphism $g_\bullet : * \rightarrow G_\bullet$ is a retract of $z_\bullet : * \rightarrow Z_\bullet$. In order to show that G_\bullet is n -cofibrant, let us find lifting on any commutative square

$$\begin{array}{ccc} * & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow g_\bullet & \nearrow D_\bullet & \downarrow p_\bullet \\ G_\bullet & \xrightarrow{b_\bullet} & B_\bullet \end{array}$$

with p_\bullet any n -trivial fibration. In fact, D_\bullet exists, defined as the composition $d_\bullet i_\bullet$, where d_\bullet is the lifting corresponding to the diagram

$$\begin{array}{ccc} Sk^{n+1} * & \xlongequal{\quad} & * \xrightarrow{a_\bullet} E_\bullet \\ Sk^{n+1} z_\bullet \downarrow & & \downarrow z_\bullet \nearrow d_\bullet \downarrow p_\bullet \\ Sk^{n+1} Z_\bullet & \xlongequal{\quad} & Z_\bullet \xrightarrow{b_\bullet j_\bullet} B_\bullet \end{array}$$

whose existence is justified by 5.1, because Z_\bullet is a free simplicial group and so z_\bullet is a cofibration. Particularly, $Sk^{n+1} z_\bullet = z_\bullet$ is an n -cofibration and the result follows. \square

At this point, let us recall that the non-abelian version of the classical Dold-Kan's theorem, given in [5], allowed to find, by a canonical process of truncation, a category, $\mathbf{n-HXC}(\mathbf{Gp})$, of n -hypercrossed complexes of groups, which provides algebraic models for n -simplicial groups and so, for connected $(n + 1)$ -types of spaces. This category is equivalent to the reflexive subcategory of $\mathbf{Simp}(\mathbf{Gp})$ whose objects are those simplicial groups with vanishing Moore complex in dimensions greater than

n . This last category is denoted $n - \mathbf{Hypgd}(\mathbf{Gp})$ because it is just the category of n -hypergroupoids of groups in the sense of Duskin-Glenn, [12].

Now, the category $n - \mathbf{Hypgd}(\mathbf{Gp})$ inherits a closed model structure from the structure of $\mathbf{Simp}(\mathbf{Gp})$, via the pair of adjoint functors J and \mathbb{P} , the inclusion functor and the reflector one,

$$\mathbb{P} : \mathbf{Simp}(\mathbf{Gp}) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} n - \mathbf{Hypgd}(\mathbf{Gp}) : J, \quad \mathbb{P} \dashv J$$

by defining fibrations (weak equivalences) as those morphisms f_\bullet such that $J(f_\bullet)$ is a fibration (weak equivalence) of simplicial groups, and defining cofibrations by the LLP with respect to trivial fibrations (see [3], or [4]). The category $n - \mathbf{HXC}(\mathbf{Gp})$ supports then a closed model structure as well.

Following Quillen, [18], for any closed model category \mathcal{C} the notion of homotopy category, $Ho(\mathcal{C})$, will be considered as extended by the notion of homotopy theory associated to \mathcal{C} . The Quillen's criterion to determine an equivalence of homotopy theories (see [18], Chap.I, §4, Theorem 3) will be used for the following theorem:

Theorem 5.4. *The homotopy theory of $\mathbf{Simp}(\mathbf{Gp})$ (with the closed model n -structure) and that of $n - \mathbf{Hypgd}(\mathbf{Gp})$ (with the closed model structure given in [3]) are equivalent.*

Proof. Let us recall that the reflector functor \mathbb{P} is given by

$$\mathbb{P}(G_\bullet) = \text{cosk}^{n+1} \left(\begin{array}{ccccccc} G_{n+1} & \xrightarrow{d_0} & G_n & \xrightarrow{d_0} & G_{n-1} & \xrightarrow{d_0} & \dots & G_1 & \xrightarrow{d_0} & G_0 \\ \overline{H}_{n+1} & \xrightarrow{d_{n+1}} & d_{n+1}(N_{n+1}G_\bullet) & \xrightarrow{d_n} & G_{n-1} & \xrightarrow{d_{n-1}} & \dots & G_1 & \xrightarrow{d_1} & G_0 \end{array} \right)$$

where H_{n+1} is the normal subgroup of G_{n+1} formed by those $x \in G_{n+1}$ such that $d_i x \in d_{n+1}(N_{n+1}G_\bullet)$, for $0 \leq i \leq n + 1$ (compare with that given in [12]).

Since \mathbb{P} preserves both cofibrations and weak equivalences, so does J with fibrations and weak equivalences. In addition, the unit of the adjunction is a weak equivalence (i.e., an n -weak equivalence of simplicial groups). Consequently, the adjoint situation

$$\mathbb{P} : Ho_n(\mathbf{Simp}(\mathbf{Gp})) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} Ho(n - \mathbf{Hypgd}(\mathbf{Gp})) : J, \quad \mathbb{P} \dashv J$$

provides an equivalence of categories.

Since both $\mathbf{Simp}(\mathbf{Gp})$ and $n - \mathbf{Hypgd}(\mathbf{Gp})$ are pointed, this equivalence turns out to be an equivalence of homotopy theories. \square

It is clear now that

Corollary 5.5. *The closed model categories $\mathbf{Simp}(\mathbf{Gp})$ with the n -structure, and $n - \mathbf{HXC}(\mathbf{Gp})$, with the structure given in [3], have equivalent homotopy theories.*

References

- [1] R.Brown and M.Golasinski, *A model structure for the homotopy category of crossed complexes*, Cahiers de Top. et Géom. Diff. Cat. 30 (1989), 61-82.
- [2] J.G.Cabello, *Estructuras de modelos de Quillen para categorías que modelan algebraicamente tipos de homotopía de espacios*, Ph.D.thesis, Univ. Granada, (1993).
- [3] J.G.Cabello and A.R.Garzon, *Closed model structures for algebraic models of n -types*, J. of Pure and Applied Algebra 103, (1995), 287-302.
- [4] J.G.Cabello and A.R.Garzon, *Quillen's theory for algebraic model of n -types*, Extracta Math. Vol 9, No 1, (1994), 42-48.
- [5] P.Carrasco and A.M.Cegarra, *Group theoretic algebraic models for homotopy types*, J. of Pure and Applied Algebra 75, (1991), 195-235.
- [6] S.E.Crans, *Quillen closed model structures for sheaves*, to appear in J. of Pure and Applied Algebra.
- [7] E.B.Curtis, *Simplicial homotopy theory*, Advances in Math. 6 (1971), 107-209.
- [8] S.Eilenberg and S.Mac Lane, *Relations between homology and homotopy groups*, Proc. Nat. Acad. Sci. USA 29, (1943).

- [9] C.Elvira, L.J.Hernández, *Closed model categories for the n -type of spaces and simplicial sets*, Math. Proc. Camb. Phil. Soc., (1995).
- [10] J.I.Extremiana, L.J.Hernández and M.T.Rivas, *A closed model category for $(n - 1)$ -connected spaces*, preprint, (1995).
- [11] P.Gabriel and M.Zisman, *Calculus of Fractions and Homotopy theory*, Ergebnisse der Mathematik, vol. 35, Springer-Verlag, Berlin (1967).
- [12] P.Glenn, *Realization of cohomolgy classes in arbitrary exact categories*, J. of Pure and Applied Algebra, 25, (1982), 33-107.
- [13] P.J. Higgins, *Groups with multiple operators*, Proc. London Math. Soc. 6, (1956), 366-416.
- [14] L.Hernández and T.Porter, *Categorical models of n -types for pro-crossed complexes and J_n -spaces*, L.N. in Math. 1509, (1992), 148-185.
- [15] D.M.Kan, *On homotopy theory and CSS groups*, Ann. of Math. 68, (1958), 38-53.
- [16] J.P.May, *Simplicial objects in algebraic topology*, Nostrand, New York, (1976).
- [17] T.Porter, *n -types of simplicial groups and crossed n -cubes*, Topology 32, 1 (1993), 5-24.
- [18] D.Quillen, *Homotopical Algebra*, L.N. in Math 43, Springer (1967).
- [19] D.Quillen, *Rational Homotopy Theory*, Ann. of Math. 90, (1969).
- [20] J.H.C.Whitehead, *Combinatorial Homotopy I and II*, Bull. Amer. Soc. 55, (1949), 213-245 and 496-543.

J.G.Cabello,
 Dpto. Algebra, Facultad de Ciencias, Granada, 18071, España.
 cabello@goliat.ugr.es