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METRIZABILITY OF $\sigma$-FRAMES

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1 Preliminaries

Here, we recall some definitions from [1], [2], [4], [6], and [12].

1.1 A frame ($\sigma$-frame) is a lattice $L$ which has arbitrary (countable) joins and satisfies the arbitrary (countable) distributive law $x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$, for all $x \in L$, and arbitrary (countable) subset $S \subseteq L$. A frame ($\sigma$-frame) map $h : L \to M$ is a lattice morphism preserving arbitrary (countable) joins. The resulting category is denoted by $\text{Frm} (\sigma\text{Frm})$.

1.2 A cover of a frame ($\sigma$-frame) is any (countable) subset $A \subseteq L$ such that $\bigvee A = 1$. The set of all covers of $L$ is denoted by $\text{Cov}L$. If $A, B$ are covers we say that $A$ refines $B$, and write $A \leq B$, if, for each $a \in A$, there exists $b \in B$ such that $a \leq b$. For a cover $A$ of $L$ and $x \in L$ we
put \( Ax = \bigvee\{a : a \in A, a \land x \neq 0\} \). Let \( L \) be a frame (\( \sigma \)-frame) and \( \mathcal{U} \subseteq \text{Cov} L \). We write \( x \prec y \) or simply \( x \prec y \) if there exists \( A \in \mathcal{U} \) such that \( \{a : a \in A, a \land x \neq 0\} \subseteq y \). We say that \( \mathcal{U} \) is an admissible system if \( x = \bigvee T \), for some (countable) subset \( T \) of \( \{y : y < x\} \), for each \( x \in L \). Note that, if \( L \) is a frame, then we have \( Ax \leq y \) if and only if \( \{a : a \in A, a \land x \neq 0\} \subseteq y \), for each \( A \in \text{Cov} L \).

1.3 In a bounded lattice, we say that \( a \) is rather below \( b \), and write \( a \prec b \), if there exists a separating element \( s \) of \( L \) with \( a \land s = 0 \) and \( s \lor b = 1 \). A frame (\( \sigma \)-frame) \( L \) is called regular if each of its elements is a (countable) join of elements rather below it. Notice that, in a frame, \( a \prec b \) if and only if \( a^* \lor b = 1 \), where \( a^* = \bigvee\{y : y \land a = 0\} \). An element \( x \) of a frame \( L \) is said to be a Lindelöf element if whenever \( x = \bigvee S \) for some \( S \subseteq L \), then, there exists a countable subset \( T \) of \( S \) such that \( x = \bigvee T \). A frame \( L \) is said to be a Lindelöf frame if \( 1 \) is a Lindelöf element. A frame (\( \sigma \)-frame) \( L \) is said to be paracompact if each cover has a locally finite refinement. In [2] it is shown that any regular \( \sigma \)-frame is paracompact.

1.4 A basis of a frame (\( \sigma \)-frame) \( L \) is a subset \( B \subseteq L \) such that each element of \( L \) is a (countable) join of elements of \( B \). For the elements \( a, b \) of a frame (\( \sigma \)-frame) \( L \), we say that \( a \) meets \( b \) if \( a \land b \neq 0 \). A subset \( X \) of a frame (\( \sigma \)-frame) \( L \) is said to be locally finite if, there is a cover \( W \) of \( L \) such that each \( w \in W \) meets only finitely many \( x \in X \), and it is said to be discrete if, each \( w \in W \) meets at most one \( x \in X \). The above cover \( W \) is said to witness the local finiteness respectively discreteness of \( X \). A basis is called \( \sigma \)-locally finite (\( \sigma \)-discrete) if, it is a countable union of locally finite (discrete) sets, and it is called \( \sigma \)-admissible if, it can be written as union of an admissible system of covers.

1.5 Note Let \( L \) be a frame. If \( \mathcal{U} \) is an admissible system of covers of \( L \), then \( \bigcup \mathcal{U} \) is a basis of the frame \( L \) [1]. We can show, in the same way, that this is also true for \( \sigma \)-frames. Given \( a \in L \), let \( a = \bigvee\{y_n : y_n < a\} \). Take \( B_n \in \mathcal{U} \) such that \( B_n y_n \leq a \). Then, \( a = \bigvee\{b : b \in B_n, b \land y_n \neq 0, n \in \mathbb{N}\} \).

1.6 Lemma Any \( \sigma \)-frame with a countable basis is a frame.
Proof: Let $B$ be a countable basis of a $\sigma$-frame $L$ and $X$ be an arbitrary subset of $L$. It is easy to show that $\bigvee X$ exists and it is equal to $\bigvee\{b \in B, b \leq x\}$, for some $x \in X$.

Also, for each $X \subseteq L$ and $a \in L$, $a \land \bigvee X = \bigvee\{a \land b : b \in B, b \leq x\}$, for some $x \in X \leq \bigvee\{a \land x : x \in X\} \leq a \land \bigvee X$. Hence, $L$ is a frame.

\[ \square \]

1.7 Note In [1] it is shown that, if $X \subseteq L$ is locally finite and $x < a$, for each $x \in X$, then $\bigvee X < a$. By the above lemma, if $L$ is a $\sigma$-frame with a countable basis and $X \subseteq L$ is locally finite, then $x < a$, for each $x \in X$, implies $\bigvee X < a$.

2 Some properties of bases of $\sigma$-frames

Here, we prove the following theorem, which is the counter part of the properties of bases of frames, proved in [1].

2.1 Theorem The following are equivalent for a $\sigma$-frame $L$:

(1) $L$ has a countable basis.

(2) $L$ has a $\sigma$-discrete basis.

(3) $L$ has a $\sigma$-locally finite basis.

Moreover, these are equivalent to the following, if $L$ is regular

(4) $L$ has a $\sigma$-admissible basis.

Proof: (1 $\Rightarrow$ 2) Let $B = \{b_n : n \in \mathbb{N}\}$ be a countable basis. It is enough to take $B_n = \{b_n\}$, for each $n \in \mathbb{N}$.

(2 $\Rightarrow$ 3) This follows trivially from the definitions.
(3 $\Rightarrow$ 1) Let $B = \bigcup_{n \in \mathbb{N}} B_n$, where each $B_n$ is a locally finite set. We show that each $B_n$ is a countable set. Let $W$ be the witnessing cover of $B_n$. Thus each $w \in W$ meets only finitely many $x \in B_n$. Let

$$\{b : b \in B_n, b \wedge w \neq 0\} = \{b(w, 1), \ldots, b(w, l_w)\},$$

for each $w \in W$. Let $b \in B_n$ and $b \neq 0$. There exists $w_i \in W$ such that $b \wedge w_i \neq 0$ since, if $b \wedge w = 0$, for each $w \in W$, then $b = b \wedge \bigvee W = 0$. This shows that $b \in \{b(w_i, 1), \ldots, b(w_i, l_{w_i})\}$ and so there exists $j \leq l_{w_i}$ such that $b = b(w_i, j)$. Therefore, $B_n = \bigcup\{b(w, j) : w \in W, j \in \{1, \ldots, l_w\}\}$. This gives that $B_n$ is a countable set, since $W$ is a countable set. Hence $B = \bigcup B_n$ is a countable basis of $L$.

(4 $\Rightarrow$ 3) Let \{\{B_n : n \in \mathbb{N}\} be an admissible system of covers of $L$. Regularity of $L$ implies that $L$ is paracompact, and so each $B_n$ has a locally finite refinement, say $A_n$. Now, \{\{A_n : n \in \mathbb{N}\} is an admissible system, and so $\bigcup A_n$ is a basis, by Note 1.5. Thus $A = \bigcup A_n$ is a $\sigma$-locally finite basis.

(2 $\Rightarrow$ 4) Let $B = \bigcup B_n$ be a $\sigma$-discrete basis, and $B_n$ be witnessed by $W_n$. Take $S_w = \{b : b \in B_n, w \wedge b \neq 0\}$. We have $S_w = \emptyset$ or $S_w = \{b^w\}$. For each $x \in L$, put $T = \{b : b \in B_k, b < x\}$ and $x_k = \bigvee T$. Since $T \subseteq B_k$, $T$ is a discrete subset of $L$, and so $\bigvee T < x$, by Note 1.7. If $S_w = \{b^w\}$, then $b^w_k < b^w$ implies that there exists $t(w, k) \in L$ such that $b^w_k \wedge t(w, k) = 0$ and $b^w \vee t(w, k) = 1$. Take $U = \{A_{nk : n, k \in \mathbb{N}}\}$, where

$$A_{nk} = \{w \wedge s : w \in W_n, s \in \{b^w, t(w, k)\} \text{ if } S_w = \{b^w\}, \text{ and } s = 1 \text{ if } S_w = \emptyset\}.$$

It is easy to show that $U \subseteq CovL$. We claim that $A_{nk} b_k \leq b$, for any $b \in B_n$. Let $x \in A_{nk}$, and $x \wedge b_k \neq 0$. We show that $x = w \wedge b$, for some $w \in W_n$. We have, $x \wedge b_k \neq 0$ implies $w \wedge b_k \neq 0$ and so $w \wedge b \neq 0$. Thus $S_w \neq \emptyset$ and also $s \neq t(w, k)$, since if $x = w \wedge t(w, k)$ then $x \wedge b_k \neq 0$ implies $w \wedge t(w, k) \wedge b_k \neq 0$ and so $t(w, k) \wedge b_k \neq 0$. This contradiction shows that $x = w \wedge b^w$. Now, we have $x \wedge b_k \neq 0$ implies $w \wedge b \neq 0$ which gives $b = b^w$. Therefore $x = w \wedge b \leq b$. Hence $\{x : x \in A_{nk}, x \wedge b_k \neq 0\} \subseteq \downarrow b$ and so $b_k \lesssim_U b$. Also, by regularity of $L$.
we have that
\[ b = \bigvee \{ c \in B : c < b \} = \bigvee \{ b_k : k \in N \}. \]

Thus, for each \( x \in L \), \( x = \bigvee \{ b \in B : b \leq x \} = \bigvee \{ \bigvee \{ b_k : k \in N \} : b \leq x \} \), where \( b_k < b \leq x \), and hence \( x = \bigvee \{ b_k : b_k \leq x \} \). Therefore \( \mathcal{U} \) is an admissible system, and thus \( \bigcup \mathcal{U} \) is a \( \sigma \)-admissible basis. \( \square \)

Note that any countable basis \( B \) of a regular \( \sigma \)-frame \( L \) is in fact \( \sigma \)-admissible. Since, by the above theorem, \( L \) has a \( \sigma \)-admissible basis \( A = \bigcup A_n \). Now, putting \( B_1 = B \), \( B_{n+1} = \{ b : b \in B, b \leq a \text{ for some } a \in A_n \} \), one can show that \( B = \bigcup B_n \).

3 Metrization Theorems for \( \sigma \)-frames

In this section, interpreting the definition of a metric diameter on a frame given in [1], we prove the counterparts of the metrization theorems for \( \sigma \)-frames.

3.1 Definition A metric diameter on a \( \sigma \)-frame \( L \) is a monotone zero-preserving map \( d : L \to \mathbb{R}_+ \) such that

(1) for all \( a, b \), \( d(a \lor b) \leq d(a) + d(b) \), whenever \( a \land b \neq 0 \),

(2) for each \( \varepsilon > 0 \), there is a countable subset \( S \) of \( D_{\varepsilon}^L = \{ a : d(a) < \varepsilon \} \) such that \( \bigvee S = 1 \),

(3) for all \( a \in L \), there is a countable subset \( T \) of \( \{ y : y \ll a \} \) such that \( \bigvee T = a \), where \( y \ll a \) means that, there exists \( \varepsilon > 0 \) such that \( \{ b : b \in D_{\varepsilon}, b \land y \neq 0 \} \subseteq \downarrow a \).

(4) for all \( a \in L \), and \( \varepsilon > 0 \), \( d(a) = \sup \{ d(x \lor y) : x, y \in D_{\varepsilon} \cap \downarrow a \} \), whenever \( d(a) \geq \varepsilon \).

A \( \sigma \)-frame that admits a metric diameter is said to be metrizable. Also, a \( \sigma \)-frame map \( f : L \to M \) between metric \( \sigma \)-frames is said to be uniform if, for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( D_{\delta}^M \leq f[D_{\varepsilon}^L] \).
Metric frames (σ-frames) together with uniform frame (σ-frame) maps form a category, denoted by $\text{M Frm}$ ($\text{Mo Frm}$). Metric Lindelöf frames together with uniform frame maps form a category, denoted by $\text{ML Frm}$, which is a full subcategory of $\text{M Frm}$. See [7], [9], [10], [11] for details.

Having Theorem 2.1, we prove the remaining parts of the following.

3.2 Theorem For a regular σ-frame, the following statements are equivalent.

1. $L$ is metrizable.
2. $L$ has a σ-locally finite basis. (Nagata-Smirnov)
3. $L$ has a σ-discrete basis. (Bing)
4. $L$ has a σ-admissible basis.
5. $L$ has a countable basis.
6. $L$ has a countable admissible system of covers. (Moore)

Proof: ($5 \Rightarrow 1$) Let $L$ be a regular σ-frame with a countable basis $B$. By Lemma 1.6, $L$ is a frame and the regularity of $L$ as a σ-frame implies the regularity of $L$ as a frame. Thus, by Urysohn metrization theorem for frames [1], $L$ is a metrizable frame. Let $d : L \to \overline{R}_+$ be a metric diameter on the frame $L$.

To show that $d$ is a metric diameter on the σ-frame $L$, it is enough to check the conditions (2) and (3) of the definition. For each $\varepsilon > 0$, the set $\{b \in B : b \leq x, \text{ for some } x \in D_\varepsilon\}$ is a countable subcover of $D_\varepsilon$. Also, for $x \in L$, $x = \vee\{b \in B : b \prec x\}$. Hence, $L$ is a metric σ-frame.

($1 \Rightarrow 6$) Let $L$ be a metric σ-frame with metric diameter $d$. For each $n \in \mathbb{N}$, let $A_n$ be a countable subcover of $D_{1/n}$. Take $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$. We show that $\mathcal{U}$ is an admissible system. It is enough to show that $x \prec_d y$ implies $x \prec_\mathcal{U} y$. Let $x \prec_d y$. Take $\varepsilon > 0$ such that $\{a : a \in D_\varepsilon, a \wedge x \neq 0\} \subseteq y$. We choose $n > 1/\varepsilon$. Then, $\{a : a \in A_n, a \wedge x \neq 0\} \subseteq y$, and...
so \( x \triangleleft_{\mathcal{U}} y \), since \( A_n \in \mathcal{U} \). Hence, \( \mathcal{U} \) is an admissible system of covers.

\[(6 \Rightarrow 5)\] Let \( L \) be a \( \sigma \)-frame and \( \mathcal{U} = \{A_n : n \in \mathbb{N}\} \) be a countable admissible system of covers of \( L \). Then, by Note 1.5, \( B = \bigcup A_n \) is a countable basis. \( \Box \)

Compare the following with \([1, 4.3]\).

3.3 Proposition A regular Lindelöf frame is metrizable if and only if it has a countable basis.

Proof: By Urysohn metrization theorem \([1]\), a regular frame with countable basis is metrizable. Conversely, let \( L \) be a metric Lindelöf frame with a metric diameter \( d \). By Lindelöfness of \( L \), there exists a countable cover \( A_n \) such that \( A_n \subseteq D_{1/n} \), for each \( n \in \mathbb{N} \), since \( \bigvee D_{1/n} = 1 \). Take \( B = \bigcup A_n \). We claim that \( B \) is a countable basis of \( L \). Given \( x \in L \), we have \( x = \bigvee \{y : y \triangleleft x\} \). Consider \( C = \{y : y \triangleleft x\} \). For each \( y \in C \), there exists \( \varepsilon_y > 0 \) such that \( \{a : a \in D_{\varepsilon_y}, a \wedge y \neq 0\} \subseteq \downarrow x \). We choose \( n_y > 1/\varepsilon_y \). It is easy to show that

\[ x = \bigvee \{a \in A_{n_y} : a \wedge y \neq 0, y \in C\}. \]

Hence, \( B \) is a countable basis of \( L \). \( \Box \)

4 Characterization of metric \( \sigma \)-frames

In this section we characterize the category \( \text{M}\sigma\text{Frm} \), of metric \( \sigma \)-frames, as the intersection of the categories \( \text{ML}\text{Frm} \), of metric Lindelöf frames, and \( \text{R}\sigma\text{Frm} \), of regular \( \sigma \)-frames.

4.1 Lemma Any metric \( \sigma \)-frame is a metric Lindelöf frame.

Proof: Let \( L \) be a metric \( \sigma \)-frame with metric diameter \( d \). By Theorem 3.2, \( L \) has a countable basis, say \( B \), and so it is a frame, by Lemma 1.6. Clearly \( L \) is a metric frame with metric diameter \( d \). It is enough to show that \( L \) is a Lindelöf frame.
Let $\forall S = 1$, for some $S \subseteq L$. We have $1 = \forall S = \forall \{b \in B : b \leq s_b, \text{ for some } s_b \in S\}$. Thus, $1 = \forall \{s_b : b \in B\}$. Hence $L$ is a Lindelöf frame.

4.2 Note Let $L$ be a $\sigma$-frame with a countable basis $B$. Then, any $\sigma$-frame map $f : L \to M$ preserves arbitrary joins. Given $S \subseteq L$, then

$$f(\forall S) \leq f(\forall \{s : s \leq b, \text{ for some } b \in B\}) \leq \forall f(S) \leq f(\forall S).$$

4.3 Proposition The category $\mathbf{M\sigma Frm}$ is a full subcategory of $\mathbf{ML Frm}$.

Proof: By Lemma 4.1, it is enough to show that any uniform $\sigma$-frame map between metric $\sigma$-frames is a uniform frame map. Let $f : L \to M$ be a uniform $\sigma$-frame map. By Theorem 3.2, $L$ has a countable basis and so, by the above note, $f$ is a frame map. Uniformity of $f$ as a $\sigma$-frame map implies the uniformity of $f$ as a frame map. Thus, $\mathbf{M\sigma Frm} \subseteq \mathbf{ML Frm}$. Also, any uniform frame map is a uniform $\sigma$-frame map. Thus, $\mathbf{M\sigma Frm}$ is a full subcategory of $\mathbf{ML Frm}$. □

4.4 Lemma Let $L$ be a Lindelöf frame with countable regularity property (each element of $L$ is a countable join of elements rather below it). Then, each element of $L$ is Lindelöf.

Proof: Let $x = \forall S$, for some subset $S$ of $L$. By countable regularity of $L$, we have $x = \forall \{y_n : y_n < x\}$. For each $n \in \mathbb{N}$, we have $1 = y_n^* \lor x = y_n^* \lor \forall S = \forall \{y_n^* \lor s : s \in S\}$. Lindelöfness of $L$ implies that there exists a countable subset $T_n \subseteq S$ such that $\forall \{y_n^* \lor s : s \in T_n\} = 1$ and so $y_n < \forall T_n$, for each $n \in \mathbb{N}$. Take $T = \bigcup T_n$. Then, $x = \forall \{y_n : y_n < x\} \leq \forall T \leq x$. Therefore, $x = \forall T$ and $T$ is a countable subset of $S$. □

4.5 Lemma $\mathbf{ML Frm} \cap \mathbf{R\sigma Frm} \subseteq \mathbf{M\sigma Frm}$.

Proof: Let $L$ be a metric Lindelöf frame, with a metric diameter $d$, as well as a regular $\sigma$-frame. We show that $d$ is a metric diameter on the $\sigma$-frame $L$. Lindelöfness of $L$ as a frame gives a countable cover $A$, for each $D_e$. Given $x \in L$, we have $x = \forall \{y \in L : y < x\}$. 

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By Lemma 4.4, $x$ is a Lindelöf element and so there exists a countable subset $T$ of $\{y : y \prec x\}$ such that $x = \bigvee T$. Thus $L$ is a metric $\sigma$-frame. Therefore $\text{Ob}(\text{MLFrm}) \cap \text{Ob}(\text{RoFrm}) \subseteq \text{Ob}(\text{MoFrm})$.

Also, clearly any uniform frame map is a uniform $\sigma$-frame map. Thus $\text{MLFrm} \cap \text{RoFrm} \subseteq \text{MoFrm}$. □

4.6 Note Any metric frame ($\sigma$-frame) is a regular frame ($\sigma$-frame). To see this, it is enough to show that $x \prec y$ implies $x \preceq y$. Let $x \prec y$. Take $\varepsilon > 0$ such that $\{a : a \in D, a \wedge x \neq 0\} \subseteq y$. Then, there exists a (countable) subset $S \subseteq D$ such that $\bigvee S = 1$. Take $t = \bigvee\{s : s \in S, s \wedge x = 0\}$. It is easy to show that $x \wedge t = 0$ and $y \vee t = 1$, and so $x \prec y$.

4.7 Theorem The category $\text{M}\sigma\text{Frm}$ is exactly the intersection of the categories $\text{MLFrm}$ and $\text{RoFrm}$.

Proof: By Lemma 4.5, we have $\text{MLFrm} \cap \text{RoFrm} \subseteq \text{MoFrm}$, and by Proposition 4.3, $\text{MoFrm}$ is a full subcategory of $\text{MLFrm}$. Also, any metric $\sigma$-frame is a regular $\sigma$-frame, by the above note. Therefore the category $\text{M}\sigma\text{Frm}$ is a subcategory of the category $\text{RoFrm}$. Hence $\text{MLFrm} \cap \text{RoFrm} = \text{M}\sigma\text{Frm}$. □

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