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A construction of 2-filtered bicolimits of categories


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A CONSTRUCTION OF 2-FILTERED BICOLIMITS OF CATEGORIES

by Eduardo J. DUBUC and Ross STREET

A la mémoire de Charles Ehresmann

RESUME. Nous définissons la notion de 2-catégorie 2-filtrante et donnons une construction explicite de la bicolimite d'un 2-foncteur à valeurs dans les catégories. Une catégorie considérée comme étant une 2-catégorie triviale est 2-filtrante si et seulement si c'est une catégorie filtrante, et notre construction conduit à une catégorie équivalente à la catégorie qui s'obtient par la construction usuelle des colimites filtrantes de catégories. Pour cette construction des axiomes plus faibles suffisent, et nous appelons la notion correspondante 2-catégorie pré 2-filtrante. L'ensemble complet des axiomes est nécessaire pour montrer que les bicolimites 2-filtrantes ont les propriétés correspondantes aux propriétés essentielles des colimites.

Introduction.

We define the notion of 2-filtered 2-category and give an explicit construction of the bicolimit of a category valued 2-functor. A category considered as a trivial 2-category is 2-filtered if and only if it is a filtered category, and our construction yields a category equivalent to the category resulting from the usual construction of filtered colimits of categories. Weaker axioms suffice for this construction, and we call the corresponding notion pre 2-filtered 2-category. The full set of axioms is necessary to prove that 2-filtered bicolimits have the properties corresponding to the essential properties of filtered bicolimits.

In [3] Kennison already considers filterness conditions on a 2-category under the name of bifiltered 2-category. It is easy to check that a bifiltered 2-category is 2-filtered, so our results apply to bifiltered 2-categories. Actually Kennison’s notion is equivalent to our’s, but the other direction of this equivalence is not entirely trivial.
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1. Pre 2-Filtered 2-Categories and the construction LL

1.1 Definition. A 2-category $\mathcal{A}$ is defined to be pre-$2$-filtered when it satisfies the following two axioms:

F1. Given there exists invertible

Given \[ E \xrightarrow{g} B \]
there exists with invertible 2-cells $a, B$ such that

\[ \begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (E) at (1,0) {$E$};
  \node (B) at (0,-1) {$B$};
  \node (C) at (1,-1) {$C$};

  \draw[->] (A) to node [left] {$f$} (E);
  \draw[->] (E) to node [right] {$g$} (B);
  \draw[->] (E) to node [above] {$a$} (C);
  \draw[->] (B) to node [below] {$B$} (C);
\end{tikzpicture} \]

F2. Given any 2-cells

Given \[ E \xrightarrow{g} B \]
there exists with invertible 2-cells $\alpha, \beta$ such that

\[ \begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (E) at (1,0) {$E$};
  \node (B) at (0,-1) {$B$};
  \node (C_1) at (1,-1) {$C_1$};
  \node (C_2) at (2,-1) {$C_2$};

  \draw[->] (A) to node [left] {$f$} (E);
  \draw[->] (E) to node [right] {$g$} (B);
  \draw[->] (E) to node [above] {$\gamma_1$} (C_1);
  \draw[->] (E) to node [below] {$\gamma_2$} (C_2);
  \draw[->] (B) to node [below] {$v_1$} (C_1);
  \draw[->] (B) to node [below] {$v_2$} (C_2);

  \draw[->] (C_1) to node [left] {$w_1$} (C_2);
  \draw[->] (C_1) to node [right] {$w_2$} (C_2);
\end{tikzpicture} \]
1.3 Notation (the LL equation). Given two pairs of 2-cells \( (\gamma_1, \alpha) \), \( (\gamma_2, \beta) \) as in F2, we shall call the equation 1.2 the equation LL.

Axioms F1 and F2 can be weakened if we add a third axiom:

WF1. Same axiom F1 but not requiring \( \gamma \) to be invertible.

WF2. Same axiom F2 but only for invertible \( \gamma_1 \) and \( \gamma_2 \).

WF3.

We leave to the reader the proof of the following:

1.4 Proposition. The set of axioms F1, F2 is equivalent to the set WF1 WF2 and WF3.

When \( \mathcal{A} \) is a trivial 2-category (the only 2-cells are the identities), our axiom F1 corresponds to axiom PS1 in the definition of pseudofiltered category (cf [1] Exposé I), while axiom PS2 may not hold. Thus, a category which is pre 2-filtered as a trivial 2-category may not be pseudofiltered. Notice that our axioms F2 and WF3 are vacuous in this case.

Construction LL

Let \( \mathcal{A} \) be a pre-2-filtered 2-category and \( F : \mathcal{A} \to \text{Cat} \) a category valued 2-functor. We shall now construct a category which is to be the bicolimit (in the sense made precise in theorem 1.19)
of $F$. This construction generalizes Grothendieck's construction of the category $\operatorname{Lim}_A F$ for a filtered category $A$ (cf [2], Exposé VI).

### 1.5 Definition (Quasicategory $\mathcal{L}(F)$).

i) An object is a pair $(x, A)$ with $x \in FA$.

ii) A premorphism $(x, A) \to (y, B)$ between two objects is a triple $(u, \xi, v)$, where and $A \xrightarrow{u} C$, $B \xrightarrow{u} C$ and $\xi : F(u)(x) \to F(v)(y)$ in $FC$.

iii) A homotopy between two premorphisms is a quadruple $(w_1, w_2, \alpha, \beta) : (u_1, \xi_1, v_1) \Rightarrow (u_2, \xi_2, v_2)$, where $C_1 \xrightarrow{w_1} C$, $C_2 \xrightarrow{w_2} C$ and $\alpha : w_1v_1 \xrightarrow{\cong} w_2v_2$, $\beta : w_1u_1 \xrightarrow{\cong} w_2u_2$ are invertible 2-cells such that

$$
\begin{array}{ccc}
F(u_1)F(u_1)(x) = F(w_1u_1)(x) & \xrightarrow{F(\beta)x} & F(w_2u_2)(x) = F(w_2)F(u_2)(x) \\
F(w_1)(\xi_1) & \downarrow & F(w_2)(\xi_2) \\
F(w_1)F(v_1)(y) = F(w_1v_1)(y) & \xrightarrow{F(\alpha)y} & F(w_2v_2)(y) = F(w_2)F(v_2)(y)
\end{array}
$$

commutes in $FC$.

We shall formally introduce now an abuse of notation.

### 1.6 Notation.

i) We omit the letter $F$ in denoting the action of $F$ on its arguments. Thus $A \xrightarrow{\alpha \Downarrow \beta} B$ indicates a 2-cell in $A$ as well as the corresponding natural transformation $FA \xrightarrow{F(\alpha) \Downarrow F(\beta)} FB$ in $\mathbf{Cat}$. In this way, the above commutative square becomes

$$
\begin{array}{ccc}
w_1u_1x & \xrightarrow{\beta x} & w_2u_2x \\
\downarrow w_1\xi_1 & & \downarrow w_2\xi_2 \\
w_1v_1y & \xrightarrow{\alpha y} & w_2v_2y
\end{array}
$$
ii) We write $F \xrightarrow{x} A$ in $(\mathcal{C}at^A)^{op}$ for the natural transformation $\mathcal{A}[A, -] \rightarrow F$ defined by $xC(A \xrightarrow{u} C) = F(u)(x) \in FC$.

Notice that the notation in i) $F(u)(x) = ux$ is consistent with this since juxtaposition denotes composition. Also, in the same vein, given an object $x \in FA$ and a functor $FA \xrightarrow{h} \mathcal{X}$ into any other category, the composite $F \xrightarrow{x} A \xrightarrow{h} \mathcal{X}$ makes sense and we have $h(F(x)) = hx$.

In this notation then, a premorphism $(x, A) \rightarrow (y, B)$ is a triple $(u, \xi, v)$, where $F \xrightarrow{\xi} C$; that is $\xi : ux \rightarrow vy$ in FC. A homotopy between two premorphisms $F \xrightarrow{\xi_1} C_1, F \xrightarrow{\xi_2} C_2$ is a pair of invertible 2-cells satisfying the LL equation:

We shall simply write $(\alpha, \beta) : \xi_1 \Rightarrow \xi_2$ for all the data involved in an homotopy.

At this point it is convenient to introduce the following notation:
1.7 Notation (LL-composition of 2-cells). Given three 2-cells $\alpha$, $\beta$, and $\gamma$ that fit into a diagram as it follows, we write:

$$\beta \circ_\gamma \alpha$$

Thus, $\beta \circ_\gamma \alpha$ is our notation for the 2-cell between the top and the bottom composites of arrows. It should be thought of as the "composite of $\beta$ with $\alpha$ over $\gamma$".

Homotopies compose: Consider a third premorphism $F \xrightarrow{\xi_3 \psi} C_3$ and an homotopy axiom F1 to obtain an invertible 2-cell This determines a pair of 2-cells:

and an homotopy $B \xrightarrow{\alpha' \psi} C'$, $A \xrightarrow{\beta' \psi} C'$, $(\alpha', \beta') : \xi_2 \Rightarrow \xi_3$. Use axiom F1 to obtain an invertible 2-cell $C_2 \xrightarrow{\gamma \psi} H$. This determines a pair of 2-cells:
which defines a homotopy $\xi_1 \Rightarrow \xi_3$. The corresponding LL equation follows easily from the LL equations for $(\alpha, \beta)$ and $(\alpha', \beta')$.

Using notation 1.7, we have:

1.8 Proposition (vertical composition of homotopies). Given $(\alpha, \beta) : \xi_1 \Rightarrow \xi_2$ and $(\alpha', \beta') : \xi_2 \Rightarrow \xi_3$, there exists an appropriate $\gamma$ and a homotopy $(\alpha' \circ_{\gamma} \alpha, \beta' \circ_{\gamma} \beta) : \xi_1 \Rightarrow \xi_3$. □

Homotopies are generated by composition out of two basic ones. The proof of the following is immediate:

1.9 Proposition. Every pair of premorphisms $\xi_1, \xi_2$ and pair of 2-cells $\alpha, \beta$ that fit as follows determine two basic homotopies:

\[
(\alpha, \text{id}_1) : F_{\xi_1} \Rightarrow C_1 \quad \Rightarrow \quad (\beta, \text{id}_2) : F_{\xi_2} \Rightarrow C_2
\]

where $\text{id}_1$ and $\text{id}_2$ are the identity 2-cells corresponding to the arrows $w_1u_1$ and $w_2v_2$ respectively. When the pair $(\alpha, \beta)$ satisfies the LL equation, then the composite (over the identity 2-cell of the identity arrow of $C_2$) of these basic homotopies is defined, and it is equal to the homotopy determined by $(\alpha, \beta)$. □
Premorphisms compose: Given two premorphisms

\[(x, A) \xrightarrow{\xi} (y, B) \xrightarrow{\zeta} (z, C)\]

axiom F1 to obtain invertible \( B \xrightarrow{\gamma} H \). According to 1.7 this determines a premorphism \( \zeta \circ \gamma \xi \) between \((x, A)\) and \((z, C)\) that we take as a composite of \( \xi \) with \( \zeta \). Thus:

We have:

1.10 Proposition (horizontal composition of premorphisms).
Given \( \xi : (x, A) \to (y, B) \) and \( \zeta : (y, B) \to (z, C) \), there exists an appropriate \( \gamma \) and \( \zeta \circ \gamma \xi : (x, A) \to (z, C) \).

Homotopies also compose horizontally:

1.11 Proposition (horizontal composition of homotopies).
Consider composable premorphisms and homotopies as follows:

\[(x, A) \xrightarrow{\xi_1} (y, B) \xrightarrow{\zeta_1} (z, C) \]

Then, given any two composites \( \zeta_1 \circ \gamma_1 \xi_1 \) and \( \zeta_2 \circ \gamma_2 \xi_2 \), there exists an homotopy \( \zeta_1 \circ \gamma_1 \xi_1 \Rightarrow \zeta_2 \circ \gamma_2 \xi_2 \).
Proof. Consider the given homotopies and their LL equations:

\[
\begin{align*}
\begin{array}{c}
\text{Proof. Consider the given homotopies and their LL equations:}
\end{array}
\end{align*}
\]

and consider the horizontal composites of the premorphisms:

\[
\begin{align*}
\begin{array}{c}
\text{and consider the horizontal composites of the premorphisms:}
\end{array}
\end{align*}
\]

We shall produce a homotopy between these composites.
First use axiom F1 to obtain $\phi_1$, $\phi_2$ as follows:

![Diagram 1]

Then use axiom F2 to obtain $\theta_1$, $\theta_2$ satisfying the LL equation:

![Diagram 2]

The homotopy is given by the following pair of 2-cells:

![Diagram 3]
The corresponding LL equation is:

\[ \text{To pass from the left side to the right side of this equation use the LL equations of } (E, 6), (01, B2) \text{ and } (a, (3)), \text{ in this order.} \]

1.12 Definition (equivalence of premorphisms). Two premorphisms \( \xi_1, \xi_2 \) are said to be equivalent when there exists a homotopy \( (a, B) : \xi_1 \Rightarrow \xi_2 \). We shall write \( \xi_1 \sim \xi_2 \).

Equivalence is indeed an equivalence relation. Proposition 1.8 shows transitivity, the inverse 2-cells define an homotopy \( (\alpha^{-1}, \beta^{-1}) : \xi_2 \Rightarrow \xi_1 \) in the opposite direction, which shows symmetry, while reflexivity is obvious.

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1.13 Definition (Category \( \mathcal{L}(F) \)). We define a category \( \mathcal{L}(F) \) with objects pairs \((x, A), x \in FA\). Morphisms are equivalence classes of premorphisms, and composition is defined by composing representative premorphisms.

It follows from 1.11 that composition is, up to equivalence, independent of the choice of representatives, and independent of the choice of the 2-cells given by axiom F1 when composing each pair of representatives. Since associativity holds and identities exist, this construction actually does define a category.

**Some lemmas on pre-2-filtered categories**

We establish now a few lemmas which are useful when proving the fundamental properties of the construction \( LL \).

1.14 Lemma. Given any pair of equivalent premorphisms

\[
\begin{array}{c}
F \overset{\xi_1}{\Rightarrow} C_1 \sim F \overset{\xi_2}{\Rightarrow} C_2,
\end{array}
\]

if \( u_1 = v_1 \) and \( u_2 = v_2 \), then we can choose an homotopy \((\alpha, \beta) : \xi_1 \Rightarrow \xi_2 \) with \( \alpha = \beta \).

Proof. It follows immediately from axiom F2.

1.15 Lemma. Given a finite family of 2-cells

\[
E \xrightarrow{\gamma_i} C_i,
\]

\( i = 1 \ldots n \), there exists \( A \overset{u}{\rightarrow} C, B \overset{v}{\rightarrow} C, C_i \overset{w_i}{\rightarrow} C, i = 1 \ldots n \), with invertible 2-cells \( \alpha_i, \beta_i \), such that the 2-cells

\[
\begin{array}{c}
E \xrightarrow{\gamma_i} C_i \overset{w_i}{\rightarrow} C, i = 1 \ldots n
\end{array}
\]

are all equal.
Given a second family of 2-cells $H \xrightarrow{\delta_i \psi} C_i$ (with same $u_i, v_i, C_i$), we can assume that the same $u, v, w_i, \alpha_i, \beta_i$, also equalize the 2-cells of the second family.

Proof. Axiom F2 provides the case $n = 2$ with $u = w_1 u_1$, $v = w_2 v_2$, $\alpha_1 = \alpha$, $\beta_1 = id$, $\alpha_2 = id$, and $\beta_2 = \beta$. Using this case, induction is straightforward. For the second part, if the 2-cells of the second family are not yet equalized, use the lemma again (and patch the new 2-cells also into the first family).

From axiom F2 we deduce

1.16 Lemma. Given any 2-cells $E \xrightarrow{\gamma_1 \psi} C_1$, $E \xrightarrow{\gamma_2 \psi} C_2$ and an object $F \xrightarrow{x} E$, the premorphisms $F \xrightarrow{\gamma_1 x \psi} C_1$, $F \xrightarrow{\gamma_2 x \psi} C_2$ are equivalent.

From proposition 1.9 it follows that

1.17 Lemma. Given a pair of premorphisms $\xi_1$, $\xi_2$ and a pair of invertible two cells $\alpha$, $\beta$ that fit as follows, we have:

\[ F \xrightarrow{x} A \xrightarrow{u_1} C_1 \sim F \xrightarrow{x} A \xrightarrow{u_1} C_1 \xrightarrow{w_1} C \quad \text{and} \quad F \xrightarrow{y} B \xrightarrow{v_1} C_1 \xrightarrow{w_1} C \]
When the pair $\alpha, \beta$ satisfy the LL equation, then transitivity applied to these two equivalences yields the equivalence $\xi_1 \sim \xi_2$.

From this lemma and transitivity of equivalence we deduce

1.18 Lemma. Given a premorphism $\xi$, and a pair of invertible two cells $\alpha, \beta$ that fit as follows, we have:

The universal property of the construction LL

A pseudocone for a 2-functor $F$ with vertex the category $\mathcal{X}$ is a pseudonatural transformation $F \Rightarrow h \mathcal{X}$ from $F$ to the 2-functor which is constant at $\mathcal{X}$. Explicitly, it consists of a family of functors $(h_A : FA \to \mathcal{X})_{A \in A}$, and a family of invertible natural transformations $(h_u : h_B \circ u \to h_A)_{(A \xrightarrow{u} B) \in \mathcal{I}}$. A morphism $h \Rightarrow l$ of pseudocones (with same vertex) is a modification; as such, it consists of a family of natural transformations $(h_A \Rightarrow l_A)_{A \in A}$. In accordance with notation 1.6, we have

This data is subject to the equations:
Given a pseudo cone $F \Rightarrow h Z$ and a 2-functor $Z \Rightarrow s X$, it is clear and straightforward how to define a pseudocone $F \Rightarrow L(F)$ which induces by composition an equivalence of categories $\text{Cat}[F, X] \rightleftharpoons \text{PC}[F, X]$, and this equivalence is actually an isomorphism. We have:

1.19 Theorem. Let $u A \Rightarrow B$ in $FA$ and $x \Rightarrow y$ in $A$. The following formulas define a pseudocone $F \Rightarrow L(F)$:

$$
\lambda_A(x) = F \Rightarrow A, \quad \lambda_A(\xi) = F \Rightarrow A, \quad \lambda_u(x) = F \Rightarrow B
$$

which induces by composition an equivalence of categories $\text{Cat}[L(F), X] \overset{\cong}{\longrightarrow} \text{PC}[F, X]$, and this equivalence is actually an isomorphism. We have:
for all $h$ there exists a unique $\sim h$ such that $\sim h \lambda = h : F \xrightarrow{\lambda} \mathcal{L}(F)$.

Proof. Functoriality of $\lambda_A$, naturality of $\lambda_u$ and equation PC1 hold tautologically. The validity of PC2 means the following equivalence of premorphisms:

$$
\begin{array}{ccc}
F \xrightarrow{v} B & \xrightarrow{id} B & \xrightarrow{id} B \\
\downarrow{\gamma_x \downarrow} & \downarrow{id} & \downarrow{id} \\
A & \xrightarrow{z} B & \xrightarrow{id} B
\end{array}
\sim
\begin{array}{ccc}
F & \xrightarrow{id} B & \xrightarrow{id} B \\
\downarrow{\gamma_x \downarrow} & \downarrow{id} & \downarrow{id} \\
A & \xrightarrow{z} B & \xrightarrow{id} B
\end{array}
$$

which is given by lemma 1.16.

We pass now to prove the universal property. Given $F \xrightarrow{h} \mathcal{X}$, define $\sim h$ by the formulas: for $F \xrightarrow{\xi \downarrow} C$ in $\mathcal{L}(F)$,

$$
\sim h(x) = (F \xrightarrow{x} A \xrightarrow{h_A} \mathcal{X}), \quad \sim h(\xi) = (F \xrightarrow{\xi \downarrow} C \xrightarrow{h_C} \mathcal{X})
$$

We have to show that the definition of $\sim h(\xi)$ is compatible with the equivalence of premorphisms. It is enough to consider the two cases in
lemma 1.17. For the first case we have to show the equation

Consider the following equation which follows from PC1:

The right hand sides of equations (1) and (2) are equal by PC2, while the left hand sides are clearly equal. The second case in lemma 1.17 is treated in a similar manner. Functoriality of $\tilde{\lambda}$ follows from PC1 and PC2 with the same type of techniques as above. Finally, the equation $\tilde{\lambda} = h$ is immediate for the whole cone structure.

We finish this section with a lemma which follows from lemma 1.14

1.20 Lemma. Given two arrows $x \xrightarrow{\xi_1} y$ in $FA$, if $\lambda_A(x) = \lambda_A(y)$ in $L(F)$, then there exists $A \xrightarrow{w} C$ such that $wx = wy$ in $FC$.

2. 2-Filtered 2-Categories

2.1 Definition. A 2-category $\mathcal{A}$ is defined to be pseudo 2-filtered when it is pre 2-filtered and satisfies the stronger form of axiom F1:
Given there exists \( E_1 \), \( E_2 \) with \( f_1 \) and \( f_2 \) invertible 2-cells.

It is defined to be 2-filtered when it is pseudo 2-filtered, non empty, and satisfies in addition the following axiom.

**F0.**

\[
\begin{array}{ccc}
A & \xrightarrow{u} & C \\
\downarrow & & \downarrow \\
B & \xrightarrow{v} & B
\end{array}
\]

Given there exists \( C \).

As was the case for axiom F1, in the presence of axiom WF3, axiom FF1 can be replaced by the weaker version in which we do not require the 2-cells \( \gamma_1 \) and \( \gamma_2 \) to be invertible.

When \( A \) is a trivial 2-category (the only 2-cells are the identities), axiom F0 is the usual axiom in the definition of filtered category, while our axiom FF1 is equivalent to the conjunction of the two axioms PSI and PS2 in the definition of pseudofiltered category (cf [1] Exposé I).

Two properties of the construction LL that follow for pseudo 2-filtered 2-categories and not for pre 2-filtered 2-categories are the following:

**2.2 Lemma.** Given any morphism \( (x, A) \to (y, A) \) in \( \mathcal{L}(F) \), we can choose a representative premorphism \( F \xrightarrow{\xi \psi} C \) with \( u = v \).

\[
\begin{array}{ccc}
x & \xrightarrow{u} & A \\
y & \xrightarrow{v} & A
\end{array}
\]

Proof. Consider \( F \xrightarrow{\xi \psi} C \) and apply axiom FF1 to obtain invertible 2-cells.
2-cells $\alpha$, $\beta$ as follows:

\[
\begin{array}{c}
A \xrightarrow{id} \ x \xrightarrow{\alpha} \ D \\
A \xrightarrow{\beta} \ D \\
C \\
\end{array}
\quad
\begin{array}{c}
F \xrightarrow{\xi} \ C \\
C \xrightarrow{\alpha} \ C \\
A \xrightarrow{\beta} \ D \\
\end{array}
\]

The proof follows by lemma 1.18.

\[\Box\]

### 2.3 Lemma

*Given a finite family of premorphisms $F \xrightarrow{\xi, \psi} C_1$, $i = 1 \ldots n$, there exists $A \xrightarrow{u} C$, $B \xrightarrow{v} C$, $C_i \xrightarrow{w} C$, $i = 1 \ldots n$, with invertible 2-cells $\alpha_i$, $\beta_i$ as in the diagram:

\[
\begin{array}{c}
F \xrightarrow{\xi} \ C \\
C \xrightarrow{\alpha} \ C \\
A \xrightarrow{\beta} \ C \\
\end{array}
\]

When $A = B$ we can assume $u = v$.

*Proof.* Given $\xi_1$, $\xi_2$, apply FF1 to obtain invertible $\alpha$, $\beta$ to fit as follows:

\[
\begin{array}{c}
F \xrightarrow{\xi} \ C_1 \\
C_1 \xrightarrow{w_1} \ C \\
B \xrightarrow{\alpha} \ C \\
\end{array}
\quad
\begin{array}{c}
F \xrightarrow{\xi} \ C_2 \\
C_2 \xrightarrow{w_2} \ C \\
B \xrightarrow{\alpha} \ C \\
\end{array}
\]

This gives the case $n = 2$ with $u = w_1u_1$, $v = w_2v_2$, $\alpha_1 = \alpha$, $\beta_1 = id$, $\alpha_2 = id$, and $\beta_2 = \beta$. Using this case, induction is straightforward. For the second part, do as in the proof of lemma 2.2.
2.4 Theorem. Let $\mathcal{A}$ be a pre 2-filtered 2-category, $F : \mathcal{A} \rightarrow \text{Cat}$ a 2-functor, and $\mathcal{P}$ a finite category. Consider the 2-functor $F^\mathcal{P} : \mathcal{A} \rightarrow \text{Cat}$ defined by $F^\mathcal{P}(A) = (FA)^\mathcal{P}$, and the canonical functor:

$$\diamond : \mathcal{L}(F^\mathcal{P}) \rightarrow \mathcal{L}(F)^\mathcal{P} \quad (\text{given by theorem 1.19}).$$

Then, $\diamond$ is an equivalence of categories provided that $\mathcal{A}$ is 2-filtered or that $\mathcal{A}$ is pseudo 2-filtered and $\mathcal{P}$ is connected.

Proof. Notice that an object $F^\mathcal{P} \rightarrow A$ in $\mathcal{L}(F^\mathcal{P})$ is by definition a diagram $\mathcal{P} \rightarrow FA$. We shall prove, in turn, that $\diamond$ is (a) essentially surjective, (b) faithful, and (c) full.

(a) essentially surjective: We shall see that given a diagram $\mathcal{P} \rightarrow \mathcal{L}(F)$, there exists $A \in \mathcal{A}$ and a factorization (up to isomorphism):

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\cong} & \mathcal{L}(F) \\
\downarrow & & \downarrow \\
FA & \xrightarrow{\lambda_A} & \mathcal{L}(F)
\end{array}$$

Consider explicitly an object in $\mathcal{L}(F)^\mathcal{P}$:

$$F \xrightarrow{x_k} A_k, \; k \in \mathcal{P}, \quad F \xrightarrow{\varphi_f} A_f, \; p \xrightarrow{\gamma} q \in \mathcal{P}.$$

satisfying equations $\varphi_{fog} \sim \varphi_f \circ \gamma \varphi_g$ for all composable pairs $f, g$.

Let $Q \subset \mathcal{P}$ be a part of $\mathcal{P}$ for which there exists $A, w_k, \psi_f$ such that:

$$F \xrightarrow{x_k} A_k, \; k \in Q, \quad F \xrightarrow{\psi_f} A_f \sim F \xrightarrow{\psi_g} A, \; p \xrightarrow{\gamma} q \in Q.$$

$Q$ is not necessarily a subcategory, but we agree that if $p \xrightarrow{\gamma} q \in Q$, then we consider $p \in Q$ and $q \in Q$. The equations $\psi_{fog} \sim \psi_f \circ \gamma \psi_g$.
hold for all composable pairs \( f, g \) in \( Q \) with \( f \circ g \) also in \( Q \). By lemma 1.20 we can assume strict equality \( \psi_{f \circ g} = \psi_f \circ \psi_g \) in \( FA \).

We shall see that if \( p \xrightarrow{g} q \) is not in \( Q \), we can add it to \( Q \) in such a way that the enlarged part retains the same property. In what it follows we can use without mention lemma 1.18.

1) \( p \in Q, q \notin Q \) or \( q \in Q, p \notin Q \): In the first case apply axiom F1 to obtain an invertible 2-cell \( \beta \) as follows:

![Diagram 1](image1)

The new \( A \) is \( H \), the new \( w_k \) are \( hw_k \), all \( k \in Q \), the new \( \psi_f \) are \( h\psi_f \), all \( f \in Q \), and, finally, \( w_q = lv_g \) and \( \psi_g \) is the 2-cell above. The other case can be proved in the same way.

2) \( p \in Q, q \in Q \): Apply axiom FF1 to obtain invertible 2-cells \( \alpha, \beta \) as follows:

![Diagram 2](image2)

The new \( A \) is \( H \), the new \( w_k \) are \( hw_k \), all \( k \in Q \), the new \( \psi_f \) are \( h\psi_f \), all \( f \in Q \), and, finally, \( \psi_g \) is the 2-cell above on the right. This proof holds whether \( p \neq q \) or \( p = q \).

3) \( p \notin Q, q \notin Q \): Apply axiom F0 to obtain \( A_g \xrightarrow{w} H, A \xrightarrow{h} H \). The new \( A \) is \( H \), the new \( w_k \) are \( hw_k \), all \( k \in Q \), the new \( \psi_f \) are \( h\psi_f \), all \( f \in Q \), and, finally, \( w_p = lu_g, w_q = lv_g, \psi_g = l\varphi_g \). If \( p = q \), use lemma 2.2 to assume \( u_g = v_g \), and in this way \( w_p \) is uniquely defined.
Notice that if $\mathcal{P}$ is connected, it is not necessary to consider this case since we can always choose $p \xrightarrow{g} q$ such that either $p$, or $q$, or both, are in $Q$. Thus for connected $\mathcal{P}$ axiom $F0$ is not necessary.

It is clear that any singleton $\{(k, f = id_k)\}$ serves as an initial $Q$, thus we can assume $Q = \mathcal{P}$.

To finish the proof observe that given any arrow $ux \xrightarrow{\psi} vy$ in $FA$, the square $(ux, A) \xrightarrow{(id, id, u)} (x, A)$ commutes in $\mathcal{L}(F)$.

$$
\begin{array}{ccc}
\lambda_A(\psi) & \xrightarrow{(u, \psi, v)} \\
\downarrow & & \downarrow \\
(vy, A) & \xrightarrow{(id, id, v)} & (y, B)
\end{array}
$$

Notice that if $\mathcal{P}$ is a poset, case 2) cannot happen, so axiom $FF1$ is not necessary. For posets $\mathcal{P}$ the functor $\diamond$ is essentially surjective also for pre 2-filtered 2-categories.

(b) faithful: Consider two premorphisms $F^\mathcal{P} \xrightarrow{\xi} C$, $F^\mathcal{P} \xrightarrow{\eta} D$.

in $\mathcal{L}(F^\mathcal{P})$, $F \xleftarrow{\xi_k} C$, $F \xleftarrow{\eta_k} D$, $k \in \mathcal{P}$. To be equivalent in $\mathcal{L}(F)^\mathcal{P}$ means that there are homotopies $(\alpha_k, \beta_k) : \xi_k \Rightarrow \eta_k$ given by invertible 2-cells $B \xleftarrow{\alpha_k} H_k$, $A \xleftarrow{\beta_k} H_k$, $k \in \mathcal{P}$. From lemma 1.15 it readily follows that we can assume there are single invertible 2-cells $\alpha$, $\beta$ which define all the homotopies $(\alpha, \beta) : \xi_k \Rightarrow \eta_k$. But this means that $\xi$ and $\eta$ are equivalent in $\mathcal{L}(F^\mathcal{P})$.

Notice that this proof of the faithfulness of the functor $\diamond$ holds for pre 2-filtered 2-categories.

(c) full: Consider two objects $F^\mathcal{P} \xrightarrow{x} A$, $F^\mathcal{P} \xrightarrow{y} B$ in $\mathcal{L}(F^\mathcal{P})$. 

A premorphism in $\mathcal{L}(F)^P$ consists of a family $\xi_k: C_k \to F$, $k \in \mathcal{P}$. From lemma 2.3 we can assume all the $u_k, v_k, C_k$ to be equal to a single $u, v, C$. But this is the data for a premorphism in $\mathcal{L}(F^P)$. For the naturality equations we proceed as in the proof of faithfulness in (b).

Here axiom FF1 is inevitable (lemma 2.3), and plays the role of axiom PS2 in the filtered category case. Notice that a function between sets viewed as a functor between trivial categories is injective precisely when it is a full functor.

We state now an important corollary of theorem 2.4. Let $\mathcal{C}at_{\ell\ell}$ be the 2-category of finitely complete categories and finite limit preserving functors. We have:

2.5 Theorem. Let $A$ be a 2-filtered 2-category, and $A \xrightarrow{F} \mathcal{C}at_{\ell\ell}$ a 2-functor. Then, the category $\mathcal{L}(F)$ has finite limits, the pseudocone functors $FA \xrightarrow{\lambda A} \mathcal{L}(F)$ preserve finite limits and induce an equivalence of categories $\mathcal{C}at_{\ell\ell}[\mathcal{L}(F), \mathcal{X}] \xrightarrow{\simeq} \mathcal{P}C_{\ell\ell}[F, \mathcal{X}]$; this equivalence is actually an isomorphism.

Kennison notion of bifiltered 2-category

In [3], Kennison considers the following notion:

2.6 Definition (Kennison). A 2-category $\mathcal{A}$ is defined to be bifiltered when it satisfies the following three axioms:

BF0. Given two objects $A, B$, there exists $C$ and $A \to C, B \to C$.

BF1. Given two arrows $A \xrightarrow{f} B$, there exists $B \xrightarrow{u} C$ and an invertible 2-cell $\gamma : uf \cong ug$.

BF2. Given two 2-cells $A \xrightarrow{\gamma_1 \psi \gamma_2} B$, there exists $B \xrightarrow{u} C$ such that $u\gamma_1 = u\gamma_2$.

This notion of bifiltered 2-category is equivalent to our notion of 2-filtered 2-category. The proof of this is elementary although not entirely trivial, and we leave it as an interesting exercise.
References


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