E. COLEBUNDERS
A. GERLO

Firm reflections generated by complete metric spaces


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RESUME. Nous étudions des catégories concrètes où chaque objet est un sous-espace d’un produit “d’espaces métrisables”. Si une telle catégorie est munie d’un opérateur s de fermeture, nous considérons $U_s$, la classe des immersions denses. Nous traitons les questions suivantes: (1) si les espaces complètement métrisables sont des objets $U_s$-injectifs, (2) si la classe des sous-objets $s$-fermés d’un produit d’espaces complètement métrisables est $U_s$ “uniquement” reflective. Nous démontrons que dans notre contexte, ces questions sont équivalentes et nous formulons des conditions pour avoir une réponse affirmative. Le théorème principal permet de traiter un grand nombre d’exemples.

1 Introduction

The category $\text{Unif}_0$ of separated uniform spaces, endowed with the closure operator $r$ determined by the underlying topology, will be our guiding example in the study of completeness in a more general setting. Completely metrizable uniform spaces play an important role in the uniform case, since firstly they are injective objects with respect to the class $\mathcal{U}_r$ of all dense embeddings and secondly the complete uniform spaces are exactly the closed subspaces of products of completely metrizable spaces. Moreover the complete objects form a firmly $\mathcal{U}_r$-reflective subconstruct of $\text{Unif}_0$ in the sense of [3].

We will investigate to what extent these results hold in a more general setting. The general framework we will be working in is the one of metrically generated constructs as introduced in [6]. These are constructs $X$ for
which a natural functor describes the transition from (generalized) metric spaces to objects in the given category $X$. For example, with a (generalized) metric $d$ one can associate e.g. a (completely regular) topology $T_d$, a (quasi)uniformity $U_d$, a proximity $P_d$ or an approach structure $A_d$. In each of these examples, a natural functor $K$ from a suitable base category $C$ consisting of (generalized) metric spaces to the category $X$ is given. If the functor $K$ fulfills certain conditions (preserves initial morphisms and has an initially dense image) then the category $X$ is said to be metrically generated. This setting, which covers all the examples above and many others, is convenient for our purpose since in particular every object in $X$ is a subspace of a product of “metrizable” spaces. We will restrict to $T_0$-objects and a first attempt will be to endow $X_0$ with its regular closure operator $r$ and to consider the class $U_r$ of all $r$-dense embeddings. The following two questions will be investigated:

1) Are the completely metrizable objects $U_r$-injective?
2) Is the class of all $r$-closed subspaces of products of completely metrizable objects firmly $U_r$-reflective?

In fact we will show that in our setting these questions are equivalent and we will give necessary and sufficient conditions for a positive answer. Our main theorem will apply to a large collection of examples listed in the tables of the next sections. It will become clear that there exist metrically generated constructs $X$ allowing a $U_r$-firm reflective subconstruct $R$ which cannot be generated by complete metric spaces, so for which the questions above nevertheless have a negative answer.

In some cases where the answer to the questions above is negative, we still succeed in defining a smaller non-trivial closure operator for which the answers do become positive.

2 Metrically generated theories

In this section we gather some preliminary material that is needed to introduce the setting of this paper. We use categorical terminology as developed in [1] or [17] and we refer to [9] for material on closure operators. In [6] it was shown that every metrically generated construct can be isomorphically described as a subconstruct of a certain model category. It will be
convenient to deal with these isomorphic copies. So we recall the material on the model categories and fix some notation.

We call a function \( d : X \times X \rightarrow [0, \infty] \) a quasi-pre-metric if it is zero on the diagonal, we will drop “pre” if \( d \) satisfies the triangle inequality and we will drop “quasi” if \( d \) is symmetric. Note that we do not ask these quasi-pre-metrics to be realvalued or separated. If \( d \) is a quasi-metric we denote by \( d^* \) its symmetrization \( d \vee d^{-1} \).

Denote by \( \text{Met} \) the construct of quasi-pre-metrics and contractions. Recall that a map \( f : (X, d) \rightarrow (X', d') \) is a contraction (also called a nonexpansive map) if for every \( x \in X \) and \( y \in X \) one has \( d'(f(x), f(y)) \leq d(x, y) \) (or shorty if \( d' \circ f \times f \leq d \)). Further denote by \( \text{Met}(X) \) the fiber of \( \text{Met} \) structures on \( X \).

The particular full subcategory of \( \text{Met} \) consisting of all quasi-metric spaces [12] will be denoted by \( C^\Delta \). Other subconstructs that will be considered are \( C^\Delta_s \), the construct of metric spaces, \( C^{\Delta_{\text{tbd}}} \), the construct of totally bounded metric spaces and \( C^\Delta_u \), the construct of ultrametric spaces.

The order on \( \text{Met}(X) \) is defined pointwise and as usual a downset in \( \text{Met}(X) \) is a non-empty subset \( S \) such that if \( d \in S \) and \( e \) is a quasi-pre-metric, \( e \leq d \) then \( e \in S \). For any collection \( B \) of quasi-pre-metrics we put \( B \downarrow := \{ e \in \text{Met}(X) \mid \exists d \in B : e \leq d \} \). We say that \( B \) is a basis for \( M \) if \( B \downarrow = M \).

\( M \) is the construct with objects, pairs \((X, M)\) where \( X \) is a set and \( M \) is a downset in \( \text{Met}(X) \). \( M \) is called a meter (on \( X \)) and \((X, M)\) a metered space. If \((X, M)\) and \((X', M')\) are metered spaces and \( f : (X, M) \rightarrow (X', M') \) then we say that \( f \) is a contraction if

\[ \forall d' \in M' : d' \circ f \times f \in M. \]

It is easily verified that \( M \) is a well fibred topological construct. We refer to [6] for the detailed constructions of initial and final structures.

A base category \( C \) is a full and isomorphism-closed concrete subconstruct of \( \text{Met} \) which satisfies certain stability conditions as formulated in [6].

In this paper we will only consider base categories \( C \) that are contained in \( C^\Delta \) and that satisfy some supplementary conditions from [5] ensuring some results on separation.

In order to deal with completions we will add one more condition which will be assumed on all base categories we encounter.
[B] \( C \) is said to be closed under “r-dense” extensions in \( C^\Delta \) whenever \( f : (X, d) \rightarrow (Y, d') \) is a \( \mathcal{T}_d \)-dense embedding in \( C^\Delta \) with \((X, d)\) belonging to \( C \) then also \((Y, d')\) belongs to \( C \).

The subconstructs of \( \text{Met} \) introduced earlier, \( C^\Delta, C^{\Delta_s}, C^{\Delta_s \emptyset} \) and \( C^\mu \) are base categories and as we know from [5] the results on separation go through. Note that all of them satisfy [B].

Given a base category \( C \), one considers \( C \)-meters, these are meters having a basis consisting of \( C \)-metrics. The full reflective subconstruct of \( \text{M} \), consisting of all metered spaces with meters having a basis consisting of \( C \)-metrics is denoted by \( \text{M}^C \) and the fiber of \( \text{M}^C \) structures on \( X \) is denoted by \( \text{M}^C(X) \).

An expander \( \xi \) on \( \text{M}^C \) provides us for every set \( X \) with a function

\[
\text{M}^C(X) \rightarrow \text{M}^C(X) : \mathcal{M} \mapsto \xi(\mathcal{M})
\]

such that the following properties are fulfilled:

- **[E1]** \( \mathcal{M} \subset \xi(\mathcal{M}) \),
- **[E2]** \( \mathcal{M} \subset \mathcal{N} \Rightarrow \xi(\mathcal{M}) \subset \xi(\mathcal{N}) \),
- **[E3]** \( \xi(\xi(\mathcal{M})) = \xi(\mathcal{M}) \),
- **[E4]** if \( f : Y \rightarrow X \) and \( \mathcal{M} \in \text{M}^C(X) \), then: \( \xi(\mathcal{M}) \circ f \times f \subset \xi(\mathcal{M} \circ f \times f \downarrow) \)

Given an expander \( \xi \) on \( \text{M}^C \), then \( \text{M}^C_\xi \) is the full coreflective subconstruct of \( \text{M}^C \) with objects, those metered spaces \((X, \mathcal{M})\) for which \( \xi(\mathcal{M}) = \mathcal{M} \).

The main result of [6] states that \( \text{M}^C \) provides a model for all \( C \)-metrically generated theories in the sense that a topological construct \( X \) is \( C \)-metrically generated (meaning that there is a functor \( K : C \rightarrow X \) preserving initial morphisms and having an initially dense image) if and only if \( X \) is concretely isomorphic to \( \text{M}^C_\xi \) for some expander \( \xi \) on \( \text{M}^C \). Again in order to apply some results on separation we assume two extra technical assumptions [E5],[E6] on the expanders:

- **[E5]** \( \xi(\{0\}) = \{0\} \), where 0 denotes the zero-metric,
- **[E6]** \( \xi(\mathcal{M}) \) is saturated for taking finite suprema, for every \( \mathcal{M} \in \text{M}^C(X) \).

**Without explicit mentioning, we will only consider expanders that satisfy the conditions [E1] up to [E6] from [6] and [5].**

For a \( C \)-meter \( \mathcal{D} \) on a set \( X \), denote \( \xi^C(\mathcal{D}) = \{ d \in \xi(\mathcal{D}) \mid d \text{ \( C \)-metric} \} \). If we consider the following examples for \( \xi \), we obtain expanders \( \xi^\mathcal{A}_f, \xi^\mathcal{C}_f, \xi^\mathcal{A}_f, \xi^\mathcal{C}_f \),
and \( t^C \) on \( M^C \), which will yield important constructs within the framework of metrically generated theories.

\[
\begin{align*}
\xi_F^C \subseteq \xi_G^C \subseteq \xi_D^C \\
\xi_F^C \subseteq \xi_A^C \subseteq \xi_U^C \\
\xi_U^C \subseteq \xi_G^C \subseteq \xi_D^C \\
\xi_G^C \subseteq \xi_D^C
\end{align*}
\]

Whenever it is clear from the context what base category is involved, we will drop the superscript \( C \) in the notations above. We capture many known topological constructs, considering the above expanders on categories \( M^C \), for different base categories \( C \).

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\( \text{Top} \), \( \text{Creg} \) and \( \text{ZDim} \) consist of all topological spaces, of all completely regular and of all zero dimensional topological spaces respectively, with continuous maps as morphisms.

\( \text{Ap} \) and \( \text{UAp} \) consist of all approach spaces and uniform approach spaces in the sense of [13], with contractions as morphisms. \( \text{ZDAp} \) is the full subconstruct consisting of all zero dimensional approach spaces. These are approach spaces with a gauge basis consisting of ultrametrics or could be equivalently defined as those approach spaces that are subspaces of products in \( \text{Ap} \) of ultrametric spaces.

\( \text{qUnif} \) consists of all quasi-uniform spaces [12], [8], \( \text{Unif} \) of all uniform spaces, with uniformly continuous maps as morphisms, \( \text{Prox} \) of all proximity spaces and proximally continuous maps [17] and \( \text{naUnif} \) is the full subconstruct of \( \text{Unif} \) consisting of all non-Archimedean uniform spaces in the sense of [16].

\( \text{qUG} \) consists of all quasi-uniform gauge spaces [7], \( \text{UG} \) of all uniform gauge spaces [14], with uniform contractions, \( \text{efGap} \) of all Effremovic-gap
spaces in the sense of [10] with associated maps and \( \text{tUG} \) is the full subconstruct of \( \text{UG} \) consisting of all transitive uniform gauge spaces.

3 Cogeneration by completely metrizable spaces

Recall that an object \((X,d)\) in \( C^A \) is said to be bicomplete if \((X,d^*)\) is complete. \((Y,q)\) is a bicompletion of a \( C^A \)-object \((X,d)\) if \((Y,q)\) is a bicomplete space in which \((X,d)\) is \( q^* \)-densely embedded. For objects in a base category \( C \), we will use the following analogous definition for completeness and completion.

**Definition 3.1.**

- A \( C \)-object \((X,d)\) is called bicomplete if \((X,d^*)\) is complete.
- \((Y,q)\) is a \( C \)-completion of a \( C \)-object \((X,d)\) if \((Y,q)\) is a bicompletion of \((X,d)\) in \( C^A \) and \((Y,d)\) belongs to \( C \).

As usual we denote by \( X_0 \) the class of \( T_0 \)-objects in \( X \) [15]. In particular \( C_0 \) is the subconstruct of \( C \) consisting of its \( T_0 \)-objects.

It is well known that every \( T_0 \) quasi-metric space has an (up to isometry) unique \( C_0^0 \)-completion. It easily follows from our assumptions on the base categories that for \((X,d)\) a \( T_0 \) \( C \)-object, the \( C_0^0 \)-completion of \((X,d)\) is also the unique \( C_0 \)-completion.

Recall from [4] that a (complete) construct is said to be \( \text{Emb-cogenerated} \) by a subclass \( P \) if every object is embedded in a product of \( P \)-objects.

**Proposition 3.2.** Assume \( C \) is a base category and let \( \xi \) be an expander on \( M_C \). Let
\[
P = \{(Z,\xi(\{e\} \downarrow)) : (Z,e) \text{ is a bicomplete } C_0 - \text{space}\}
\]
Then \( P \) is an \( \text{Emb-cogenerating class for } (M_{\xi}^C)_0 \).

**Proof.** Case 1) of the proof deals with the expander \( t_C \). Let \((X,D)\) be an arbitrary \( (M_{t_C}^C)_0 \)-object, with a base \( Q \) of \( C \)-metrics.

Note that the source
\[
(1_X : (X,D) \rightarrow (X,\{q\} \downarrow))_{q \in Q}
\]
is initial in \( M^C_{\mathcal{C}} \). Recall that the \( T_0 \)-quotient reflection of a quasi-metric space \((X,d)\) is given by the morphism
\[
\tau_d : (X,d) \longrightarrow (X_{\overline{d}}, \overline{d}) : x \longmapsto \overline{x}
\]
where \( \overline{x} = \{ y \in X \mid d(x,y) = d(y,x) = 0 \} \), \( X_{\overline{d}} = \{ \overline{x} \mid x \in X \} \) and \( \overline{d}(\overline{x}, \overline{y}) = d'(x,y) \) for \( x, y \in X \). Using the standing assumptions on \( C \), the \( T_0 \)-reflection of a \( C \)-object is obtained in the same way as in \( C^\Delta \). The reflection morphism \( \tau_q : (X,q) \longrightarrow (X_q, \overline{q}) : x \longmapsto \overline{x} \) is initial, which implies that also the source
\[
\left( \tau_q : (X, \mathcal{D}) \longrightarrow (X_q, \{ \overline{q} \}) \right)_{q \in Q}
\]
is initial in \( M^C_{\mathcal{C}} \). By our standing assumptions on \( C \), for each \( q \in Q \), one can consider the \( C_0 \)-completion \((X_q, \overline{q})\) of the space \((X_q, \overline{q})\). So, for every \( q \in Q \), the map \( k_q : (X_q, \overline{q}) \longrightarrow (X_q, \overline{q}) \) is initial in \( C \). It follows that the contraction \( k_q : (X_q, \{ \overline{q} \}) \longrightarrow (X_q, \{ \overline{q} \}) \) is initial in \( M^C_{\mathcal{C}} \). Finally one obtains the following initial source in \( M^C_{\mathcal{C}} \):
\[
\left( k_q \circ \tau_q : (X, \mathcal{D}) \longrightarrow (X_q, \{ \overline{q} \}) \right)_{q \in Q}
\]
Due to the \( T_0 \) property of \((X, \mathcal{D})\), which means that for any \( x, y \in X, x \neq y \), there exists \( d \in \mathcal{M} : d(x,y) \neq 0 \) or \( d(y,x) \neq 0 \), this source turns out to be point-separating. Moreover for every \( q \in Q \), the \( C \)-space \((X_q, \{ \overline{q} \})\) is a \( \mathcal{P} \)-object.

For case 2) of the proof, let \((X, \mathcal{D})\) be an arbitrary \((M^C_{\mathcal{C}})_{T_0} \)-object. It suffices to apply the coreflector \( \xi : M^C_{\mathcal{C}} \longrightarrow M^C_{\delta} : (Y, \mathcal{G}) \longmapsto (Y, \xi(\mathcal{G})) \) to the source \((k_q \circ \tau_q)_{q \in Q}\).

We capture some well known results like \text{Unif}_0 being Emb-cogenerated by the class
\[
\{(Z, \mathcal{U}_d) \mid d \text{ a complete Hausdorff metric on } Z\}
\]
and the construct \text{UAp}_0 being Emb-cogenerated by the class
\[
\{(Z, \delta_d) \mid d \text{ a complete Hausdorff metric on } Z\}.
\]
The previous theorem implies analogous results for all the constructs in table of section 2. Note that $\text{Top}_0$ and $\text{Ap}_0$ are cogenerated by a single object. $\text{Top}_0$ is Emb-cogenerated by the Sierpinski space $S_2$ which is quasi-metrizable by a $T_0$ bicomplete quasi-metric. $\text{Ap}_0$ is cogenerated by the object $\mathbb{P}$. This object $\mathbb{P}$ however is not (bicompletely) quasi-metrizable. We will come back to these examples in section 5.

4 Construction of complete objects from completely metrizable spaces

In this section we tackle our main problem. We will endow $(M^\xi)_0$ with a closure operator $s$ and we will consider the class $\mathcal{U}_s$ of all $s$-dense embeddings. The following two questions will be investigated:

1) Are the completely metrizable objects $\mathcal{U}_s$-injective?

2) Is the class of all $s$-closed subspaces of products of completely metrizable objects firmly $\mathcal{U}_s$-reflective?

For explicit definitions on firmness we refer to [4] and [3]. Here we briefly recall that, given a class $\mathcal{U}$ of $\mathcal{X}$-morphisms, a reflective subconstruct with reflector $R$ is said to be subfirmly $\mathcal{U}$-reflective if it is $\mathcal{U}$-reflective and if for every morphism $u$ in $\mathcal{U}$ the reflection $R(u)$ is an isomorphism. If $\mathcal{U}$ coincides with the class of morphisms for which $R(u)$ is an isomorphism, the subconstruct is said to be firmly $\mathcal{U}$-reflective. Among other things $\mathcal{U}$-firmness implies uniqueness of completion with respect to the class $\mathcal{U}$.

Since the class $\mathcal{U}_s$ we will be dealing with consists of certain embeddings, $\mathcal{U}_s$-firmness will imply that $\mathcal{U}_s$ is contained in the class of all epimorphic embeddings. In all the examples in section 6. we will be dealing with closure operators on $(M^\xi)_0$ that are (pointwise) smaller than the regular closure operator $r$, describing the epimorphisms. In order to satisfy the standing assumptions on stability of $\mathcal{U}$ with respect to compositions, as put forward in [3], we will assume that the closure operator $s$ is idempotent. The class of $\mathcal{U}_s$-injective objects is denoted by $\text{Inj}\mathcal{U}_s$. The proof of the next result uses standard techniques, see for instance [4].

**Proposition 4.1.** If $s$ is a weakly hereditary, idempotent closure operator on $\mathcal{X}$, then $\text{Inj}\mathcal{U}_s$ is closed for taking $s$-closed subspaces of products in $(M^\xi)_0$. 

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In [5] the closure operator \( r \) has been explicitly formulated in the following way. For an \( (M^C_\xi)_0 \)-object \((X, D)\)

\[
x \in r_X(M) \iff \forall d \in D : \inf_{m \in M} d(x,m) + d(m,x) = 0.
\]

The closure operator \( r \) is known to be idempotent and was shown to be hereditary on \((M^C_\xi)_0\) for all the expanders listed in section 2, i.e. for arbitrary \( C \) in cases where \( \xi \) equals any of the expanders \( \xi^C_U, \xi^C_G \) or \( \xi^C_D \), and for \( C \subset \mathcal{C}^\Delta \) and \( \mathcal{C}^\Delta \) in the cases \( \xi_D^\Delta, \xi_A^\Delta \).

**Theorem 4.2.** Assume \( C \) is a base category and let \( \xi \) be an expander on \( M^C \). On \((M^C_\xi)_0\) let \( s \) be a weakly hereditary, idempotent closure operator and let \( U_s \) be the class of all \( s \)-dense embeddings in \((M^C_\xi)_0\).

The following are equivalent:

1. For every \( j : (X, \mathcal{H}) \longrightarrow (Y, D) \) with \( j \in U_s \):

\[
j \in U_r \text{ and } \mathcal{H} = D \circ j \times j \\
\]

2. The class \( \mathcal{P} = \{(Z, \xi(\{e\})): (Z, e) \text{ is a bicomplete } C_0\text{-object}\} \) is \( U_s \)-injective in \((M^C_\xi)_0\) and \( U_s \subset U_r \);

3. The class \( \mathcal{R}_s \) of \( s \)-closed subobjects of products of \( \mathcal{P} \)-objects is a sub-firm \( U_s \)-reflective subcategory of \((M^C_\xi)_0\).

**Proof.** To prove that 1. implies 2. let \((Z, \xi(\{e\})))\) be an arbitrary \( \mathcal{P} \)-object, \( j : (X, \mathcal{H}) \longrightarrow (Y, D) \) belong to \( U_s \) and \( f : (X, \mathcal{H}) \longrightarrow (Z, \xi(\{e\})) \) be a contraction in \( M^C_\xi \). Since \( e \circ f \times f \) belongs to \( \mathcal{H} \) and since by 1. \( \mathcal{H} = D \circ j \times j \), we can choose a \( C \)-metric \( d \in D \) such that \( e \circ f \times f \leq d \circ j \times j \). Consider the following situation in \( \mathcal{C}^\Delta \). The map \( j : (X, d \circ j \times j) \longrightarrow (Y, d) \) is a \( d^* \)-dense embedding and \( f : (X, d \circ j \times j) \longrightarrow (Z, e) \) is a contraction. Since \( (Z, e) \) is bicomplete, it is injective in \( \mathcal{C}^\Delta \) with respect to \( r \)-dense embeddings, and hence there is a contraction \( \tilde{f} : (Y, d) \longrightarrow (Z, e) \) such that \( \tilde{f} \circ j = f \). Clearly \( \tilde{f} : (Y, D) \longrightarrow (Z, \{e\}) \) is a contraction in \( M^C \) and since \( (Y, D) \) belongs to \( M^C_\xi \) the map \( f : (Y, D) \longrightarrow (Z, \xi(\{e\})) \) is a contraction in \( M^C_\xi \).
To prove that 2. implies 3., we follow the lines of proof of theorem 1.6 in [4]. First note that by 3. $\mathcal{P} \subseteq \text{Inj}\, \mathcal{U}_s$. Hence, from proposition 4.1 we have that $\mathcal{R}_s \subseteq \text{Inj}\, \mathcal{U}_s$. Next we show that $\mathcal{R}_s$ is a $\mathcal{U}_s$-reflective subconstruct.

Let $X$ be an arbitrary $(M_s^i)_0$-object. Proposition 3.2 ensures that there exist objects $P_i \in \mathcal{P}$ ($i \in I$) such that we have an embedding $j : X \hookrightarrow \prod_{i \in I} P_i$. Consider its $(\mathcal{E}^s, \mathcal{M}^s)$-factorization $j = m \circ e$ where $X \xrightarrow{e} M \xrightarrow{m} \prod_{i \in I} P_i$, with $e \in \mathcal{E}^s$ and $m \in \mathcal{M}^s$. Since $j$ is an embedding, so is $e$. So we get that $e \in \mathcal{U}_s$ and $M \in \mathcal{R}_s$.

For $Y \in \mathcal{R}_s$ and $f : X \longrightarrow Y$ an arbitrary contraction, using the $\mathcal{U}_s$-injectivity of $Y$, we can construct a contraction $f^*$ such that $f^* \circ e = f$ which is unique by the fact that $e$ is an epimorphism.

Moreover, $\mathcal{R}_s$ is subfirmly $\mathcal{U}_s$-reflective. For $(M_s^i)_0$-objects $X$ and $Z$ suppose $g : X \longrightarrow Z$ belongs to $\mathcal{U}_s$. Denote by $r_Z : Z \longrightarrow RZ$ and $r_X : X \longrightarrow RX$ the $\mathcal{R}_s$-reflection morphisms. Using the $\mathcal{U}_s$-injectivity of $RX$ and the fact that $g$, $r_Z$ and $r_X$ belong to $\mathcal{U}_s$, we can conclude that there exists a contraction $h : RZ \longrightarrow RX$ such that $h$ and $Rg$ are each others inverses. Finally $Rg$ is an isomorphism.

To prove that 3. implies 1. suppose $\mathcal{R}_s$ is subfirmly $\mathcal{U}_s$-reflective. Then the results in [3] already imply that $\mathcal{R}_s = \text{Inj}\, \mathcal{U}_s$ and that $\mathcal{U}_s \subseteq \mathcal{U}_r$.

Let $j : (X, \mathcal{H}) \longrightarrow (Y, \mathcal{D})$ belong to $\mathcal{U}_s$ and consider an arbitrary $C$-metric $e \in \mathcal{H}$. Then, as in the proof of proposition 3.2, the map

$$\alpha_e : (X, \mathcal{H}) \longrightarrow (X_e, \hat{\mathcal{H}}) : x \longmapsto \hat{x}$$

is a contraction in $M^C$ and therefore $\alpha_e : (X, \mathcal{H}) \longrightarrow (\hat{X}_e, \hat{\mathcal{H}}(\{\hat{e}\} \downarrow))$ is a contraction in $M_s^C$. Since $(\hat{X}_e, \hat{\mathcal{H}}(\{\hat{e}\} \downarrow))$ is $\mathcal{U}_s$-injective, there exists a contraction $\alpha_e : (Y, \mathcal{D}) \longrightarrow (\hat{X}_e, \hat{\mathcal{H}}(\{\hat{e}\} \downarrow))$, such that $\alpha_e \circ j = \alpha_e$. Composing $\alpha_e$ with the $M^C$-morphism

$$j' : (X, \mathcal{D} \circ j \times j \downarrow) \longrightarrow (Y, \mathcal{D}) : x \longmapsto j(x)$$

we get that

$$\alpha_e \circ j' : (X, \mathcal{D} \circ j \times j \downarrow) \longrightarrow (\hat{X}_e, \hat{\mathcal{H}}(\{\hat{e}\} \downarrow))$$

is a morphism in $M^C$. Consequently: $e = \hat{e} \circ (\alpha_e \circ j') \times (\alpha_e \circ j')$ belongs to $\mathcal{D} \circ j \times j \downarrow$. \qed
If moreover we assume the closure operator $s$ to be hereditary, we can strengthen 3. in the equivalences of theorem 4.2.

**Corollary 4.3.** Assume $C$ is a base category and let $\xi$ be any expander on $M^C$. On $(M^C)_{\xi}$ let $s$ be a hereditary, idempotent closure operator and let $U_s$ be the class of all $s$-dense embeddings in $(M^C)_{\xi}$. The following are equivalent:

1. For every $j : (X, \mathcal{H}) \to (Y, \mathcal{D})$ with $j \in U_s$:
   \[ j \in U_r \text{ and } \mathcal{H} = \mathcal{D} \circ j \times j \]

2. $P = \{(Z, \xi(\{e\}^\perp)) : (Z, e) \text{ is a bicomplete } C_0\text{-object}\}$ is $U_s$-injective in $(M^C)_{\xi}$ and $U_s \subseteq U_r$;

3. The class $R_s$ of $s$-closed subobjects of products of $P$-objects is a firm $U_s$-reflective subcategory of $(M^C)_{\xi}$.

**Proof.** The only non-trivial implication is 2. implies 3. In view of the fact that by theorem 4.2 the class $R_s$ is already subfirmly $U_s$-reflective, it is sufficient to show that $U_r$ is coessential [3]. Suppose both $u$ and $u \circ f$ belong to $U_s$ then clearly $f$ is an embedding. The hereditariness of $s$ and the fact that $u \circ f$ is $s$-dense imply that $f$ is $s$-dense.

\[ \square \]

5 **Examples**

Remark that if one of the equivalent claims of propositions 4.2 or 4.3 holds for the regular closure operator $r$ of $(M^C)_{\xi}$, then it also holds for every idempotent, (weakly) hereditary closure $s$ on $(M^C)_{\xi}$ with $s \leq r$. For this reason we start investigating concrete situations of categories endowed with the regular closure $r$. 

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5.1 $\mathcal{U}_r$-firmly reflective subconstructs: the case of the expanders $\xi$ equal to $1^C$, $\xi^C_U$, $\xi^C_U G$ or $\xi^C_D$.

Let $C$ be any base category. As was shown in [5] the regular closure $r$ on $(\mathcal{M}_\xi^C)_0$, built with the expanders listed above, is idempotent and hereditary. We will show that the first claim in 4.3 (and thus also property 2. and 3.) holds.

**Proposition 5.1.** For any expander listed in the subtitle 6.1., let $j : (X, \mathcal{H}) \to (Y, \mathcal{D})$ a morphism in $(\mathcal{M}_\xi^C)_0$ such that $j \in \mathcal{U}_r$, then we have

$$\mathcal{H} = \mathcal{D} \circ j \times j \downarrow.$$

**Proof.** Remark that the proof of the statement for the expanders $\xi^C_U$ and $1^C$ is based on the fact that in both cases subobjects in $\mathcal{M}_\xi^C$ coincide with subobjects in $\mathcal{M}_\xi^C$.

We give an explicit proof for the case $\xi$ equal to $\xi^C_U G$. The remaining case where $\xi$ equals $\xi^C_U$ will follow from it, since $\mathcal{M}_\xi^C$ is a bireflective subconstruct of $\mathcal{M}_\xi^C$. Let $j : (X, \mathcal{H}) \to (Y, \mathcal{D})$ a morphism in $(\mathcal{M}_\xi^C)_0$, and suppose $j \in \mathcal{U}_r$. First apply the symmetrizer in the sense of [5] to $(X, \mathcal{H}), (Y, \mathcal{D})$ and to $j$. It is a coreflector in this case. Then compose it with the restriction of the uniform coreflector. Using isomorphic descriptions of the objects we denote $\mathcal{U}(\mathcal{H}^*)$ and $\mathcal{U}(\mathcal{D}^*)$ for the objects obtained and again $j : (X, \mathcal{U}(\mathcal{H}^*)) \to (Y, \mathcal{U}(\mathcal{D}^*))$ for the image through the composed functor. $j$ now is a dense embedding in $\text{Unif}_0$.

Let $e \in \mathcal{H}$ be an arbitrary $C$-metric. Then $e$ is uniformly continuous on $X \times X$ endowed with the product of the uniformities $\mathcal{U}(\mathcal{H}^*)$. In view of the density assumption, there is a unique uniformly continuous quasimetric $g$ on $Y \times Y$ endowed with the product structure of $\mathcal{U}(\mathcal{D}^*)$ and satisfying $g \circ j \times j = e$.

An explicit formulation of $g$ is given by

$$g : Y \times Y \to [0, \infty] : (y, y') \mapsto \sup_{d \in \mathcal{D}, \varepsilon > 0} e(j^{-1}(B_{d^*}(y, \varepsilon)), j^{-1}(B_{d^*}(y', \varepsilon))).$$

Since we have that $j : (X, e) \hookrightarrow (Y, g)$ is an $r$-dense embedding in $C^\Delta$ the quasi-metric $g$ is a $C$-metric.
The only thing left to prove is that \( g \) belongs to \( \mathcal{D} \).

Let \( \varepsilon > 0 \) and \( \omega < \infty \) be arbitrary. Since \( \mathcal{H} = \mathcal{C} \mathcal{U}_\mathcal{G} (\mathcal{D} \circ j \times j \downarrow) \) there exists a \( \mathcal{C} \)-metric \( d \in \mathcal{D} \) such that \( e(z, w) \wedge \omega \leq d \circ j \times j(z, w) + \frac{\varepsilon}{3} \) for every \( z, w \in X \).

Take \( y, y' \in Y \) arbitrarily. We will show that \( g(y, y') \wedge \omega \leq d(y, y') + \varepsilon \).

Let \( p \in \mathcal{D} \), \( \zeta > 0 \) be arbitrary. Choose \( x, x' \in X \) such that \( (p \vee d)^*(y, j(x)) < \zeta \wedge \frac{\varepsilon}{3} \) and \( (p \vee d)^*(y', j(x')) < \zeta \wedge \frac{\varepsilon}{3} \). Then we have

\[
e(j^{-1}(B_{p^*}(y, \zeta)), j^{-1}(B_{p^*}(y', \zeta))) \wedge \omega \leq e(x, x') \wedge \omega \leq d(y, y') + \varepsilon.
\]

The previous results imply that for a metrically generated construct \( X_0 \), which is one of the examples \( \text{qUnif}_0, \text{Unif}_0, \text{Prox}_0, \text{naUnif}_0, \text{qUG}_0, \text{UG}_0, \text{efGap}_0, \text{tUG}_0, C_0 \), or \( (M_1^C)_0 \), there exists a \( \mathcal{U}_r \)-firmly reflective subcategory \( \mathcal{R}_r \) of complete objects. Moreover the complete objects are “generated” by the completely metrizable objects in the construct, meaning that an object in \( X_0 \) is complete if and only if it is an \( r \)-closed subset of a product of objects in the image of the class of bicompact \( C_0 \)-objects under the functor \( K : C \rightarrow X \).

In the table below we associate to each subconstruct \( \mathcal{R}_r \) in the list of examples some known subconstruct of complete objects described in the literature.

<table>
<thead>
<tr>
<th>( \mathcal{R}_r ) is generated by</th>
<th>( \mathcal{R}_r ) is generated by bicompletely metrizable objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{qUnif}_0</td>
<td>bicomplete ( T_0 ) quasi-uniform spaces</td>
</tr>
<tr>
<td>\text{Unif}_0</td>
<td>complete Hausdorff uniform spaces</td>
</tr>
<tr>
<td>\text{Prox}_0</td>
<td>Effremovic proximity spaces with compact Hausdorff underlying topology</td>
</tr>
<tr>
<td>\text{naUnif}_0</td>
<td>complete non-Archimedean uniform spaces</td>
</tr>
<tr>
<td>\text{UG}_0</td>
<td>complete ( T_0 )-Uniform Gauge spaces</td>
</tr>
<tr>
<td>\text{efGap}_0</td>
<td>Gap-spaces with compact Hausdorff underlying topology</td>
</tr>
<tr>
<td>\text{tUG}_0</td>
<td>complete transitive ( T_0 )-Uniform Gauge spaces</td>
</tr>
<tr>
<td>( C^\Delta_0 )</td>
<td>bicomplete ( T_0 ) quasi-metric spaces</td>
</tr>
<tr>
<td>( C^{\Delta_s}_0 )</td>
<td>complete Hausdorff metric spaces</td>
</tr>
<tr>
<td>( C^{\Delta_0}_0 )</td>
<td>compact metric spaces</td>
</tr>
<tr>
<td>( C^\mu_0 )</td>
<td>complete ( T_0 ) ultrametric spaces</td>
</tr>
</tbody>
</table>

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5.2 \( \mathcal{U}_r \)-firmly reflective subconstructs: the case of the expanders \( \xi_T^C \) and \( \xi_A^C \).

In case \( \xi \) equals \( \xi_T^C \) or \( \xi_A^C \), things do not work in the same way as in the previous examples.

We first deal with base categories \( C \) contained in \( C^{As} \) and we refer to table in section 2 for the isomorphic descriptions of the constructs.

It is well known that in \( \text{Creg}_0 \) there doesn’t exist a \( \mathcal{U}_r \)-subfirm subconstruct \( \mathcal{R}_e \). It is shown in [4] that \( \text{Creg}_0 \) does not have \( \mathcal{U}_r \)-injective objects except for the singleton spaces. The argument uses the \( r \)-dense embedding \( j : (\mathbb{N}, \mathcal{T}) \rightarrow (\mathbb{N}^*, \mathcal{T}^*) \) of the discrete space of natural numbers into its Alexandroff compactification. On \( (\mathbb{N}, \mathcal{T}) \) a two valued continuous function, which is 0 on even numbers and 1 on odd numbers, has no continuous extension to \( (\mathbb{N}^*, \mathcal{T}^*) \). Since both \( (\mathbb{N}, \mathcal{T}) \) and \( (\mathbb{N}^*, \mathcal{T}^*) \) are zero dimensional, the same argument shows that in \( \text{ZDim}_0 \) there cannot exist a \( \mathcal{U}_r \)-subfirm subconstruct either. Considering \( (\mathbb{N}, \mathcal{T}) \) and \( (\mathbb{N}^*, \mathcal{T}^*) \) as topological approach spaces gives the same negative result for \( \text{UA}_0 \). Showing that these spaces are moreover zero dimensional approach spaces, yields that there is no \( \mathcal{U}_r \)-subfirm subconstruct in \( \text{ZDA}_0 \) either.

Next we deal with the base category \( C^A \). The expanders \( \xi_T \) and \( \xi_A \) provide isomorphic descriptions of the constructs \( \text{Top} \) and \( \text{Ap} \) respectively. It is well known that the construct \( \text{TSob} \) of sober topological spaces is a \( \mathcal{U}_r \)-firmly reflective subconstruct of \( \text{Top}_0 \). However \( \text{TSob} \) is not generated by bicompletely quasi-metrizable objects. In fact for the class

\[
P = \{(Z, \mathcal{T}_e) \mid e \ T_0 \text{ bicomplete quasi-metric}\}
\]

we have that \( P \not\subseteq \text{TSob} \).

In order to illustrate this, consider the quasi-metric \( e \) on \( \mathbb{N} \) given by \( e(n,m) = 0 \) and \( e(m,n) = \infty \) if \( n < m \). Note that \( e \) is a \( T_0 \) quasi-metric such that \( e^* \) is discrete and therefore complete. For \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) we have \( B_e(n, \varepsilon) = \{n, n+1, \ldots\} \). It now easily follows that \( \mathbb{N} \) is irreducible and that it can’t be written as the closure of a singleton.

An analogous situation appears in \( \text{Ap}_0 \). In [11] it was shown that the construct \( \text{ASob} \) of sober approach spaces is \( \mathcal{U}_r \)-firm in \( \text{Ap}_0 \). Again

\[
P = \{(Z, \delta_e) \mid e \ T_0 \text{ bicomplete quasi-metric}\} \not\subseteq \text{ASob}
\]
and by corollary 4.3 this implies that $\text{ASob}$ is not generated by bicompletely \-{} quasi-metrizable objects. Indeed, consider the same bicomplete $T_0$ quasi-metric space $(\mathbb{N}, e)$ as in the previous argument. The fact that $(\mathbb{N}, T_e)$ is not sober as a topological space, implies that $(\mathbb{N}, S_e)$ is not sober as an approach space.

<table>
<thead>
<tr>
<th>$R_c$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Creg}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{ZDim}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{Top}_0$</td>
<td>Sober topological spaces; not generated by completely metrizable obj.</td>
</tr>
<tr>
<td>$\text{UAp}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{ZDAp}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{Ap}_0$</td>
<td>Sober approach spaces; not generated by completely metrizable obj.</td>
</tr>
</tbody>
</table>

5.3 \(\mathcal{U}_c\)-firmly reflective subconstructs for the closure operator determined by the metric coreflection

In this section, instead of considering the closure operator $r$ we look for a natural closure operator that is smaller. For $(X, D)$ an $(M_C)_0$-object, and $x, y \in X$, put

$$\varphi(x, y) = \sup_{d \in D} d(x, y).$$

Then, consider the topological closure $cl^{\varphi^*}$ associated with the symmetrization $\varphi^*$. Clearly $cl^{\varphi^*}$ is an idempotent closure operator which is smaller than the regular closure $r$.

In case $\xi = \xi^C_U$, the closure $cl^{\varphi^*}$ clearly coincides with the regular closure $r$, so the completion theory coincides with the one we investigated in 6.1.

If $\xi$ equals $\xi^C_U$ or $t^C$, then $cl^{\varphi^*}$ is the closure of the symmetrization of the coreflection into $C_0$ and $cl^{\varphi^*}$ can be seen to be hereditary. Since proposition 5.1 holds for $\xi^C_U$ (t) and the regular closure $r$, the same is true for $cl^{\varphi^*}$. It follows that the subcategory $\mathcal{R}_{cl^{\varphi^*}}$ consisting of all $cl^{\varphi^*}$-closed subobjects of products of bicompletely metrizable objects forms a $\mathcal{U}_{cl^{\varphi^*}}$-firm subconstruct of $(M_C^C)_{0} \left((M_{t^C})_{0}\right)$. Via the expander $\xi_{UG}$ we get isomorphic descriptions of $qUG_0$, $UG_0$, $efGap_0$, and $tUG_0$ for which the $\mathcal{U}_{cl^{\varphi^*}}$-completion theory was not yet considered in the literature.
Note that if $\xi$ equals $\xi_F$ or $\xi_C$, then $cl^{\phi^*}$ is the discrete closure and so the $cl^{\phi^*}$-dense embeddings coincide with the isomorphisms in $(M^c_{\xi})_0$. So the completion theory with respect to $U_{cl^{\phi^*}}$ becomes trivial in these constructs. For example, in $Top_0$, $Creg_0$, $ZDim_0$, $qUnif_0$, $Unif_0$, $Prox_0$ and $naUnif_0$, all objects are $U_{cl^{\phi^*}}$-complete.

If $\xi$ equals $\xi_A$ then $cl^{\phi^*}$ is the closure of the symmetrization of the coreflection into $G_0$ and $cl^{\phi^*}$ is hereditary. We consider the constructs $UAp_0$, $ZDAp_0$ for which the completion theory with respect to the regular closure failed and $Ap_0$ for which the firm $U_{\phi}$-reflective subconstruct $ASob$ is not generated by bicompletely metrizable objects. The subconstruct $cUAp_0$ consisting of complete objects in $UAp_0$, as introduced in [13], is firm with respect to $U_{cl^{\phi^*}}$, as can be deduced from the result on uniqueness of completion there. Moreover it also follows from [13] that the completely metrizable objects are $U_{cl^{\phi^*}}$-injective. So by corollary 4.3 we can conclude that the objects in $cUAp_0$ are $cl^{\phi^*}$-closed subobjects of products of complete metric approach spaces. Similar results can easily be obtained for the objects in $cZDAp_0$, the construct of all completely zero dimensional approach spaces.

In [2] a bicompletion theory for $Ap_0$ was developed. A subconstruct $bicAp_0$ of so called bicomplete approach spaces was constructed which was shown to be $U_{cl^{\phi^*}}$-firm and the bicomplete quasi-metric spaces were shown to be $U_{cl^{\phi^*}}$-injective. Again this yields the conclusion that the objects in $bicAp_0$ are generated by bicomplete quasi-metric spaces.

<table>
<thead>
<tr>
<th>$\mathcal{R}_{cl^{\phi^*}}$</th>
<th>is generated by bicompletely metrizable objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>$UAp_0$</td>
<td>$cUAp_0$</td>
</tr>
<tr>
<td>$ZDAp_0$</td>
<td>$cZDAp_0$</td>
</tr>
<tr>
<td>$Ap_0$</td>
<td>$bicAp_0$</td>
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</tbody>
</table>

References


Vrije Universiteit Brussel, Department of Mathematics - Pleinlaan 2, 1050 Brussel, Belgium - evacoleb@vub.ac.be, agerlo@vub.ac.be