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Firm reflections generated by complete metric spaces


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RESUME. Nous étudions des catégories concrètes où chaque objet est un sous-espace d’un produit “d’espaces métrisables”. Si une telle catégorie est munie d’un opérateur $s$ de fermeture, nous considérons $U_s$, la classe des immersions denses. Nous traitons les questions suivantes: (1) si les espaces complètement métrisables sont des objets $U_r$-injectifs, (2) si la classe des sous-objets $s$-fermés d’un produit d’espaces complètement métrisables est $U_s$ “uniquement” reflective. Nous démontrons que dans notre contexte, ces questions sont équivalentes et nous formulons des conditions pour avoir une réponse affirmative. Le théorème principal permet de traiter un grand nombre d’exemples.

1 Introduction

The category $\text{Unif}_0$ of separated uniform spaces, endowed with the closure operator $r$ determined by the underlying topology, will be our guiding example in the study of completeness in a more general setting. Completely metrizable uniform spaces play an important role in the uniform case, since firstly they are injective objects with respect to the class $\mathcal{U}_r$ of all dense embeddings and secondly the complete uniform spaces are exactly the closed subspaces of products of completely metrizable spaces. Moreover the complete objects form a firmly $\mathcal{U}_r$-reflective subconstruct of $\text{Unif}_0$ in the sense of [3].

We will investigate to what extent these results hold in a more general setting. The general framework we will be working in is the one of metrically generated constructs as introduced in [6]. These are constructs $X$ for
which a natural functor describes the transition from (generalized) metric spaces to objects in the given category $\mathcal{X}$. For example, with a (generalized) metric $d$ one can associate e.g. a (completely regular) topology $\mathcal{T}_d$, a (quasi)uniformity $\mathcal{U}_d$, a proximity $\mathcal{P}_d$ or an approach structure $\mathcal{A}_d$. In each of these examples, a natural functor $K$ from a suitable base category $\mathcal{C}$ consisting of (generalized) metric spaces to the category $\mathcal{X}$ is given. If the functor $K$ fulfills certain conditions (preserves initial morphisms and has an initially dense image) then the category $\mathcal{X}$ is said to be metrically generated. This setting, which covers all the examples above and many others, is convenient for our purpose since in particular every object in $\mathcal{X}$ is a subspace of a product of “metrizable” spaces. We will restrict to $T_0$-objects and a first attempt will be to endow $\mathcal{X}_0$ with its regular closure operator $r$ and to consider the class $\mathcal{U}_r$ of all $r$-dense embeddings. The following two questions will be investigated:

1) Are the completely metrizable objects $\mathcal{U}_r$-injective?  
2) Is the class of all $r$-closed subspaces of products of completely metrizable objects firmly $\mathcal{U}_r$-reflective?

In fact we will show that in our setting these questions are equivalent and we will give necessary and sufficient conditions for a positive answer. Our main theorem will apply to a large collection of examples listed in the tables of the next sections. It will become clear that there exist metrically generated constructs $\mathcal{X}$ allowing a $\mathcal{U}_r$-firm reflective subconstruct $\mathcal{R}$ which cannot be generated by complete metric spaces, so for which the questions above nevertheless have a negative answer.

In some cases where the answer to the questions above is negative, we still succeed in defining a smaller non-trivial closure operator for which the answers do become positive.

2 Metrically generated theories

In this section we gather some preliminary material that is needed to introduce the setting of this paper. We use categorical terminology as developed in [1] or [17] and we refer to [9] for material on closure operators. In [6] it was shown that every metrically generated construct can be isomorphically described as a subconstruct of a certain model category. It will be
convenient to deal with these isomorphic copies. So we recall the material on the model categories and fix some notation.

We call a function \( d : X \times X \rightarrow [0, \infty] \) a quasi-pre-metric if it is zero on the diagonal, we will drop "pre" if \( d \) satisfies the triangle inequality and we will drop "quasi" if \( d \) is symmetric. Note that we do not ask these quasi-pre-metrics to be realvalued or separated. If \( d \) is a quasi-metric we denote by \( d^* \) its symmetrization \( d \vee d^{-1} \).

Denote by \( \text{Met} \) the construct of quasi-pre-metrics and contractions. Recall that a map \( f : (X, d) \rightarrow (X', d') \) is a contraction (also called a nonexpansive map) if for every \( x \in X \) and \( y \in X \) one has \( d'(f(x), f(y)) \leq d(x, y) \) (or shortly if \( d' \circ f \times f \leq d \)). Further denote by \( \text{Met}(X) \) the fiber of \( \text{Met} \) structures on \( X \).

The particular full subcategory of \( \text{Met} \) consisting of all quasi-metric spaces \([12]\) will be denoted by \( \mathcal{C}^\Delta \). Other subconstructs that will be considered are \( \mathcal{C}^\Delta_{tb} \) the construct of metric spaces, \( \mathcal{C}^\Delta_{ul} \) the construct of totally bounded metric spaces and \( \mathcal{C}' \) the construct of ultrametric spaces.

The order on \( \text{Met}(X) \) is defined pointwise and as usual a downset in \( \text{Met}(X) \) is a non-empty subset \( S \) such that if \( d \in S \) and \( e \) is a quasi-pre-metric, \( e \leq d \) then \( e \in S \). For any collection \( \mathcal{B} \) of quasi-pre-metrics we put \( \mathcal{B} \downarrow:= \{ e \in \text{Met}(X) | \exists d \in \mathcal{B} : e \leq d \} \). We say that \( \mathcal{B} \) is a basis for \( \mathcal{M} \) if \( \mathcal{B} \downarrow = \mathcal{M} \).

\( \mathcal{M} \) is the construct with objects, pairs \( (X, \mathcal{M}) \) where \( X \) is a set and \( \mathcal{M} \) is a downset in \( \text{Met}(X) \). \( \mathcal{M} \) is called a meter (on \( X \)) and \( (X, \mathcal{M}) \) a metered space. If \( (X, \mathcal{M}) \) and \( (X', \mathcal{M}') \) are metered spaces and \( f : (X, \mathcal{M}) \rightarrow (X', \mathcal{M}') \) then we say that \( f \) is a contraction if

\[
\forall d' \in \mathcal{M}' : d' \circ f \times f \in \mathcal{M}.
\]

It is easily verified that \( \mathcal{M} \) is a well fibred topological construct. We refer to [6] for the detailed constructions of initial and final structures.

A base category \( \mathcal{C} \) is a full and isomorphism-closed concrete subconstruct of \( \text{Met} \) which satisfies certain stability conditions as formulated in [6].

In this paper we will only consider base categories \( \mathcal{C} \) that are contained in \( \mathcal{C}^\Delta \) and that satisfy some supplementary conditions from [5] ensuring some results on separation.

In order to deal with completions we will add one more condition which will be assumed on all base categories we encounter.
[B] $C$ is said to be closed under "$r$-dense" extensions in $C^\Delta$ whenever $f : (X,d) \to (Y,d')$ is a $T_{\mu}$-dense embedding in $C^\Delta$ with $(X,d)$ belonging to $C$ then also $(Y,d')$ belongs to $C$.

The subconstructs of $\text{Met}$ introduced earlier, $C^\Delta$, $C^{\Delta \delta}$, $C^{\Delta \delta \delta}$ and $C^{\mu}$ are base categories and as we know from [5] the results on separation go through. Note that all of them satisfy [B].

Given a base category $C$, one considers $C$-meters, these are meters having a basis consisting of $C$-metrics. The full reflective subconstruct of $\mathcal{M}$, consisting of all metered spaces with meters having a basis consisting of $C$-metrics is denoted by $\mathcal{M}^C$ and the fiber of $\mathcal{M}^C$ structures on $X$ is denoted by $\mathcal{M}^C(X)$.

An expander $\xi$ on $\mathcal{M}^C$ provides us for every set $X$ with a function

$$\mathcal{M}^C(X) \to \mathcal{M}^C(X) : \mathcal{M} \mapsto \xi(\mathcal{M})$$

such that the following properties are fulfilled:

- $\mathcal{E}_1 \ M \subset \xi(\mathcal{M})$,
- $\mathcal{E}_2 \ \mathcal{M} \subset \mathcal{N} \Rightarrow \xi(\mathcal{M}) \subset \xi(\mathcal{N})$,
- $\mathcal{E}_3 \ \xi(\xi(\mathcal{M})) = \xi(\mathcal{M})$,
- $\mathcal{E}_4 \ \text{if } f : Y \to X \text{ and } \mathcal{M} \in \mathcal{M}^C(X), \text{ then: } \xi(\mathcal{M}) \circ f \times f \subset \xi(\mathcal{M} \circ f \times f \downarrow)$

Given an expander $\xi$ on $\mathcal{M}^C$, then $\mathcal{M}^C_\xi$ is the full coreflective subconstruct of $\mathcal{M}^C$ with objects, those metered spaces $(X,\mathcal{M})$ for which $\xi(\mathcal{M}) = \mathcal{M}$.

The main result of [6] states that $\mathcal{M}^C$ provides a model for all $C$-metrically generated theories in the sense that a topological construct $X$ is $C$-metrically generated (meaning that there is a functor $K : C \to X$ preserving initial morphisms and having an initially dense image) if and only if $X$ is concretely isomorphic to $\mathcal{M}^C_\xi$ for some expander $\xi$ on $\mathcal{M}^C$. Again in order to apply some results on separation we assume two extra technical assumptions $\mathcal{E}_5, \mathcal{E}_6$ on the expanders:

- $\mathcal{E}_5 \ \xi(\{0\}) = \{0\}$, where $0$ denotes the zero-metric,
- $\mathcal{E}_6 \ \xi(\mathcal{M})$ is saturated for taking finite suprema, for every $\mathcal{M} \in \mathcal{M}^C(X)$.

Without explicit mentioning, we will only consider expanders that satisfy the conditions $\mathcal{E}_1$ up to $\mathcal{E}_6$ from [6] and [5].

For a $C$-meter $\mathcal{D}$ on a set $X$, denote $\xi^C(\mathcal{D}) = \{d \in \xi(\mathcal{D}) \mid d \text{ C-metric}\} \downarrow$. If we consider the following examples for $\xi$, we obtain expanders $\xi^C, \xi^C, \xi^C, \xi^C_\mathcal{D}$. 

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\( \xi^C_{\mathcal{U}_G}, \xi^C_{\mathcal{D}} \) and \( 1^C \) on \( M^C \), which will yield important constructs within the framework of metrically generated theories.

- \( d \in \xi^r(\mathcal{D}) \) iff \( \forall x \in X, \forall \varepsilon > 0, \exists d_1, \ldots, d_n \in \mathcal{D}, \exists \delta > 0 : \sup_{i=1}^n d(x, y) < \delta \Rightarrow d(x, y) < \varepsilon \)
- \( d \in \xi^A(\mathcal{D}) \) iff \( \forall x \in X, \forall \varepsilon > 0, \forall \omega < \infty, \exists d_1, \ldots, d_n \in \mathcal{D} : d(x, y) \land \omega \leq \sup_{i=1}^n d_i(x, y) + \varepsilon \)
- \( d \in \xi^U(\mathcal{D}) \) iff \( \forall \varepsilon > 0, \exists d_1, \ldots, d_n \in \mathcal{D}, \exists \delta > 0 : \sup_{i=1}^n d_i(x, y) < \delta \Rightarrow d(x, y) < \varepsilon \)
- \( d \in \xi^{U_G}(\mathcal{D}) \) iff \( \forall \varepsilon > 0, \forall \omega < \infty, \exists d_1, \ldots, d_n \in \mathcal{D} : d(x, y) \land \omega \leq \sup_{i=1}^n d_i(x, y) + \varepsilon \)
- \( d \in \xi^D(\mathcal{D}) \) iff \( d \leq \sup_{\varepsilon \in \mathcal{E}} \varepsilon \).
- \( d \in t(\mathcal{D}) \) iff \( d \leq \sup_{\varepsilon \in \mathcal{E}} \varepsilon \), for a finite \( \mathcal{E} \subset \mathcal{D} \).

Whenever it is clear from the context what base category is involved, we will drop the superscript \( C \) in the notations above. We capture many known topological constructs, considering the above expanders on categories \( M^C \), for different base categories \( \mathcal{C} \).

<table>
<thead>
<tr>
<th>( \xi^C_{\mathcal{U}<em>G}, \xi^C</em>{\mathcal{D}} )</th>
<th>( C^\Delta )</th>
<th>( C^{\Delta^s} )</th>
<th>( C^{\Delta^s\theta} )</th>
<th>( C^\Upsilon )</th>
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<tbody>
<tr>
<td>( \xi^C_{\mathcal{T}} )</td>
<td>Top</td>
<td>Creg</td>
<td>Creg</td>
<td>ZDim</td>
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<tr>
<td>( \xi^C_{\mathcal{A}} )</td>
<td>Ap</td>
<td>UAp</td>
<td>UAp</td>
<td>ZDAp</td>
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<tr>
<td>( \xi^C_{\mathcal{S}} )</td>
<td>qUnif</td>
<td>Unif</td>
<td>Prox</td>
<td>naUnif</td>
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<td>( \xi^C_{\mathcal{U}_G} )</td>
<td>qUG</td>
<td>UG</td>
<td>efGap</td>
<td>tUG</td>
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<tr>
<td>( \xi^C_{\mathcal{D}} )</td>
<td>( C^\Delta )</td>
<td>( C^{\Delta^s} )</td>
<td>( C^{\Delta^s\theta} )</td>
<td>( C^\Upsilon )</td>
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</table>

**Top**, **Creg** and **ZDim** consist of all topological spaces, of all completely regular and of all zero dimensional topological spaces respectively, with continuous maps as morphisms.

**Ap** and **UAp** consist of all approach spaces and uniform approach spaces in the sense of [13], with contractions as morphisms. **ZDAp** is the full subconstruct consisting of all zero dimensional approach spaces. These are approach spaces with a gauge basis consisting of ultrametrics or could be equivalently defined as those approach spaces that are subspaces of products in **Ap** of ultrametric spaces.

**qUnif** consists of all quasi-uniform spaces [12], **Unif** of all uniform spaces, with uniformly continuous maps as morphisms, **Prox** of all proximity spaces and proximally continuous maps [17] and **naUnif** is the full subconstruct of **Unif** consisting of all non-Archimedean uniform spaces in the sense of [16].

**qUG** consists of all quasi-uniform gauge spaces [7], **UG** of all uniform gauge spaces [14], with uniform contractions, **efGap** of all Effremovic-gap
spaces in the sense of [10] with associated maps and $\mathsf{tUG}$ is the full subconstruct of $\mathsf{UG}$ consisting of all transitive uniform gauge spaces.

3 Cogeneration by completely metrizable spaces

Recall that an object $(X,d)$ in $\mathcal{C}^\Delta$ is said to be bicompact if $(X,d^*)$ is complete. $(Y,q)$ is a bicompactification of a $\mathcal{C}^\Delta$-object $(X,d)$ if $(Y,q)$ is a bicompact space in which $(X,d)$ is $q^*$-densely embedded. For objects in a base category $\mathcal{C}$, we will use the following analogous definition for completeness and completion.

**Definition 3.1.**

- A $\mathcal{C}$-object $(X,d)$ is called bicompact if $(X,d^*)$ is complete.
- $(Y,q)$ is a $\mathcal{C}$-completion of a $\mathcal{C}$-object $(X,d)$ if $(Y,q)$ is a bicompactification of $(X,d)$ in $\mathcal{C}^\Delta$ and $(Y,d)$ belongs to $\mathcal{C}$.

As usual we denote by $\mathcal{X}_0$ the class of $T_0$-objects in $\mathcal{X}$ [15]. In particular $\mathcal{C}_0$ is the subconstruct of $\mathcal{C}$ consisting of its $T_0$-objects. It is well known that every $T_0$ quasi-metric space has an (up to isometry) unique $\mathcal{C}^\Delta_0$-completion. It easily follows from our assumptions on the base categories that for $(X,d)$ a $T_0$ $\mathcal{C}$-object, the $\mathcal{C}^\Delta_0$-completion of $(X,d)$ is also the unique $\mathcal{C}_0$-completion.

Recall from [4] that a (complete) construct is said to be Emb-cogenerated by a subclass $\mathcal{P}$ if every object is embedded in a product of $\mathcal{P}$-objects.

**Proposition 3.2.** Assume $\mathcal{C}$ is a base category and let $\xi$ be an expander on $\mathcal{M}_C^\mathcal{C}$. Let

$$\mathcal{P} = \{(Z,\xi(\{e\}^\downarrow)) : (Z,e) \text{ is a bicompact } \mathcal{C}_0-\text{space}\}$$

Then $\mathcal{P}$ is an Emb-cogenerating class for $(\mathcal{M}_\xi^\mathcal{C})_0$.

**Proof.** Case 1) of the proof deals with the expander $\mathcal{C}^\mathcal{C}$. Let $(X,D)$ be an arbitrary $(\mathcal{M}_\mathcal{C}^\mathcal{C})_0$-object, with a base $Q$ of $\mathcal{C}$-metrics. Note that the source

$$(1_X : (X,D) \rightarrow (X,\{q\}^\downarrow))_{q \in Q}$$
is initial in $\mathbf{M}^C_\mathcal{C}$. Recall that the $T_0$-quotient reflection of a quasi-metric space $(X, d)$ is given by the morphism

$$\tau_d : (X, d) \rightarrow (X, \overline{d}) : x \mapsto \overline{x}$$

where $\overline{x} = \{y \in X \mid d(x, y) = d(y, x) = 0\}$, $X_d = \{\overline{x} \mid x \in X\}$ and $\overline{d}(\overline{x}, \overline{y}) = d(x, y)$ for $x, y \in X$. Using the standing assumptions on $\mathcal{C}$, the $T_0$-reflection of a $\mathcal{C}$-object is obtained in the same way as in $\mathcal{C}^\Delta$. The reflection morphism $\tau_q : (X, q) \rightarrow (X, \overline{q}) : x \mapsto \overline{x}$ is initial, which implies that also the source

$$\left(\tau_q : (X, D) \rightarrow (X, \{\overline{q}\})\right)_{q \in Q}$$

is initial in $\mathbf{M}^C_\mathcal{C}$. By our standing assumptions on $\mathcal{C}$, for each $q \in Q$, one can consider the $C_0$-completion $(\overline{X}_q, \overline{\xi})$ of the space $(X, q)$. So, for every $q \in Q$, the map $k_q : (X_q, \overline{q}) \rightarrow (\overline{X}_q, \overline{\xi})$ is initial in $\mathcal{C}$. It follows that the contraction $k_q : (X_q, \{\overline{q}\}) \rightarrow (\overline{X}_q, \{\overline{\xi}\})$ is initial in $\mathbf{M}^C_\mathcal{C}$. Finally one obtains the following initial source in $\mathbf{M}^C_\mathcal{C}$:

$$\left(k_q \circ \tau_q : (X, D) \rightarrow (\overline{X}_q, \{\overline{\xi}\})\right)_{q \in Q}$$

Due to the $T_0$ property of $(X, D)$, which means that for any $x, y \in X, x \neq y$, there exists $d \in \mathcal{M} : d(x, y) \neq 0$ or $d(y, x) \neq 0$, this source turns out to be point-separating. Moreover for every $q \in Q$, the $\mathcal{C}$-space $(\overline{X}_q, \{\overline{\xi}\})$ is a $\mathcal{P}$-object.

For case 2) of the proof, let $(X, D)$ be an arbitrary $(\mathbf{M}^C_\mathcal{C})_0$-object. It suffices to apply the coreflector $\xi : \mathbf{M}^C_\mathcal{C} \rightarrow \mathbf{M}^\xi_\mathcal{C} : (Y, \xi(G)) \mapsto (Y, \xi(\xi(G)))$ to the source $(k_q \circ \tau_q)_{q \in Q}$.

We capture some well known results like $\text{Unif}_0$ being Emb-cogenerated by the class

$$\{(Z, \mathcal{U}_d) \mid d \text{ a complete Hausdorff metric on } Z\}$$

and the construct $\text{UAp}_0$ being Emb-cogenerated by the class

$$\{(Z, \delta_d) \mid d \text{ a complete Hausdorff metric on } Z\}.$$
The previous theorem implies analogous results for all the constructs in table of section 2. Note that \( \text{Top}_0 \) and \( \text{Ap}_0 \) are cogenerated by a single object. \( \text{Top}_0 \) is Emb-cogenerated by the Sierpinski space \( S_2 \) which is quasi-metrizable by a \( T_0 \) bicomplete quasi-metric. \( \text{Ap}_0 \) is cogenerated by the object \( P \). This object \( P \) however is not (bicompletely) quasi-metrizable. We will come back to these examples in section 5.

4 Construction of complete objects from completely metrizable spaces

In this section we tackle our main problem. We will endow \( (M^\xi)_0 \) with a closure operator \( s \) and we will consider the class \( \mathcal{U}_s \) of all \( s \)-dense embeddings. The following two questions will be investigated:
1) Are the completely metrizable objects \( \mathcal{U}_s \)-injective?
2) Is the class of all \( s \)-closed subspaces of products of completely metrizable objects firmly \( \mathcal{U}_s \)-reflective?

For explicit definitions on firmness we refer to [4] and [3]. Here we briefly recall that, given a class \( \mathcal{U} \) of \( X \)-morphisms, a reflective subconstruct with reflector \( R \) is said to be subfirmly \( \mathcal{U} \)-reflective if it is \( \mathcal{U} \)-reflective and if for every morphism \( u \) in \( \mathcal{U} \) the reflection \( R(u) \) is an isomorphism. If \( \mathcal{U} \) coincides with the class of morphisms for which \( R(u) \) is an isomorphism, the subconstruct is said to be firmly \( \mathcal{U} \)-reflective. Among other things \( \mathcal{U} \)-firmness implies uniqueness of completion with respect to the class \( \mathcal{U} \).

Since the class \( \mathcal{U}_s \) we will be dealing with consists of certain embeddings, \( \mathcal{U}_s \)-firmness will imply that \( \mathcal{U}_s \) is contained in the class of all epimorphic embeddings. In all the examples in section 6. we will be dealing with closure operators on \( (M^\xi)_0 \) that are (pointwise) smaller than the regular closure operator \( r \), describing the epimorphisms. In order to satisfy the standing assumptions on stability of \( \mathcal{U} \) with respect to compositions, as put forward in [3], we will assume that the closure operator \( s \) is idempotent. The class of \( \mathcal{U}_s \)-injective objects is denoted by \( \text{Inj}\mathcal{U}_s \). The proof of the next result uses standard techniques, see for instance [4].

**Proposition 4.1.** If \( s \) is a weakly hereditary, idempotent closure operator on \( X \), then \( \text{Inj}\mathcal{U}_s \) is closed for taking \( s \)-closed subspaces of products in \( (M^\xi)_0 \).
In [5] the closure operator $r$ has been explicitely formulated in the following way. For an $(M^C_{\xi})_0$-object $(X, \mathcal{D})$

$$x \in r_X(M) \iff \forall \ d \in \mathcal{D} : \inf_{m \in M} d(x,m) + d(m,x) = 0.$$

The closure operator $r$ is known to be idempotent and was shown to be hereditary on $(M^C_{\xi})_0$ for all the expanders listed in section 2, i.e. for arbitrary $C$ in cases where $\xi$ equals any of the expanders $1^C, \xi_{U^C}, \xi_{U^G}$ or $\xi_{D^C}$, and for $C \subset C^\Delta$ and $C^\Delta$ in the cases $\xi_{F^\Delta}, \xi_{A^\Delta}$.

**Theorem 4.2.** Assume $C$ is a base category and let $\xi$ be an expander on $M^C$. On $(M^C_{\xi})_0$ let $s$ be a weakly hereditary, idempotent closure operator and let $\mathcal{U}_s$ be the class of all $s$-dense embeddings in $(M^C_{\xi})_0$.

The following are equivalent:

1. For every $j : (X, \mathcal{H}) \rightarrow (Y, \mathcal{D})$ with $j \in \mathcal{U}_s$:
   $$j \in \mathcal{U}_r \text{ and } \mathcal{H} = \mathcal{D} \circ j \times j \downarrow$$

2. The class $\mathcal{D} = \{(Z, \xi(\{e\} \downarrow)) : (Z, e) \text{ is a bicomplete } C_0\text{-object}\}$ is $\mathcal{U}_s$-injective in $(M^C_{\xi})_0$ and $\mathcal{U}_s \subset \mathcal{U}_r$;

3. The class $\mathcal{R}_s$ of $s$-closed subobjects of products of $\mathcal{P}$-objects is a sub-firm $\mathcal{U}_s$-reflective subcategory of $(M^C_{\xi})_0$.

**Proof.** To prove that 1. implies 2. let $(Z, \xi(\{e\} \downarrow))$ be an arbitrary $\mathcal{P}$-object, $j : (X, \mathcal{H}) \rightarrow (Y, \mathcal{D})$ belong to $\mathcal{U}_s$ and $f : (X, \mathcal{H}) \rightarrow (Z, \xi(\{e\} \downarrow))$ be a contraction in $M^C_{\xi}$. Since $e \circ f \times f$ belongs to $\mathcal{H}$ and since by 1. $\mathcal{H} = \mathcal{D} \circ j \times j \downarrow$, we can choose a $C$-metric $d \in \mathcal{D}$ such that $e \circ f \times f \leq d \circ j \times j$. Consider the following situation in $C^\Delta$. The map $j : (X, d \circ j \times j) \rightarrow (Y, d)$ is a $d^*\text{-dense embedding}$ and $f : (X, d \circ j \times j) \rightarrow (Z, e)$ is a contraction. Since $(Z, e)$ is bicomplete, it is injective in $C^\Delta$ with respect to $r\text{-dense embeddings}$, and hence there is a contraction $\tilde{f} : (Y, d) \rightarrow (Z, e)$ such that $\tilde{f} \circ j = f$. Clearly $\tilde{f} : (Y, \mathcal{D}) \rightarrow (Z, \xi(\{e\} \downarrow))$ is a contraction in $M^C$ and since $(Y, \mathcal{D})$ belongs to $M^C_{\xi}$ the map $f : (Y, \mathcal{D}) \rightarrow (Z, \xi(\{e\} \downarrow))$ is a contraction in $M^C_{\xi}$.
To prove that 2. implies 3., we follow the lines of proof of theorem 1.6 in [4]. First note that by 3. $\mathcal{P} \subseteq \text{Inj} \mathcal{U}_s$. Hence, from proposition 4.1 we have that $\mathcal{R}_s \subseteq \text{Inj} \mathcal{U}_s$. Next we show that $\mathcal{R}_s$ is a $\mathcal{U}_s$-reflective subconstruct.

Let $X$ be an arbitrary $(\mathcal{M}_\xi^C)_0$-object. Proposition 3.2 ensures that there exist objects $P_i \in \mathcal{P} (i \in I)$ such that we have an embedding $j : X \hookrightarrow \prod_{i \in I} P_i$. Consider its $(\mathcal{E}^s, \mathcal{M}^s)$-factorization $j = m \circ e$ where $X \xrightarrow{e} \mathcal{M} \xrightarrow{m} \prod_{i \in I} P_i$, with $e \in \mathcal{E}^s$ and $m \in \mathcal{M}^s$. Since $j$ is an embedding, so is $e$. So we get that $e \in \mathcal{U}_s$ and $\mathcal{M} \in \mathcal{R}_s$.

For $Y \in \mathcal{R}_s$ and $f : X \rightarrow Y$ an arbitrary contraction, using the $\mathcal{U}_s$-injectivity of $Y$, we can construct a contraction $f^*$ such that $f^* \circ e = f$ which is unique by the fact that $e$ is an epimorphism.

Moreover, $\mathcal{R}_s$ is subfirmly $\mathcal{U}_s$-reflective. For $(\mathcal{M}_\xi^C)_0$-objects $X$ and $Z$ suppose $g : X \rightarrow Z$ belongs to $\mathcal{U}_s$. Denote by $r_Z : Z \rightarrow RZ$ and $r_X : X \rightarrow RX$ the $\mathcal{R}_s$-reflection morphisms. Using the $\mathcal{U}_s$-injectivity of $RX$ and the fact that $g$, $r_Z$ and $r_X$ belong to $\mathcal{U}_s$, we can conclude that there exists a contraction $h : RZ \rightarrow RX$ such that $h$ and $Rg$ are each others inverses. Finally $Rg$ is an isomorphism.

To prove that 3. implies 1. suppose $\mathcal{R}_s$ is subfirmly $\mathcal{U}_s$-reflective. Then the results in [3] already imply that $\mathcal{R}_s = \text{Inj} \mathcal{U}_s$ and that $\mathcal{U}_s \subseteq \mathcal{U}_r$.

Let $j : (X, \mathcal{H}) \rightarrow (Y, \mathcal{D})$ belong to $\mathcal{U}_s$ and consider an arbitrary $\mathcal{C}$-metric $e \in \mathcal{H}$. Then, as in the proof of proposition 3.2, the map

$$\alpha_e : (X, \mathcal{H}) \rightarrow (\widehat{X}_e, \widehat{\mathcal{H}}) : x \mapsto \bar{x}$$

is a contraction in $\mathcal{M}^C$ and therefore $\alpha_e : (X, \mathcal{H}) \rightarrow (\widehat{X}_e, \xi(\{\widehat{e}\} \downarrow))$ is a contraction in $\mathcal{M}_\xi^C$. Since $(\widehat{X}_e, \xi(\{\widehat{e}\} \downarrow))$ is $\mathcal{U}_s$-injective, there exists a contraction $\widetilde{\alpha}_e : (Y, \mathcal{D}) \rightarrow (\widehat{X}_e, \xi(\{\widehat{e}\} \downarrow))$, such that $\widetilde{\alpha}_e \circ j = \alpha_e$. Composing $\alpha_e$ with the $\mathcal{M}^C$-morphism

$$j' : (X, \mathcal{D} \circ j \times j \downarrow) \rightarrow (Y, \mathcal{D}) : x \mapsto j(x)$$

we get that

$$\widetilde{\alpha}_e \circ j' : (X, \mathcal{D} \circ j \times j \downarrow) \rightarrow (\widehat{X}_e, \xi(\{\widehat{e}\} \downarrow))$$

is a morphism in $\mathcal{M}^C$. Consequently: $e = \widehat{e} \circ (\alpha_e \circ j') \times (\widetilde{\alpha}_e \circ j')$ belongs to $\mathcal{D} \circ j \times j \downarrow$. □
If moreover we assume the closure operator $s$ to be hereditary, we can strengthen 3. in the equivalences of theorem 4.2.

**Corollary 4.3.** Assume $C$ is a base category and let $\xi$ be any expander on $M^C$. On $(M^C_\xi)_0$ let $s$ be a hereditary, idempotent closure operator and let $\mathcal{U}_s$ be the class of all $s$-dense embeddings in $(M^C_\xi)_0$.

The following are equivalent:

1. For every $j: (X, \mathcal{H}) \rightarrow (Y, \mathcal{D})$ with $j \in \mathcal{U}_s$:
   \[ j \in \mathcal{U}_r \text{ and } \mathcal{H} = \mathcal{D} \circ j \times j \downarrow \]

2. $\mathcal{P} = \{(Z, \xi(\{e\})_1) : (Z, e) \text{ is a bicomplete } C_0\text{-object}\}$ is $\mathcal{U}_s$-injective in $(M^C_\xi)_0$ and $\mathcal{U}_s \subset \mathcal{U}_r$;

3. The class $\mathcal{R}_s$ of $s$-closed subobjects of products of $\mathcal{P}$-objects is a firm $\mathcal{U}_s$-reflective subcategory of $(M^C_\xi)_0$.

**Proof.** The only non-trivial implication is 2. implies 3. In view of the fact that by theorem 4.2 the class $\mathcal{R}_s$ is already subfirmly $\mathcal{U}_s$-reflective, it is sufficient to show that $\mathcal{U}_s$ is coessential [3]. Suppose both $u$ and $u \circ f$ belong to $\mathcal{U}_s$ then clearly $f$ is an embedding. The hereditariness of $s$ and the fact that $u \circ f$ is $s$-dense imply that $f$ is $s$-dense.

\[ \square \]

5 Examples

Remark that if one of the equivalent claims of propositions 4.2 or 4.3 holds for the regular closure operator $r$ of $(M^C_\xi)_0$, then it also holds for every idempotent, (weakly) hereditary closure $s$ on $(M^C_\xi)_0$ with $s \leq r$. For this reason we start investigating concrete situations of categories endowed with the regular closure $r$. 

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5.1 \( \mathcal{U}_r \)-firmly reflective subconstructs: the case of the expanders \( \xi \) equal to \( 1^C, \xi_U^C, \xi_U^G \) or \( \xi_D^C \)

Let \( C \) be any base category. As was shown in [5] the regular closure \( r \) on \( (M_\xi)^0 \), built with the expanders listed above, is idempotent and hereditary. We will show that the first claim in 4.3 (and thus also property 2. and 3.) holds.

**Proposition 5.1.** For any expander listed in the subtitle 6.1., let 

\[ j : (X, \mathcal{H}) \rightarrow (Y, \mathcal{D}) \]

a morphism in \( (M_\xi)^0 \) such that \( j \in \mathcal{U}_r \), then we have 

\[ \mathcal{H} = \mathcal{D} \circ j \times j \downarrow. \]

**Proof.** Remark that the proof of the statement for the expanders \( \xi_U^C \) and \( 1^C \) is based on the fact that in both cases subobjects in \( M_\xi^C \) coincide with subobjects in \( M^C \).

We give an explicit proof for the case \( \xi \) equal to \( \xi_U^C \). The remaining case where \( \xi \) equals \( \xi_U^G \) will follow from it, since \( M_\xi^C \) is a bireflective subconstruct of \( M_\xi^U \). Let \( j : (X, \mathcal{H}) \rightarrow (Y, \mathcal{D}) \) a morphism in \( (M_\xi^C)^0 \), and suppose \( j \in \mathcal{U}_r \). First apply the symmetrizer in the sense of [5] to \( (X, \mathcal{H}), (Y, \mathcal{D}) \) and to \( j \). It is a coreflector in this case. Then compose it with the restriction of the uniform coreflector. Using isomorphic descriptions of the objects we denote \( \mathcal{U}(\mathcal{H}^*) \) and \( \mathcal{U}(\mathcal{D}^*) \) for the objects obtained and again \( j : (X, \mathcal{U}(\mathcal{H}^*)) \rightarrow (Y, \mathcal{U}(\mathcal{D}^*)) \) for the image through the composed functor. \( j \) now is a dense embedding in \( \text{Unif}_0 \).

Let \( e \in \mathcal{H} \) be an arbitrary \( C \)-metric. Then \( e \) is uniformly continuous on \( X \times X \) endowed with the product of the uniformities \( \mathcal{U}(\mathcal{H}^*) \). In view of the density assumption, there is a unique uniformly continuous quasimetric \( g \) on \( Y \times Y \) endowed with the product structure of \( \mathcal{U}(\mathcal{D}^*) \) and satisfying \( g \circ j \times j = e \).

An explicit formulation of \( g \) is given by 

\[
g : Y \times Y \rightarrow [0, \infty] : (y, y') \longmapsto \sup_{d \in \mathcal{D}, \varepsilon > 0} e(j^{-1}(B_d, (y, \varepsilon)), j^{-1}(B_d, (y', \varepsilon))).
\]

Since we have that \( j : (X, e) \hookrightarrow (Y, g) \) is an \( r \)-dense embedding in \( C^\Delta \) the quasi-metric \( g \) is a \( C \)-metric.
The only thing left to prove is that $g$ belongs to $\mathcal{D}$.

Let $\varepsilon > 0$ and $\omega < \infty$ be arbitrary. Since $\mathcal{H} = \xi_{\text{UBG}}(\mathcal{D} \circ j \times j)$ there exists a $C$-metric $d \in \mathcal{D}$ such that $e(z, w) \wedge \omega \leq d \circ j \times j(z, w) + \frac{\varepsilon}{3}$ for every $z, w \in X$.

Take $y, y' \in Y$ arbitrarily. We will show that $g(y, y') \wedge \omega \leq d(y, y') + \varepsilon$.

Let $p \in \mathcal{D}$, $\zeta > 0$ be arbitrary. Choose $x, x' \in X$ such that $(p \vee d)^*(y, j(x)) < \zeta \wedge \frac{\varepsilon}{3}$ and $(p \vee d)^*(y', j(x')) < \zeta \wedge \frac{\varepsilon}{3}$. Then we have

$$e(j^{-1}(B_{p^*}(y, \zeta)), j^{-1}(B_{p^*}(y', \zeta))) \wedge \omega \leq e(x, x') \wedge \omega \leq d(y, y') + \varepsilon.$$ 

\[ \blacksquare \]

The previous results imply that for a metrically generated construct $X_0$, which is one of the examples $\text{qUnif}_0, \text{Unif}_0, \text{Prox}_0, \text{naUnif}_0, \text{qUG}_0, \text{UG}_0, \text{efGap}_0, \text{tUG}_0, \mathcal{C}_0$, or $(\mathcal{M}_1^C)_0$, there exists a $U_r$-firmly reflective subcategory $\mathcal{R}_x$ of complete objects. Moreover the complete objects are “generated” by the completely metrizable objects in the construct, meaning that an object in $X_0$ is complete if and only if it is an $r$-closed subset of a product of objects in the image of the class of bicomplete $\mathcal{C}_0$-objects under the functor $K : C \rightarrow X$.

In the table below we associate to each subconstruct $\mathcal{R}_x$ in the list of examples some known subconstruct of complete objects described in the literature.

| $\mathcal{R}_x$ is generated by bicompletely metrizable objects |
|-----------------|-----------------|
| $\text{qUnif}_0$ | bicomplete $T_0$ quasi-uniform spaces |
| $\text{Unif}_0$ | complete Hausdorff uniform spaces |
| $\text{Prox}_0$ | Effremovic proximity spaces with compact Hausdorff underlying topology |
| $\text{naUnif}_0$ | complete non-Archimedian uniform spaces |
| $\text{UG}_0$ | complete $T_0$-Uniform Gauge spaces |
| $\text{efGap}_0$ | Gap-spaces with compact Hausdorff underlying topology |
| $\text{tUG}_0$ | complete transitive $T_0$-Uniform Gauge spaces |
| $\mathcal{C}_0^\Delta$ | bicomplete $T_0$ quasi-metric spaces |
| $\mathcal{C}_0^{\Delta s}$ | complete Hausdorff metric spaces |
| $\mathcal{C}_0^{\Delta \alpha}$ | compact metric spaces |
| $\mathcal{C}_0^\mu$ | complete $T_0$ ultrametric spaces |
5.2 \( U_r \)-firmly reflective subconstructs: the case of the expanders \( \xi_T^C \) and \( \xi_A^C \).

In case \( \xi \) equals \( \xi_T^C \) or \( \xi_A^C \), things do not work in the same way as in the previous examples.

We first deal with base categories \( C \) contained in \( C^{\Delta s} \) and we refer to table in section 2 for the isomorphic descriptions of the constructs.

It is well known that in \( C_{\text{reg}}^0 \) there doesn't exist a \( U_r \)-subfirm subconstruct \( R_{\xi} \). It is shown in [4] that \( C_{\text{reg}}^0 \) does not have \( U_r \)-injective objects, except for the singleton spaces. The argument uses the \( r \)-dense embedding \( j : (\mathbb{N}, T) \rightarrow (\mathbb{N}^*, T^*) \) of the discrete space of natural numbers into its Alexandroff compactification. On \( (\mathbb{N}, T) \) a two valued continuous function, which is 0 on even numbers and 1 on odd numbers, has no continuous extension to \( (\mathbb{N}^*, T^*) \). Since both \( (\mathbb{N}, T) \) and \( (\mathbb{N}^*, T^*) \) are zero dimensional, the same argument shows that in \( Z\Delta m_0 \) there cannot exist a \( U_r \)-subfirm subconstruct either. Considering \( (\mathbb{N}, T) \) and \( (\mathbb{N}^*, T^*) \) as topological approach spaces gives the same negative result for \( U\Delta p_0 \). Showing that these spaces are moreover zero dimensional approach spaces, yields that there is no \( U_r \)-subfirm subconstruct in \( Z\Delta p_0 \) either.

Next we deal with the base category \( C^{\Delta} \). The expanders \( \xi_T \) and \( \xi_A \) provide isomorphic descriptions of the constructs \( \text{Top} \) and \( \text{Ap} \) respectively. It is well known that the construct \( T\text{Sob} \) of sober topological spaces is a \( U_r \)-firmly reflective subconstruct of \( \text{Top}_0 \). However \( T\text{Sob} \) is not generated by bicompletely quasi-metrizable objects. In fact for the class

\[ \mathcal{P} = \{(Z, T_e) \mid e \ T_0 \text{ bicomplete quasi-metric}\} \]

we have that \( \mathcal{P} \not\subseteq T\text{Sob} \).

In order to illustrate this, consider the quasi-metric \( e \) on \( \mathbb{N} \) given by \( e(n, m) = 0 \) and \( e(m, n) = \infty \) if \( n < m \). Note that \( e \) is a \( T_0 \) quasi-metric such that \( e^* \) is discrete and therefore complete. For \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) we have \( B_{e}(n, \varepsilon) = \{n, n+1, \ldots\} \). It now easily follows that \( \mathbb{N} \) is irreducible and that it can't be written as the closure of a singleton.

An analogous situation appears in \( \text{Ap}_0 \). In [11] it was shown that the construct \( A\text{Sob} \) of sober approach spaces is \( U_r \)-firm in \( \text{Ap}_0 \). Again

\[ \mathcal{P} = \{(Z, \delta_e) \mid e \ T_0 \text{ bicomplete quasi-metric}\} \not\subseteq A\text{Sob} \]
and by corollary 4.3 this implies that $\text{ASob}$ is not generated by bicompletely quasi-metrizable objects. Indeed, consider the same bicomplete $T_0$ quasi-metric space $(\mathbb{N}, e)$ as in the previous argument. The fact that $(\mathbb{N}, T_e)$ is not sober as a topological space, implies that $(\mathbb{N}, S_e)$ is not sober as an approach space.

<table>
<thead>
<tr>
<th>$\mathcal{R}$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Creg}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{ZDim}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{Top}_0$</td>
<td>Sober topological spaces; not generated by completely metrizable obj.</td>
</tr>
<tr>
<td>$\text{UAp}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{ZDAp}_0$</td>
<td>non existing</td>
</tr>
<tr>
<td>$\text{Ap}_0$</td>
<td>Sober approach spaces; not generated by completely metrizable obj.</td>
</tr>
</tbody>
</table>

### 5.3 $\mathcal{U}_r$-firmly reflective subconstructs for the closure operator determined by the metric coreflection

In this section, instead of considering the closure operator $r$ we look for a natural closure operator that is smaller. For $(X, D)$ an $\left(\text{M}^C_{\xi}\right)_0$-object, and $x, y \in X$, put

$$\varphi(x, y) = \sup_{d \in D} d(x, y).$$

Then, consider the topological closure $cl^{\varphi^*}$ associated with the symmetrization $\varphi^*$. Clearly $cl^{\varphi^*}$ is an idempotent closure operator which is smaller than the regular closure $r$.

In case $\xi = \xi^{C}_{D}$, the closure $cl^{\varphi^*}$ clearly coincides with the regular closure $r$, so the completion theory coincides with the one we investigated in 6.1.

If $\xi$ equals $\xi^{C}_{U\xi}$ or $t^C$, then $cl^{\varphi^*}$ is the closure of the symmetrization of the coreflection into $C_0$ and $cl^{\varphi^*}$ can be seen to be hereditary. Since proposition 5.1 holds for $\xi_{UG}$ ($t$) and the regular closure $r$, the same is true for $cl^{\varphi^*}$. It follows that the subcategory $\mathcal{R}_{cl^{\varphi^*}}$ consisting of all $cl^{\varphi^*}$-closed subobjects of products of bicompletely metrizable objects forms a $\mathcal{U}_{cl^{\varphi^*}}$-firm subconstruct of $\left(\text{M}^C_{\xi^{C}_{UG}}\right)_0 \left(\text{M}^C_{t^C}\right)_0$. Via the expander $\xi^{C}_{UG}$ we get isomorphic descriptions of $qU\xi_0$, $UG_0$, $efGap_0$, and $tUG_0$ for which the $\mathcal{U}_{cl^{\varphi^*}}$-completion theory was not yet considered in the literature.
Note that if $\xi$ equals $\xi_F$ or $\xi_C$, then $cl^{\theta^*}$ is the discrete closure and so the $cl^{\theta^*}$-dense embeddings coincide with the isomorphisms in $(M^\xi_0)$. So the completion theory with respect to $U_{cl^{\theta^*}}$ becomes trivial in these constructs. For example, in $\text{Top}_0$, $\text{Creg}_0$, $Z\text{Dim}_0$, $q\text{Unif}_0$, $\text{Unif}_0$, $\text{Prox}_0$ and $\text{naUnif}_0$, all objects are $U_{cl^{\theta^*}}$-complete.

If $\xi$ equals $\xi_A$ then $cl^{\theta^*}$ is the closure of the symmetrization of the coreflection into $G_0$ and $cl^{\theta^*}$ is hereditary. We consider the constructs $U\text{Ap}_0$, $Z\text{DAp}_0$ for which the completion theory with respect to the regular closure failed and $\text{Ap}_0$ for which the firm $U_r$-reflective subconstruct $\text{ASob}$ is not generated by bicompletely metrizable objects. The subconstruct $cU\text{Ap}_0$ consisting of complete objects in $U\text{Ap}_0$, as introduced in [13], is firm with respect to $U_{cl^{\theta^*}}$, as can be deduced from the result on uniqueness of completion there. Moreover it also follows from [13] that the completely metrizable objects are $U_{cl^{\theta^*}}$-injective. So by corollary 4.3 we can conclude that the objects in $cU\text{Ap}_0$ are $cl^{\theta^*}$-closed subobjects of products of complete metric approach spaces. Similar results can easily be obtained for the objects in $cZ\text{DAp}_0$, the construct of all complete zero dimensional approach spaces.

In [2] a bicompletion theory for $\text{Ap}_0$ was developed. A subconstruct $\text{bicAp}_0$ of so called bicomplete approach spaces was constructed which was shown to be $U_{cl^{\theta^*}}$-firm and the bicomplete quasi-metric spaces were shown to be $U_{cl^{\theta^*}}$-injective. Again this yields the conclusion that the objects in $\text{bicAp}_0$ are generated by bicomplete quasi-metric spaces.

<table>
<thead>
<tr>
<th>$\mathcal{R}_{cl^{\theta^*}}$ is generated by bicompletely metrizable objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U\text{Ap}_0$</td>
</tr>
<tr>
<td>$Z\text{DAp}_0$</td>
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<td>$\text{Ap}_0$</td>
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References


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