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A NOTE ON KIEBOOM'S PULLBACK THEOREM FOR COFIBRATIONS

by Afework SOLOMON

RESUME Le but de cet article est de montrer que le théorème de Kieboom sur les produits fibrés pour les cofibrations (Kieboom's Pullback Theorem for Cofibrations [8]) a de nombreuses applications et généralisations de certains résultats classiques bien connus de théorie homotopique. Kieboom a montré que le théorème sur les produits fibrés de Strom (Strøm's Pullback Theorem [14, Theorem 12], [2, Corollary 3]) est un cas particulier de son théorème et il a donné beaucoup d'applications de son théorème aux espaces localement équiconnexes.

Dans cet article, nous donnons des applications plus importantes du théorème de Kieboom et nous montrons qu'une version du théorème principal de l'article de Kieboom [9] est en fait une conséquence de son théorème sur les produits fibrés, montrant ainsi que la plupart des résultats bien connus de Strom sont en fait des cas particuliers du théorème sur les produits fibrés.

Throughout this paper we work in the category Top of Topological Spaces and continuous maps. See [5] and [7] for the concepts of cofibrations, fibrations and cofibrations over a space B. The author wishes to thank Peter Booth for his useful suggestions as well as continuous encouragement during this work. Information regarding Lemma 1 was obtained from Peter Booth.
Next, we'll state Kieboom's "Pullback Theorem for Cofibrations" for the ensuing remarks and examples.

Theorem 1. *Consider the following commutative diagram in Top:*

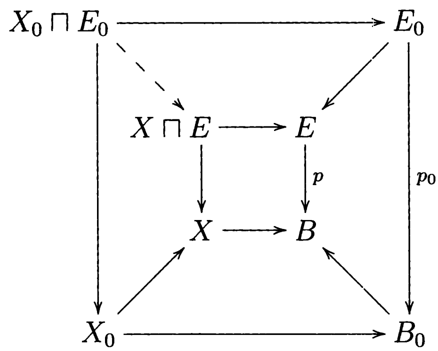


fig. 1

in which the inclusions $B_0 \to B$, $E_0 \to E$ and $X_0 \to X$ are closesd cofibrations and if $p : E \to B$ and $p_0 : E_0 \to B_0$ are fibrations, then the inclusion $X_0 \sqcap E_0 \to X \sqcap E$ is also a closed cofibration. See [10] for the proof.(Note: $X \sqcap E$ denotes the pullback of $X \to B \leftarrow E$).

Kieboom's proof makes use of the following proposition which is contained in a paper by Heath and Kamps [5, Example 1.3].

Proposition 1. *If $i : A \to X$ is a closed cofibration in Top and if further $p_A : A \to B$ and $p_X : X \to B$ are Hurewicz fibrations, then $i : p_A \to p_X$ is a cofibration over B.*

The proof given by Heath and Kamps depends on a result of J. E . Arnold, Jr; [1, Lemma 3.8] which roughly states the following:

Proposition 2. *Let $E_0 = E_1 \cap E_2$ and assume further that (E_i, E_0) i = 1,2 are closed cofibrations and $p_i : E_i \to B$ are fibrations with $p_i = p|E_i$ (p restricted to the E_i) for i = 0,1,2, then $p : E_1 \cup E_2 \to B$ is a fibration.*

Proposition 2 follows from [9, Proposition] once we show that proposition 1 is a consequence of a more general lemma which we state below. That is, Proposition 1 can be proved with no recourse to proposition 2.

Lemma 1. *Let $f \in C_B(X,Y)$. That is, f is a morphism in Top_B (the category of topological spaces over some fixed topological space B). Suppose $p_X : X \to B$ and $p_Y : Y \to B$ are Hurewicz fibrations. Then, $M_f \to B$ is a Hurewicz fibration. (M_f is the mapping cylinder of f)*

Proof. See [11, Corollary I] □

Now with the help of Lemma 1, we can easily prove Proposition 1.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} X\times 0\cup A\times I & \xrightarrow{1_{X\times 0\cup A\times I}} & X\times 0\cup A\times I\cong M_i \\ \downarrow & \overset{R}{\nearrow} & \downarrow \\ X\times I & \xrightarrow[p_X pr_X]{} & B \end{array}$$

Since $A \hookrightarrow X$ is closed, it follows that $M_i \cong X\times 0\cup A\times I$ and by Lemma 1, it follows that $X\times 0\cup A\times I \to B$ is a fibration. Now, by Strøm's relative lifting theorem, [13, Theorem 4] $\exists\ R : X\times I \to X\times 0\cup A\times I$ a retraction making the triangles above commute. Therefore, $i : p_A \to p_X$ is a cofibration over B. □

We remark that the closedness condition in Theorem 1 cannot be removed under the conditions stated as we have used the relative lifting theorem of Strøm [13, Theorem 4] in our proof of Proposition 1 and it was shown by Kiebom [8,remark 1] that Strøm's Theorem no longer holds when fibration is replaced by regular fibration and closed cofibration replaced by weak cofibration. We observe furthermore, that Furey and Heath [4, Proposition 1.7] have given a proof of the above theorem under a different set of conditions. As remarked by Kieboom, Theorem 1 generalizes the result of Furey and Heath. We also note that the proof given by Furey and Heath does not depend on the relative lifting theorem of Strøm and so the closedness condition can be removed in their proof if we replace the fibrations by regular fibrations and use the well known result of Kieboom "The pullback of a cofibration along a regular fibration is a cofibration [8, Theorem 2.1]. We now state an improved version of the result of Furey and Heath.

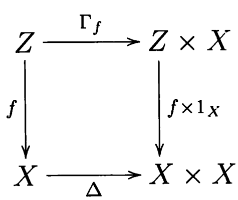
Theorem 2. *Consider fig.(i) of Theorem 1. in which the inclusions* $B_0 \to B$, $E_0 \to E$ *and* $X_0 \to X$ *are cofibrations and* $p : E \to B$, $p_0 : E_0 \to B_0$, $f : X \to B$ *and* $f_0 : X_0 \to B_0$ *are regular fibrations, then the inclusion* $X_0 \sqcap E_0 \to X \sqcap E$ *is also a cofibration. (Note:* $X \sqcap E$ *denotes the pullback of* $X \to B \leftarrow E$*).*

We will now give some examples of well known results in homotopy theory that are direct consequences of Kieboom's Pullback Theorem for cofibrations.

Definition 1. *A space* X *is Locally Equiconnected (LEC) if the inclusion of the diagonal* ΔX *in* $X \times X$ *is a cofibration. Equivalently, if the diagonal map* $\Delta : X \to X \times X$ *is a cofibration.See [6] and [3] for the definition.*

Application(i): A space X is LEC(locally–equiconnected) iff for all continuous maps $f : Z \to X$, the graph of f, $G_f = \{(z, f(z)) | z \in Z\} \subseteq Z \times X$ is a cofibration. See[6, Proposition] for definitions on LEC and the original proof of the example. We will now show that this theorem is a consequence of Theorem 1.

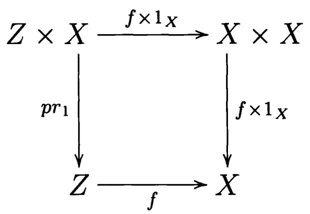
Proof. Let $\Gamma_f : Z \to Z \times X$ be defined by $\Gamma_f(\text{z}) = (z, f(z))$ and consider the following diagram below



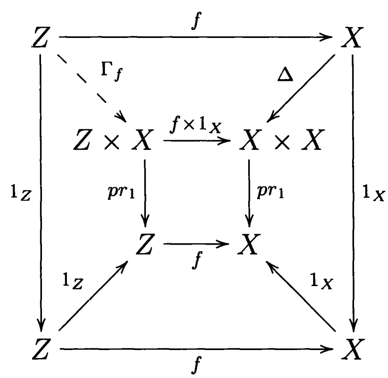
We claim that the diagram above is commutative. To see this,

$$\begin{aligned} (f \times 1_X)\ \Gamma_f\ (\text{z}) &= (f \times 1_X)\ (z, f(z)) \\ &= (f(z), f(z)) \\ &= \Delta\ (f(z)) \\ &= (\Delta f)(z) \text{ for all } z \\ &\Longrightarrow (f \times 1_X)\ \Gamma_f = \Delta f \end{aligned}$$

using basic properties of pullbacks on composite squares, it is easy to see that the diagram below



is a pullback. Finally, consider the following diagram of pullbacks:

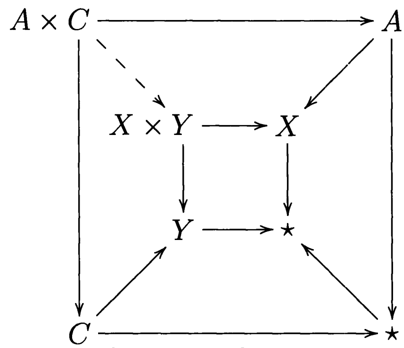


Since the inner and outer squares are pullbacks,it follows from Theorem 1 that $\Gamma_f : Z \to Z \times X$ is a cofibration.

$\square$

Application(ii): If $A \to X$ and $C \to Y$ are (closed) cofibrations, then so is $A \times C \to X \times Y$ a closed cofibration.

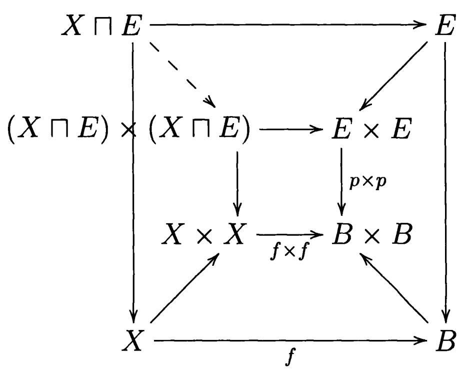
Proof. The proof follows by applying theorem 1 to the diagram below.



Note : The closedness condition can be circumvented. In the diagram above, $X \to \star$, $Y \to \star$, $A \to \star$ and $C \to \star$ are all regular fibrations and therefore, $A \times C \to X \times Y$ is a cofibration by Theorem 2.. □

Application(iii): If $p : E \to B$ is a fibration and E, B, and X are LEC, then so is $X \sqcap E$ LEC. This theorem was originally proved by Heath [6] and later noted by Kieboom as a consequence of Theorem 1. We remark that this theorem is the dual of the well known adjunction theorem of Dyer and Eilenberg (see[3, Adjunction Theorem]). A simpler proof of the Adjunction Theorem was later given by Gaunce Lewis, see[12, Theorem 2.3].

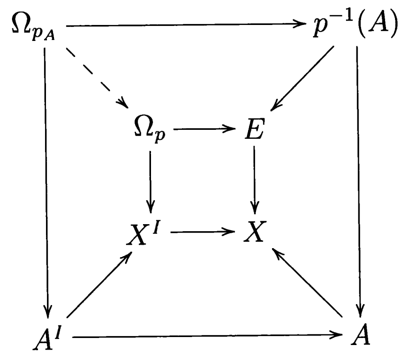
Proof. The proof actually follows from the diagram shown below:



□

Application(iv): If $p : E \longrightarrow X$ is a Hurewicz fibration denoted by $\Omega_p = \{(e,\omega) \in E \times X^I | p(e) = \omega(0)\} = E \sqcap X^I$; and if $A \subseteq X$, let p_A: $p^{-1}(A) \longrightarrow A$ and denote by $\Omega_{p_A} = \{(e,\omega) \in p^{-1}(A) \times A^I | P_A(e) = \omega(0)\}$ $= p^{-1}(A) \sqcap A^I$.
If $p : E \longrightarrow X$ is a Hurewicz fibration and $A \longrightarrow X$ is a closed cofibration, then $\Omega_{p_A} \longrightarrow \Omega_p$ is a closed cofibration.

Proof. Since the pullback of a closed cofibration along a fibration is again a closed cofibration, it follows that $p^{-1}(A) \longrightarrow E$ is a closed cofibration [14,Theorem 12]. Moreover, since I is compact, and $A \longrightarrow X$ is a cofibration, we have $A^I \longrightarrow X^I$ is a cofibration[15, Lemma 4] It is also known that $p^{-1}(A) \longrightarrow A$ is a fibration. We therefore conclude from the diagram below



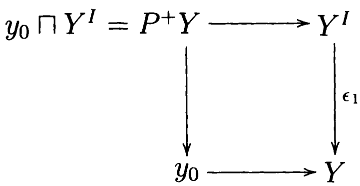
that $\Omega_{p_A} \longrightarrow \Omega_p$ is a cofibration by Theorem 1.

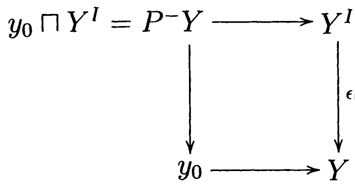
□

Application(v): Let (X, x_0) and (Y, y_0) be pointed topological spaces. We define the following two spaces:

$$P^+Y = \{\lambda \in Y^I | \lambda(1) = y_0\}$$
$$P^-Y = \{\lambda \in Y^I | \lambda(0) = y_0\}$$

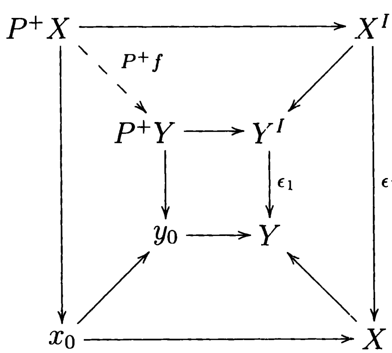
We regard P^+Y and P^-Y as subspaces of Y^I. We can also think of P^+Y and P^-Y as pullbacks of the following diagrams.





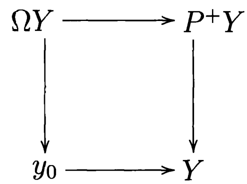
where $\epsilon_1(\lambda) = \lambda(1) = y_0$ and $\epsilon_0(\lambda) = \lambda(0) = y_0$.
Suppose $f : X \longrightarrow Y$ is a closed cofibration and $x_0 \longrightarrow X$ and $y_0 \longrightarrow Y$ are closed cofibrations, then $P^+f : P^+X \longrightarrow P^+Y$ is a closed cofibration.

Proof. Since $f : X \longrightarrow Y$ is a closed cofibration and I is compact Hausdorff, it follows that $X^I \longrightarrow Y^I$ is a closed cofibration. Clearly, $x_0 \longrightarrow y_0$ is a closed cofibration and It is well known that $\epsilon_1 : Y^I \longrightarrow Y$ and $\epsilon_1 : X^I \longrightarrow X$ are fibrations. Now apply Theorem 1 to the following diagram:



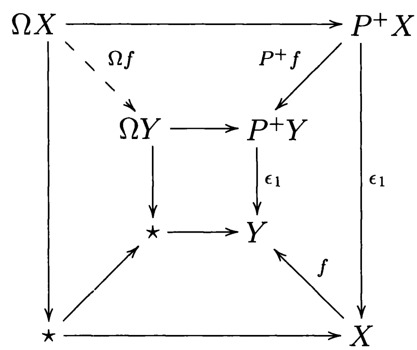
□

Application(vi): Let (X, x_0) be a based topological space such that $x_0 \longrightarrow X$ is a closed cofibration.
The space of loops $\Omega Y = \{\lambda \in Y^I \mid \lambda(0) = \lambda(1) = y_0\}$ is the pullback of the following diagram.



If $f : X \to Y$ is a closed cofibration , then $\Omega f : \Omega X \to \Omega Y$ is a closed cofibration.

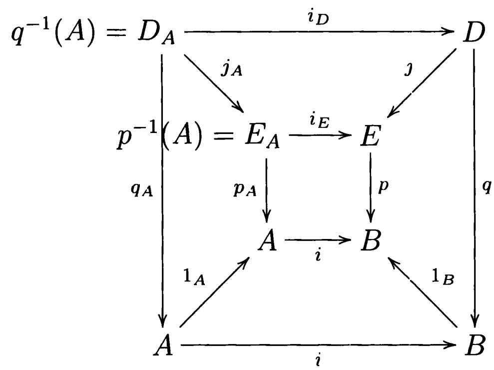
Proof. Since f is a closed cofibration, it follows from Example (vii) above that $P^{+}f : P^{+}X \to P^{+}Y$ is a closed cofibration. Moreover, $\epsilon_1 : P^{+}X \to X$ and $\epsilon_1 : P^{+}Y \to Y$ are fibrations. Hence the result follows by application of Theorem 1 to the following diagram.



□

Application(vii):In this example we will show that a version of the main theorem of a paper by Kieboom[9] is somehow a consequence of Theorem 1.

Consider the following diagram



where i, j are inclusions and i_E, i_D, j_A, p_A, q_A are induced by i, j, p, q respectively. We also denote $D_A = q^{-1}(A)$ and $E_A = p^{-1}(A)$

(a) If i, and j are closed cofibrations and p and q are fibrations, then $E_A \cup D \to E$ is a closed cofibration.

(b) If i is a cofibration, j is a closed cofibration and p, q are regular fibrations, then

$E_A \cup D \to E$ is a cofibration.

Note: Observe that we have replaced the condition j is a cofibration over B by simply requiring j to be a (closed) cofibration and adding an extra condition q a fibration in (a) and q a regular fibration in (b) in order to apply Theorem 1.(Compare with main theorem of [9])

Proof. (a) Under the stated conditions, $j_A : D_A \to E_A$ is a cofibration by Theorem 1. The remainder of the proof is exactly the same as given by Kieboom [9].

(b) Proof exactly the same as Kieboom.

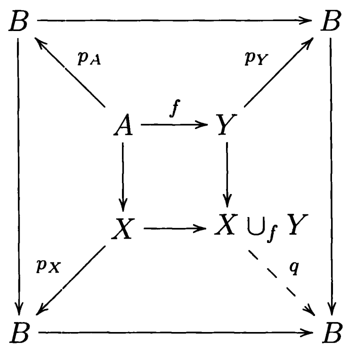
□

Application(viii): If $i : A \to X$ is a cofibration and $j : B \to Y$ is a closed cofibration, then $X \times B \cup A \times Y \to X \times Y$ is a cofibration.

Proof. Let $X = B$, $X \times Y = E$, $X \times B = D$, $A \times Y = E_A$, $A \times B = D_A$, $pr_1 = p$, and $pr_1 = q$ in Example (v) above. Hence, $X \times B \cup A \times Y \to X \times Y$ is a cofibration. □

For the remainder of our discussion, we will focus on another theorem of Kieboom which we state below:

Theorem 3. *Consider the commutative diagram in Top*



If p_A , p_X , p_Y are fibrations and if $A \to X$ is a closed cofibration, then the induced map $q : X \cup_f Y \to B$ is a fibration.

Proof. By Lemma 1 , $A \to X$ is a cofibration over B. The remainder of the proof is exactly the same as given by Kieboom. □

Note : There's no need of using Proposition 1 to show that $A \to X$ is a cofibration over B. The prooof originally given by Kieboom makes use of proposition 1 which in turn depended on proposition 2 . As explained earlier, proposition 1 is a consequence of Lemma 1, and it will soon be shown that proposition 2 follows from Theorem 3.
We will now give some applications of Theorem 3.

Application(i): **Proof of Proposition 2 :**
Take $E_1 \cap E_2 = A$, $E_1 = Y$, $E_2 = X$, $X \cup_f Y = E_1 \cup E_2$ and $q = p$ in Theorem 2.
Application(ii): Suppose $p : E \to B$ is a continuous function such that $B = B_1 \cup B_2$ and $B_0 = B_1 \cap B_2 \to B$ is a closed cofibration. Let $E_0 = p^{-1}(B_1 \cap B_2)$, $E_1 = p^{-1}(B_1)$, $E_2 = p^{-1}(B_2)$ and let $E = E_1 \cup E_2$. Suppose $p_i : E_i \to B$ for $i = 0, 1, 2$ is a fibration. Then, $p : E \to B$ is a fibration.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} E_0 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ B_1 \cap B_2 & \longrightarrow & B \end{array}$$

□

Since $B_1 \cap B_2 \to B$ is a closed cofibration and $E_2 \to B$ is a fibration, it follows that $E_0 \to E_2$ is a closed cofibration [13, Theorem 4]. Now take $E_0 = A$, $E_1 = Y$, $E_2 = X$, and $X \cup_f Y = E_0 = E_1 \cap E_2$ in Theorem 3.

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