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A UNIVERSAL CONSTRUCTION IN GOURSAT CATEGORIES

Dedicated to Jiří Adámek on the occasion of his sixtieth birthday

by Marino GRAN and Diana RODELO

Abstract

Nous démontrons que la catégorie des groupoïdes internes $\mathrm{Grd}(\mathcal{E})$ est une sous-catégorie réflexive de la catégorie $\mathrm{RGph}(\mathcal{E})$ des graphes réflexifs internes à une catégorie régulière de Goursat avec coégalisateurs: on en déduit que la catégorie $\mathrm{Grd}(\mathcal{E})$ est à son tour régulière de Goursat.

Introduction

The category $\mathrm{Grd}(\mathcal{E})$ of internal groupoids in a regular Mal'cev category $\mathcal{E}$ with coequalisers is known to be a reflective subcategory of the category of internal reflexive graphs [2]. This fact can also be seen as a consequence of the good properties of the commutator of congruences in a regular Mal'cev category: indeed, the reflection of an internal reflexive graph

$$\underline{X} = X_1 \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} X_0$$

is simply given by the quotient of X_1 by the commutator $[R[d], R[c]]$ of the kernel congruences $R[d]$ and $R[c]$ of the "domain" and the "codomain" arrows d and c.

In this note an explicit construction of the reflection is presented under the weaker assumption that $\mathcal{E}$ is a regular Goursat category with coequalisers. If regular Mal'cev categories are characterised by the 2-permutability of the composition of equivalence relations, so that $RS = SR$ for any equivalence relation R and S on an object A, regular Goursat categories satisfy the strictly weaker 3-permutability condition: $RSR = SRS$ [7]. When we started this work, we knew in advance that, in a regular Goursat category $\mathcal{E}$, the reflection could not be simply given by a regular quotient. Indeed, since the category $\mathrm{RGph}(\mathcal{E})$ of internal reflexive graphs is regular, this fact would have implied that any reflexive relation in $\mathcal{E}$ is an equivalence relation, which is one of the equivalent definitions of a Mal'cev category.

This fact partially explains why the proof is more delicate at this level of generality: it is inspired by a construction due to Bourn [4], and uses the technique of "calculus of relations" developed in [8, 7].

In the first section we recall the basic definitions and properties of equivalence relations and their composition in a regular Goursat category which will be needed in the subsequent sections. In Section 2 some important properties of internal categories and internal groupoids in a regular Goursat category are recalled. Section 3 is devoted to the main result, asserting that the category $\mathrm{Grd}(\mathcal{E})$ is reflective in $\mathrm{RGph}(\mathcal{E})$. In the last section this result is used to show that the category $\mathrm{Grd}(\mathcal{E})$ is regular Goursat whenever $\mathcal{E}$ is regular Goursat.

1 Regular Goursat categories

In this article $\mathcal{E}$ will always be a finitely complete *regular* category, that is a category endowed with a pullback-stable (regular epimorphism, monomorphism) factorization system. Regular categories provide a very natural context for working with relations since their composition exists and is associative.

A relation R from A to B is a subobject $(r_0, r_1) : R \rightarrowtail A \times B$. The opposite of R is the relation R° from B to A given by $(r_1, r_0) : R \rightarrowtail B \times A$. In particular, we can identify a morphism $f : A \to B$ with the relation $(1_A, f) : A \rightarrowtail A \times B$ and write f° for its opposite. So any relation R can be written $R = r_1 r_0^\circ$.

Remark 1.1. Consider the (regular epimorphism, monomorphism) factorization $f = i \cdot r$ of an arrow $f : A \to B$. Then:

1. $f^\circ f$ is the kernel pair of f, thus $1_A \leq f^\circ f$ and $1_A = f^\circ f$ if and only if f is a monomorphism.
2. ff° is (i, i), thus $ff^\circ \leq 1_A$ and $ff^\circ = 1_A$ if and only if f is a regular epimorphism.
3. $ff^\circ f = f$ and $f^\circ f f^\circ = f^\circ$.

Recall that an *equivalence* relation on an object A is a relation R from A to A that is *reflexive* ($1_A \leq R$), *symmetric* ($R^\circ \leq R$) and *transitive* ($RR \leq R$). An equivalence relation is called a *congruence* when it is the kernel pair of some morphism f, written $(f_0, f_1) : R[f] \rightarrowtail A \times A$. We shall usually represent a congruence as $R[r]$, where r is a regular epimorphism. A regular category is called *exact* when every equivalence relation is a congruence [1]. We denote by $\mathrm{Equiv}(A)$ the poset of equivalence relations on A and by $\mathrm{Cong}(A)$ the poset of congruences on A.

Given relations R and S from A to B, the meet of R and S, $R \wedge S$, always exists and is given by the pullback of $R \to A \times B$ along $S \to A \times B$. For arbitrary morphisms $g : X \to A$ and $h : X \to B$ we have

$$(R \wedge S)g = Rg \wedge Sg \qquad \text{and} \qquad h^{\circ}(R \wedge S) = h^{\circ}R \wedge h^{\circ}S. \tag{1}$$

Regular categories with the additional property of having a commutative composition of equivalence relations (on any fixed object) are called *regular Mal'cev* categories [8]; they have been intensively studied in the last years (see [2], and the references therein).

In this work we shall focus on the strictly weaker property of 3-permutability.

Definition 1.2. [7] *A regular category $\mathcal{E}$ is called a* regular Goursat *category when the equivalence relations in $\mathcal{E}$ are* 3*-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.*

Examples 1.3. A variety $\mathcal{V}$ of universal algebras is a Goursat category exactly when it is a 3-permutable variety: this property is known to be equivalent to a Mal'cev condition, namely the existence of two ternary terms $p(x, y, z)$ and $q(x, y, z)$ satisfying the identities $p(x, y, y) = x$, $q(x, x, y) = y$ and $p(x, x, y) = q(x, y, y)$. In particular the varieties of groups, rings, von Neumann regular rings, associative algebras, Heyting algebras and implication algebras are exact Goursat categories (see [11]).

Many interesting properties of regular Goursat categories were discovered in [8, 7, 12, 3]. In particular, the following characterization will be useful:

Theorem 1.4. *Let $\mathcal{E}$ be a regular category.*
Then the following statements are equivalent:

i. *$\mathcal{E}$ is a regular Goursat category.*

ii. *$RSR = SRS$, for all $R, S \in$* Equiv(A).

iii. *$RSR = SRS$, for all $R, S \in$* Cong(A).

iv. *$RSR \in$* Equiv(A), *thus $R \vee S = RSR$, for all $R, S \in$* Equiv(A).

When $\mathcal{E}$ is an exact Goursat category, both joins of two congruences $R[r]$ and $R[s]$ exist in Equiv(A) and in Cong(A). Consequently, given two regular epimorphisms r and s with the same domain A, their pushout

$$\begin{tikzcd} A \arrow[r, two heads, "r"] \arrow[d, two heads, "s"'] & B \arrow[d, two heads, "u"] \\ C \arrow[r, two heads, "v"'] \arrow[ur, phantom, "\ulcorner"] & D \end{tikzcd} \tag{2}$$

always exists since $R[r]R[s]R[r] = R[u \cdot r]$.

Another characterization of regular Goursat categories is given by the preservation of equivalence relations through the image by a regular epimorphism.

Theorem 1.5. [7] *A regular category $\mathcal{E}$ is a regular Goursat category if and only if for any regular epimorphism $r : A \twoheadrightarrow B$ and $S \in \mathrm{Equiv}(A)$, the image $r(S) = rSr^{\circ} \in \mathrm{Equiv}(B)$ is also an equivalence relation.*

In a regular Goursat category $\mathcal{E}$, images do not necessarily preserve congruences. That is, given regular epimorphisms r and s, the image $r(R[s]) = T$ is an equivalence relation which is not necessarily a congruence. If there exists a commutative square $u \cdot r = v \cdot s$

$$\begin{tikzcd}
R[s] \arrow[r, two heads, "\overline{r}"] \arrow[rr, bend left, "R(r)"] \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & T \arrow[r, dotted] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & R[u] \arrow[dl, shift left] \arrow[dl, shift right] \\
A \arrow[r, two heads, "r"] \arrow[d, "s"'] \arrow[dr, phantom, "A"] & B \arrow[d, "u"] & \\
C \arrow[r, two heads, "v"'] & D, &
\end{tikzcd} \tag{3}$$

then $T \leq R[u]$ and $R(r)$ is not a regular epimorphism in general. It is clear that in the exact Goursat context, we always have $T = R[u]$ for $u = \mathrm{coeq}(t_0, t_1)$ and $R(r)$ a regular epimorphism. In the non-exact regular Goursat case, we would like to know under which conditions $R(r)$ is a regular epimorphism.

Proposition 1.6. *Let $\mathcal{E}$ be a regular Goursat category and consider a commutative square of regular epimorphisms* (3)A. *The induced morphism $R(r) : R[s] \to R[u]$ is a regular epimorphism if and only if $R[r]R[s]R[r] = R[u \cdot r]$.*

Proof. We use the properties mentioned in Remark 1.1. If $R(r)$ is a regular epimorphism, then $r(R[s]) = R[u]$. So, $R[r]R[s]R[r] = r^{\circ}(rR[s]r^{\circ})r = r^{\circ}u^{\circ}ur = R[u \cdot r]$. Conversely, when $R[r]R[s]R[r] = R[u \cdot r]$, the image of $R[s]$ along r is:

$$r(R[s]) = rs^{\circ}sr^{\circ} = r(r^{\circ}rs^{\circ}sr^{\circ}r)r^{\circ} = rR[ur]r^{\circ} = rr^{\circ}u^{\circ}urr^{\circ} = R[u]. \quad \square$$

Corollary 1.7. *Consider the conditions of Proposition 1.6. Then $R(r)$ is a regular epimorphism if and only if $R(s)$ is a regular epimorphism.*

Proof. $R(s)$ is a regular epimorphism if and only if $R[v \cdot s] = R[s]R[r]R[s]$. But $v \cdot s = u \cdot r$ and the regular Goursat assumption gives

$$R[u \cdot r] = R[s]R[r]R[s] = R[r]R[s]R[r],$$

i.e. $R(r)$ is a regular epimorphism. □

We observe that the Corollary 1.7 can also be deduced from the 3-by-3 Lemma proved by Lack in [12].

Corollary 1.8. *Consider the conditions of Proposition 1.6. If the vertical morphisms are split epimorphisms and the diagram commutes with the splittings, then $R(r)$ is a regular epimorphism.*

Proof. If s and u are split epimorphisms, then $R(s)$ is also a split epimorphism. Thus, $R(r)$ is a regular epimorphism by Corollary 1.7. □

2 Internal structures

One of the well known consequences of the 3-permutability is the modularity of the lattice of equivalence relations $\mathrm{Equiv}(A)$.

Proposition 2.1. [7] *Let $\mathcal{E}$ be a regular Goursat category. If $R, S, T \in \mathrm{Equiv}(A)$ are such that $R \leq T$, then $R \vee (S \wedge T) = (R \vee S) \wedge T$.*

For a variety of universal algebras, it was shown by Gumm [11] that congruence modularity is equivalent to satisfying the so-called
Shifting Property: Let $R, S, T \in \mathrm{Equiv}(A)$ be such that $R \wedge S \leq T$. Then, from $(x, y) \in R$, $(t, z) \in R$, $(x, t) \in S$, $(y, z) \in S$ and $(x, y) \in T$

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] \arrow[d, dash, bend right=70, "T"'] & t \arrow[d, dash, "R"] \\ y \arrow[r, dash, "S"'] & z \end{tikzcd}$$

it follows that $(t, z) \in T$.

It is easy to see that any regular Goursat category is a *Gumm* category in the sense of [6]: it satisfies the categorical formulation of the Shifting Property, which can be expressed in a simple way in terms of pullbacks [5].

This implies in particular that the internal structures in a regular Goursat category have some nice properties, that we are now going to recall.

An (internal) reflexive graph (in $\mathcal{E}$) is given by a diagram

$$\underline{X} = X_1 \underset{c}{\overset{d}{\underset{\leftarrow e -}{\rightrightarrows}}} X_0$$

such that $d \cdot e = c \cdot e = 1_{X_0}$. Let p_0 and p_1 be the first and the second projections in the pullback

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] & X_1 \arrow[d, "d"] \\ X_1 \arrow[r, "c"'] & X_0. \end{tikzcd}$$

When the category $\mathcal{E}$ satisfies the Shifting Property, to have an internal category structure it suffices to have a multiplication $m : X_1 \times_{X_0} X_1 \to X_1$ such that

$$d \cdot m = d \cdot p_0 \ \text{ and } \ c \cdot m = c \cdot p_1, \tag{4}$$

$$m \cdot (e \cdot d, 1_{X_1}) = 1_{X_1} \ \text{ and } \ m \cdot (1_{X_1}, e \cdot c) = 1_{X_1}. \tag{5}$$

In other words, the usual associativity axiom follows from the other axioms. Furthermore, such a multiplication is necessarily unique, when it exists [5].

An (internal) category is a(n) (internal) groupoid when there exists an inversion morphism $i : X_1 \to X_1$ such that $d \cdot i = c$, $c \cdot i = d$, $m \cdot (i, 1_{X_1}) = e \cdot c$ and $m \cdot (1_{X_1}, i) = e \cdot d$. Among categories, groupoids are characterised by the fact that $(p_0, m) = R[d]$ or, equivalently, that $(m, p_1) = R[c]$. We write $\mathrm{RGph}(\mathcal{E})$ and $\mathrm{Grd}(\mathcal{E})$ for the categories of reflexive graphs and of groupoids in $\mathcal{E}$, respectively, and we collect the properties which will be needed later on in the following

Proposition 2.2. [5] *Let $\mathcal{E}$ be a finitely complete category satisfying the Shifting Property: in particular, $\mathcal{E}$ could be a regular Goursat category. Then*

1. *any reflexive graph in $\mathcal{E}$ equipped with a multiplication m satisfying the axioms (4) and (5) is a category in $\mathcal{E}$;*
2. *the forgetful functor $\mathrm{Grd}(\mathcal{E}) \hookrightarrow \mathrm{RGph}(\mathcal{E})$ is a full inclusion.*

3 The universal groupoid associated with a reflexive graph

In this section $\mathcal{E}$ represents a regular Goursat category with coequalisers. Given a reflexive graph $\underline{X} : X_1 \rightleftarrows X_0$ in $\mathcal{E}$, our aim is to construct the associated universal groupoid.

Let us begin with a slightly more general situation: a pair of regular epimorphisms $d : X_1 \twoheadrightarrow Y_0$ and $c : X_1 \twoheadrightarrow X_0$. We construct the following diagram

$$
\begin{tikzcd}
 & R[d_0] \arrow[r, two heads, "\overline{\delta}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, "\mu" description] \arrow[d, shift left=2, "p_1"] & R[q_0] \arrow[d, shift right=1, "\pi_0"'] \arrow[d, shift left=1, "m"] \\
 & R[d] \arrow[r, two heads, "\delta"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Q \arrow[d, shift right=2, "q_0"'] \arrow[d, shift left=2, "q_1"] \\
R[c] \arrow[r, shift left=1, "c_0"] \arrow[r, shift right=1, "c_1"'] & X_1 \arrow[u, "s_0" description] \arrow[r, two heads, "c"'] & X_0, \arrow[u, "\varepsilon" description]
\end{tikzcd}
\tag{6}
$$

where δ is the coequaliser of $(s_0 \cdot c_0, s_0 \cdot c_1)$, q_0, q_1 and ε are induced by the coequalisers δ and c and $\overline{\delta}$ is necessarily a regular epimorphism by Corollary 1.8. We obviously have a left vertical groupoid since it is the congruence $R[d]$. On the other hand, the lower right vertical diagram is a reflexive graph; we denote it by $\underline{D}$, as it is induced by $R[d]$. Note that there exists an inversion $i : R[d] \xrightarrow{\sim} R[d]$ from which we can deduce a morphism $j : Q \to Q$ on the right diagram such that $j \cdot j = 1_Q$, $j \cdot \delta = \delta \cdot i$, $q_0 \cdot j = q_1$ and $q_1 \cdot j = q_0$. Being already equipped with two morphisms π_0 and m, we would like to find a third morphism π_1 that will endow $\underline{D}$ with a category structure, which is consequently a groupoid structure with inversion j.

Lemma 3.1. *In diagram* (6) *we have* $R[p_0] \wedge R[\delta \cdot \mu] \leq R[\delta \cdot p_1]$.

Proof.

$$
\begin{array}{rcl}
R[p_0] \wedge R[\delta \cdot \mu] & = & R[d_0 \cdot p_0] \wedge R[d_1 \cdot p_0] \wedge R[\delta \cdot \mu] \\
 & = & R[d_0 \cdot p_1] \wedge R[d_0 \cdot \mu] \wedge R[\delta \cdot \mu] \\
 & \overset{(1)}{=} & R[d_0 \cdot p_1] \wedge \mu^\circ(\, R[d_0] \wedge R[\delta]\,)\mu \\
 & \leq & R[d_0 \cdot p_1] \wedge \mu^\circ(\, R[d_1](R[d_0] \wedge R[\delta])R[d_1]\,)\mu \\
 & \overset{d_1 \cdot \mu = d_1 \cdot p_1}{=} & R[d_0 \cdot p_1] \wedge p_1^\circ(\, R[d_1](R[d_0] \wedge R[\delta])R[d_1]\,)p_1 \\
 & \overset{(1), RSR = R \vee S}{=} & p_1^\circ(\, R[d_0] \wedge (\, R[d_1] \vee (R[d_0] \wedge R[\delta])\,)\,)p_1 \\
 & \overset{\text{Proposition 2.1}}{=} & p_1^\circ(\, (R[d_0] \wedge R[d_1]) \vee (R[d_0] \wedge R[\delta])\,)p_1 \\
 & = & p_1^\circ(R[d_0] \wedge R[\delta])p_1 \\
 & \leq & p_1^\circ R[\delta] p_1 \\
 & = & R[\delta \cdot p_1]
\end{array}
$$

$\square$

Proposition 3.2. *The right vertical diagram in* (6) *endows* $\underline{D}$ *with a groupoid structure.*

Proof. By the remarks above, the main difficulty in showing that $\underline{D}$ is a groupoid is to prove the existence of a morphism $\pi_1 : R\,q_0 \to Q$ such that $\delta \cdot p_1 = \pi_1 \cdot \overline{\delta}$. Then, by using $j : Q \xrightarrow{\sim} Q$, we can prove that (π_0, π_1) is the pullback of (q_0, q_1). The proof of axioms (4) and (5) are quite straightforward. Then, $\underline{D}$ is a category by Proposition 2.2 with an inversion j, thus a groupoid.

We begin by considering $w = \text{coeq}(\,(1_{R\,d}, s_0 \cdot d_1) \cdot \delta_0\,,\ (1_{R\,d}, s_0 \cdot d_1) \cdot \delta_1\,)$ and the right vertical morphisms induced from the coequalisers w and δ in

$$
\begin{tikzcd}[column sep=large, row sep=huge]
R\,\delta \arrow[r, shift left, "(1,s_0\cdot d_1)\cdot\delta_0"] \arrow[r, shift right, "(1,s_0\cdot d_1)\cdot\delta_1"'] & R\,d_0 \arrow[r, two heads, "w"] \arrow[d, shift right=3, "p_0"'] \arrow[d, "\mu"] \arrow[d, shift left=3, "p_1"] & W \arrow[d, shift right=3, "v_0"'] \arrow[d, shift left=1.5, "u"] \arrow[d, shift left=3, "v_1"] \\
& R\,d \arrow[u, shift right=1.5, "(1,s_0\cdot d_1)"] \arrow[r, two heads, "\delta"'] & Q. \arrow[u, shift right=1, "\sigma_0"]
\end{tikzcd}
$$

¿From the commutative downward squares we easily see that

$$R\,\delta \cdot p_0 = R\,w \vee R\,p_0 \quad \text{by Corollary 1.8 and Proposition 1.6} \tag{7}$$

$$R\,w \leq R\,w \vee R\,\mu = R\,\delta \cdot \mu \quad \text{by Corollary 1.8 and Proposition 1.6} \tag{8}$$

$$R\,w \leq R\,v_1 \cdot w = R\,\delta \cdot p_1 \tag{9}$$

We have

$$
\begin{array}{rcl}
R\,\overline{\delta} & = & R\,\pi_0 \cdot \overline{\delta} \wedge R\,m \cdot \overline{\delta} \\
 & = & R\,\delta \cdot p_0 \wedge R\,\delta \cdot \mu \\
 & \overset{(7)}{=} & (R\,w \vee R\,p_0\,) \wedge R\,\delta \cdot \mu \\
 & \overset{(8),\text{Proposition 2.1}}{=} & R\,w \vee (\,R\,p_0 \wedge R\,\delta \cdot \mu\,) \\
 & \overset{\text{Lemma 3.1}}{\leq} & R\,w \vee R\,\delta \cdot p_1 \\
 & \overset{(9)}{=} & R\,\delta \cdot p_1\,.
\end{array}
$$

This implies that $\delta \cdot p_1 \leq \overline{\delta}$ as regular epimorphisms, i.e. there exists a morphism $\pi_1 : R\,q_0 \to Q$ such that $\pi_1 \cdot \overline{\delta} = \delta \cdot p_1$ $\square$.

We now focus on the construction of the groupoid $\underline{D}$ when starting from an actual reflexive graph $\underline{X} : X_1 \rightleftarrows X_0$.

Proposition 3.3. *The inclusion* U : $\mathrm{Grd}(\mathcal{E}) \hookrightarrow \mathrm{RGph}(\mathcal{E})$ *admits a left adjoint functor* $F : \mathrm{RGph}(\mathcal{E}) \to \mathrm{Grd}(\mathcal{E})$ *defined by* $F(\underline{X}) = \underline{D}$ *as in diagram* (6).

Proof. By using the same notations as above, for any reflexive graph $\underline{X}$, the unit of the adjunction is given by $\eta_{\underline{X}} = (\delta \cdot (e \cdot d, 1_{X_1}), 1_{X_0})$

$$\begin{tikzcd}
\underline{X}: \arrow[d, "\eta_{\underline{X}}"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "\delta\cdot(e\cdot d, 1_{X_1})"'] & X_0 \arrow[l, "e" description] \arrow[d, equal] \\
U(F(\underline{X})): & Q \arrow[r, shift left=2, "q_0"] \arrow[r, shift right=2, "q_1"'] & X_0. \arrow[l, "\varepsilon" description]
\end{tikzcd}$$

For the universal property, let us consider an arbitrary morphism of reflexive graphs $(\alpha_1, \alpha_0) : \underline{X} \to U(\underline{X}')$. We define a morphism $(\beta_1, \alpha_0) : \underline{D} \to \underline{X}'$ of reflexive graphs (which is necessarily a morphism between the groupoids by Proposition 2.2), where β_1 is the unique morphism with the property $\beta_1 \cdot \delta = \pi_1' \cdot R(\alpha_1)$ given in

$$\begin{tikzcd}
R[c] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & X_1 \arrow[r, "s_0"] \arrow[d, "\alpha_1"'] & R[d] \arrow[r, two heads, "\delta"] \arrow[d, "R(\alpha_1)"'] & Q \arrow[d, dotted, "\beta_1"] \\
 & X_1' \arrow[r, "(1_{X_1'}, e'\cdot c')"'] & X_1' \times_{X_0'} X_1' \arrow[r, "\pi_1'"'] & X_1',
\end{tikzcd}$$

for $R(\alpha_1)$ the induced morphism from $R[d]$ to $R[d'] \cong X_1' \times_{X_0'} X_1'$. Moreover, $\beta_1 \cdot \delta \cdot (e \cdot d, 1_{X_1}) = \pi_1' \cdot R(\alpha_1) \cdot (e \cdot d, 1_{X_1}) = \pi_1' \cdot (e' \cdot d', 1_{X_1'}) \cdot \alpha_1 = \alpha_1$.
As for the uniqueness, suppose there exists another functor $(\varphi_1, \alpha_0) : \underline{D} \to \underline{X}'$ such that $\varphi_1 \cdot \delta \cdot (e \cdot d, 1_{X_1}) = \alpha_1$. The morphism $R(e \cdot d, 1_{X_1}) : R[d] \to R[d_0]$ is such that $p_1 \cdot R(e \cdot d, 1_{X_1}) = 1_{R[d]}$ and the equality $\beta_1 \cdot \delta \cdot (e \cdot d, 1_{X_1}) = \varphi_1 \cdot \delta \cdot (e \cdot d, 1_{X_1})$ implies that $\beta_1 \times_{X_0'} \beta_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1_{X_1}) = \varphi_1 \times_{X_0'} \varphi_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1_{X_1})$. Since

$$\begin{aligned}
\beta_1 \cdot \delta &= \beta_1 \cdot \delta \cdot p_1 \cdot R(e \cdot d, 1_{X_1}) \\
&= \beta_1 \cdot \pi_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1_{X_1}) \\
&= \pi_1' \cdot \beta_1 \times_{X_0'} \beta_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1_{X_1}) \\
&= \pi_1' \cdot \varphi_1 \times_{X_0'} \varphi_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1_{X_1}) \\
&= \cdots \\
&= \varphi_1 \cdot \delta
\end{aligned}$$

and δ is an epimorphism, then we get $\beta_1 = \varphi_1$. □

Remark 3.4. 1. If $\underline{X} \in \mathrm{Grd}(\mathcal{E})$, then $F(\underline{X}) \cong \underline{X}$ by Proposition 3.3.

2. Given a reflexive graph $\underline{X} : X_1 \overset{d}{\underset{c}{\rightleftarrows}} X_0$, we have $\underline{D} \cong \underline{C}$, where $\underline{C}$ denotes the groupoid induced from the kernel pair $R[c]$ of the codomain c when taking the coequaliser $\gamma = \mathrm{coeq}(s_0 \cdot d_0, s_0 \cdot d_1)$. Using the universal property of $(\delta \cdot (e \cdot d, 1_{X_1}), 1_{X_0}) : \underline{X} \to U(F(\underline{X}))$ and the morphism $(\gamma \cdot (e \cdot c, 1_{X_1}), 1_{X_0}) : \underline{X} \to U(\underline{C})$, we get a unique functor $\underline{D} \to \underline{C}$. By exchanging the roles of d and c we obtain the inverse morphism $\underline{C} \to \underline{D}$.

3. It follows from the construction of the universal groupoid associated with a reflexive graph in a regular Goursat with coequalisers category that the congruence $R[\delta]$ on $R[d]$ also determines an equivalence relation $(R(d_0), R(d_1)) : R[\delta] \rightarrowtail R[c] \times R[c]$.

$$
\begin{tikzcd}
R[\delta] \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] \arrow[d, shift right=2, "R(d_0)"'] \arrow[d, shift left=2, "R(d_1)"] & R[d] \arrow[r, two heads, "\delta"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Q \arrow[d, shift right=2, "q_0"'] \arrow[d, shift left=2, "q_1"] \\
R[c] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] \arrow[u, "R(s_0)" description] & X_1 \arrow[r, two heads, "c"'] \arrow[u, "s_0" description] & X_0. \arrow[u, "\varepsilon" description]
\end{tikzcd}
\tag{10}
$$

One then has a double equivalence relation $R[\delta]$ on $R[d]$ and $R[c]$: in universal algebra this is the well-known double congruence $\Delta_{R[d],R[c]}$ used to define the commutator $[R[d], R[c]]$ of $R[d]$ and $R[c]$ (see [11] and [13]).

4 $\mathrm{Grd}(\mathcal{E})$ is a regular Goursat category

In this section we prove that $\mathrm{Grd}(\mathcal{E})$ is a regular Goursat category for any regular Goursat category $\mathcal{E}$ with coequalisers. The crucial point is to show that $\mathrm{Grd}(\mathcal{E})$ has (regular epimorphism, monomorphism) factorizations, since $\mathrm{Grd}(\mathcal{E})$ is not closed under quotients in $\mathrm{RGph}(\mathcal{E})$ in general.

Given $(\psi_1, \psi_0) : \underline{X}' \to \underline{X} \in \mathrm{RGph}(\mathcal{E})$, for any regular category $\mathcal{E}$, it is clear that (ψ_1, ψ_0) is a regular epimorphism (a monomorphism) in $\mathrm{RGph}(\mathcal{E})$ if and only if ψ_1 and ψ_0 are regular epimorphisms (monomorphisms) in $\mathcal{E}$. If the functor $(\psi_1, \psi_0) : \underline{X}' \to \underline{X} \in \mathrm{Grd}(\mathcal{E})$ is a regular epimorphism in $\mathrm{RGph}(\mathcal{E})$, then $\psi_2 : X_1' \times_{X_0'} X_1' \to X_1 \times_{X_0} X_1$ is also a regular epimorphism in any regular Goursat category $\mathcal{E}$ by Corollary 1.8, so that (ψ_1, ψ_0) is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$. The converse statement holds in regular Mal'cev categories (Lemma 3.1 in [10]), and we shall now prove that it also holds in the regular Goursat context in Proposition 4.2 below.

Remark 4.1. For any regular Goursat category $\mathcal{E}$ with coequalisers, the functor F : $\mathrm{RGph}(\mathcal{E}) \to \mathrm{Grd}(\mathcal{E})$ preserves colimits since it is a left adjoint (Proposition 3.3). So, given a regular epimorphism $(\psi_1, \psi_0) : \underline{X}' \to \underline{X}$ in $\mathrm{RGph}(\mathcal{E})$, the image $F(\psi_1, \psi_0) = (F(\psi_1), \psi_0) : \underline{D}' \to \underline{D}$ is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$, where $F(\psi_1)$ is given by the unique morphism such that $F(\psi_1) \cdot \delta' = \delta \cdot R(\psi_1)$:

$$\begin{tikzcd}
R[c'] \arrow[r, shift left, "c'_0"] \arrow[r, shift right, "c'_1"'] \arrow[d, "R(\psi_1)"'] & X'_1 \arrow[r, "s'_0"] \arrow[d, "\psi_1"] & R[d'] \arrow[r, two heads, "\delta'"] \arrow[d, "R(\psi_1)"] & Q' \arrow[d, dotted, "F(\psi_1)"] \\
R[c] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & X_1 \arrow[r, "s_0"'] & R[d] \arrow[r, two heads, "\delta"'] & Q.
\end{tikzcd}$$

By Corollary 1.8, $R(\psi_1) : R[d'] \twoheadrightarrow R[d]$ is a regular epimorphism and, consequently, $F(\psi_1)$ is a regular epimorphism in $\mathcal{E}$.

Proposition 4.2. *Let $\mathcal{E}$ be a regular Goursat category with coequalisers and consider a functor $(\varphi_1, \varphi_0) : \underline{X}' \to \underline{X}'' \in \mathrm{Grd}(\mathcal{E})$. Then (φ_1, φ_0) is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$ if and only if it is a regular epimorphism in $\mathrm{RGph}(\mathcal{E})$.*

Proof. If (φ_1, φ_0) is a regular epimorphism in $\mathrm{RGph}(\mathcal{E})$, then it is also a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$ as already observed above.

For the converse, let (φ_1, φ_0) be a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$, and we have to prove that φ_1 and φ_0 are regular epimorphisms in $\mathcal{E}$. We begin by considering the (regular epimorphism, monomorphism) factorizations $\varphi_1 = \iota_1 \cdot \psi_1$ and $\varphi_0 = \iota_0 \cdot \psi_0$ and the induced reflexive graph $\underline{X} : X_1 \rightleftarrows X_0$ given in the following diagram

$$\begin{tikzcd}
R[\varphi_1] \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, shift left=2] \arrow[dd, shift right=2] & X'_1 \arrow[rr, "\varphi_1"] \arrow[dr, two heads, "\psi_1"] \arrow[dd, shift right=2, "d'"'] \arrow[dd, shift left=2, "c'"] & & X''_1 \arrow[dd, shift right=2, "d''"'] \arrow[dd, shift left=2, "c''"] \\
& & X_1 \arrow[ur, tail, "\iota_1"'] \arrow[dd, shift right=2, "d"'] \arrow[dd, shift left=2, "c"] & \\
R[\varphi_0] \arrow[r, shift left] \arrow[r, shift right] \arrow[uu] & X'_0 \arrow[rr, "\varphi_0"] \arrow[dr, two heads, "\psi_0"'] \arrow[uu, "e'"] & & X''_0. \arrow[uu, "e''"] \\
& & X_0 \arrow[ur, tail, "\iota_0"'] \arrow[uu, "e"] &
\end{tikzcd} \tag{11}$$

Since $\underline{X}'$ and $\underline{X}''$ are groupoids, then the images $F(\psi_1, \psi_0) : \underline{X}' \to \underline{D}$ and $F(\iota_1, \iota_0) : \underline{D} \to \underline{X}''$ (see Remark 3.4.1.) provide the commutative diagrams

$$\begin{tikzcd}
X'_1 \arrow[r, two heads, "\psi_1"] \arrow[rr, bend left, "F(\psi_1)"] \arrow[d, shift right=2] \arrow[d, shift left=2] & X_1 \arrow[r, "\delta\cdot(e\cdot d, 1_{X_1})"] \arrow[d, shift right=2] \arrow[d, shift left=2] & Q \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X'_0 \arrow[r, two heads, "\psi_0"'] \arrow[u] & X_0 \arrow[r, equal] \arrow[u] & X_0 \arrow[u]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_1 \arrow[r, two heads, "\delta\cdot(e\cdot d, 1_{X_1})"] \arrow[rr, bend left, "\iota_1"] \arrow[d, shift right=2] \arrow[d, shift left=2] & Q \arrow[r, "F(\iota_1)"] \arrow[d, shift right=2] \arrow[d, shift left=2] & X''_1 \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X_0 \arrow[r, equal] \arrow[u] & X_0 \arrow[r, tail, "\iota_0"'] \arrow[u] & X''_0. \arrow[u]
\end{tikzcd}$$

With the first diagram, we see that $\delta \cdot (e \cdot d, 1_{X_1})$ is a regular epimorphism since $F(\psi_1)$ is a regular epimorphism (Remark 4.1) and with the second diagram we see that it is also a monomorphism. Hence, $\underline{X} \cong \underline{D}$, i.e. $\underline{X}$ is actually a groupoid. To finish, we have both $(\varphi_1, \varphi_0) = (\iota_1, \iota_0) \cdot (\psi_1, \psi_0)$ and (ψ_1, ψ_0) regular epimorphisms in $\mathrm{Grd}(\mathcal{E})$, implying that $(\varphi_1, \varphi_0) \cong (\psi_1, \psi_0)$. □

Proposition 4.3. *Let $\mathcal{E}$ be a regular Goursat category with coequalisers. Then* $\mathrm{Grd}(\mathcal{E})$ *is a regular Goursat category.*

Proof. The category $\mathrm{Grd}(\mathcal{E})$ is finitely complete because the limits are computed as in $\mathcal{E}$, and this latter category is finitely complete. Moreover, regular epimorphisms in $\mathrm{Grd}(\mathcal{E})$ are stable under pullbacks since they are composed by pairs of regular epimorphisms in $\mathcal{E}$ (Proposition 4.2) which are stable under pullbacks in $\mathcal{E}$. For any functor $(\varphi_1, \varphi_0) : \underline{X'} \to \underline{X}'' \in \mathrm{Grd}(\mathcal{E})$, we can consider the diagram (11) and we see that (ψ_1, ψ_0) is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$. Hence, $\mathrm{Grd}(\mathcal{E})$ has (regular epimorphism, monomorphism) factorizations, and is a regular category.
To see that $\mathrm{Grd}(\mathcal{E})$ is a regular Goursat category it suffices to check that the image of an equivalence relation along a regular epimorphism is still an equivalence relation (see Proposition 1.5). This fact easily follows from the description of regular epimorphisms, by using the fact that the regular image of an equivalence relation in $\mathcal{E}$ is an equivalence relation in $\mathcal{E}$. □

Remark 4.4. This last result obviously implies that the category $\mathrm{Grd}^2(\mathcal{E})$ of double groupoids in a regular Goursat category $\mathcal{E}$ is again regular Goursat, since $\mathrm{Grd}^2(\mathcal{E})$ is nothing but the category of groupoids in the regular Goursat category $\mathrm{Grd}(\mathcal{E})$. By iterating this construction we conclude that the category $\mathrm{Grd}^n(\mathcal{E})$ of n-fold groupoids in $\mathcal{E}$ is regular Goursat, for any natural number $n \geq 1$.

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