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CONSTRUCTION OF AN HOMOLOGY AND A COHOMOLOGY THEORY

ASSOCIATED TO A FIRST ORDER FORMULA

by René GUITART

RESUME - On montre comment chaque formule $\phi$ d'un langage $\mathcal{L}$ détermine une théorie d'homologie (et une théorie de cohomologie) sur la catégorie des interprétations de $\mathcal{L}$, dont la valeur sur chaque interprétation $I$ de $\mathcal{L}$ est une obstruction à $I \models \phi$ "à des co-équations près" (et "à des équations près").

This paper is a sequel of [7].

1. PROPOSITION. Let $\mu(x_1,\ldots,x_n)$ be a first order formula of $\mathcal{L}$, and let $\text{Mod}_\mu \phi$ be the category with objects the models of $\phi$, and with morphisms from $M$ to $M'$ the morphisms (of models of $\phi$) $m : M \to M'$ such that

$$\forall x_1,\ldots,x_n [ \mu(m(x_1),\ldots,m(x_n)) \to \mu(x_1,\ldots,x_n) ]$$

Then there is a small mixed sketch $\sigma$ such that $\text{Mod}_\mu \phi \cong \text{Mod}_\sigma$. 

The existence of $\sigma$ is proved by the juxtaposition of proposition 3 p.8 of [6], théorème 2.1 p.26 of [5], and proposition 3 p.301 of [7],II. In fact, this juxtaposition shows more than our proposition here.

2. For $C$ a category, let $BC = |NC|$ be the geometric realization of the nerve of $C$. $BC$ is a $\text{cw}$-complex, and $\pi_1 BC \cong C(C^{-1})$ (the category of fractions of $C$). Of course if $C$ is a class, $BC$ is a class too. But, if $C = \operatorname{Mod}\phi$ for a small sketch $\sigma$, then in $BC$ we can construct a set $g\sigma$ such that the inclusion $g\sigma \longrightarrow B\operatorname{Mod}\phi$ is an equivalence of homotopy. In particular we get

PROPOSITION. $\operatorname{Mod}\phi((\operatorname{Mod}\phi)^{-1})$ is a small groupoid, up to equivalence. We call it the fundamental groupoid of $\sigma$, and we denote it by $\pi_1 g\sigma$.

The existence of the set $g\sigma$ comes from [7].

3. Let $\operatorname{Mod}_\mu \phi/1$ be the category with objects the morphisms (of interpretations of $\mathcal{L}$) $f : M \longrightarrow I$ where $M$ is a model of $\phi$, and with morphisms, from $f : M \longrightarrow I$ to $f' : M' \longrightarrow I$, the morphisms of models (morphisms of $\operatorname{Mod}_\mu \phi$) $g : M \longrightarrow M'$ such that $f'.g = f$.

Then

PROPOSITION. There is a small sketch $\sigma = \sigma(\mathcal{L}, I, \phi, \mu)$ such that $\operatorname{Mod}_\mu \phi/1 \cong \operatorname{Mod}\phi$.

So we get a small $\text{cw}$-complex $g\sigma(\mathcal{L}, I, \phi, \mu)$, which is a geometric description of the position of $I$ with respect to $\operatorname{Mod}_\mu \phi$.

4. Let $\textbf{Ab}$ be the category of small abelian groups, and let $F : \operatorname{Mod}_\mu \phi \longrightarrow \textbf{Ab}$ be a functor. (In particular $F$ could be the constant functor on a fixed abelian group $A$, or it could be a "canonical" functor if $\mathcal{L}$ is a language over the language of abelian groups, etc). The André’s homology measures "how $I$ is far from $\operatorname{Mod}_\mu \phi$, from the point of view of $F$". In order to do that we consider the chain complexe
which is

\[ \cdots \to \sum FM_2 \xrightarrow{d_2} \sum FM_1 \xrightarrow{d_1} \sum FM_0 \xrightarrow{d_0} 0 \]

with \( d_1 = s_0 - s_1 \), where

\[
\begin{align*}
\ s_0 : (FM_1)_{\alpha} : M_0 & \to I \\
\ x_0 & \to (FM_1)_{\beta'} \to I \\
\ s_1 : (FM_1)_{\alpha} : M_0 & \to I \\
\ x_1 & \to (FM_0)_{\beta} \to I
\end{align*}
\]

and so on, and we define

\[
H_0(I, F) = \ker d_0 / \text{Im } d_1 = \text{coker } d_1, \quad H_1(I, F) = \ker d_1 / \text{Im } d_2, \quad \text{and, for every } n \geq 0, \quad H_n(I, F) = \ker d_n / \text{Im } d_{n+1}.
\]

**Proposition.** \( H_n(I, F) \) is a function of \( F, I, \mu, \phi \), which in fact depends only of the homotopy type of \( \text{Mod } \mu/\phi \) and of \( F \) and could be denoted by \( H_n(\text{Mod } \mu/\phi, F) \).

Let \( \text{Int}L \) be the category of interpretations of \( L \), let \( J : \text{Mod} \phi \to \text{Int}L \) be the canonical inclusion. Then the inductive Kan extension of \( F \) along \( J \) is given by

\[
[\text{Ext}_J F](I) = \lim_{M_0 \to I} F(M_0)
\]

and we have

\[
H_0(I, F) = [\text{Ext}_J F](I).
\]

If \( I \models \phi \), then \( H_n(\text{Mod } \mu/\phi, I, F) = \left\{ \begin{array}{ll} F(I) & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{array} \right. \)

5. Now, the point is that, because of the results hereover (§§ 1 to 4), we get
PROPOSITION. The tools of [1] and of [3], available in the situation where a full and small category \( \mathcal{M} \) (called a category of "models") lives inside a big category of "spaces", are also available in the situation where a (possibly big and not necessarily full) category \( \text{Mod}_\mathcal{M} \) of models of a theory lives inside a big category of interpretations of a language \( \mathcal{L} \) (compare with the idea of "paires adéquates" p. 43 of [1]). Precisely here we get the fact that the \( H_n(\text{Mod}_\mathcal{M}/\mathcal{L},F) \) are small.

6. After the existence of \( g \) proved in [7], the theorem hereunder §9 is just a second stone for a work to be pursued. Theoretically the computation of our \( H_n \) is based on the effective construction of a "locally cofree diagram", and more precisely on the construction of a "relatively cofiltered locally cofree diagram" (r.cf.l.cf.d.) (see [5] and [6]) (in the category \( \text{Mod}_\mathcal{M} \)) generated by I. This r.cf.l.cf.d. contains all the information we need, and it will be the starting point of an absolute calculus. But for concrete situations we need a relative calculus, by the way of comparisons between various \( H_n \). For that it will be essential to go toward effective relative calculation of these small \( H_n \), and especially we need a description of the link between these calculations and the theory of demonstrations. For example we need relations among \( H_n(\text{Mod}_\mathcal{M}/\mathcal{L},F) \), \( H_n(\text{Mod}_{\mathcal{L}/\mathcal{M}}/\mu,G) \), \( H_n(\text{Mod}_{\mathcal{M}/\mathcal{L}}/\mu,\phi) \), \( H_n(\text{Mod}_{\mathcal{M}/\mathcal{L}}/\mu,\phi\land\gamma) \), \( H_n(\text{Mod}_{\mathcal{M}/\mathcal{L}}/\mu,\phi\Rightarrow\gamma) \), \( H_n(\text{Mod}_{\mathcal{M}/\mathcal{L}}/\mu,\phi) \) (for convenient \( \mu \) and \( \mathcal{L} \)).

For that it will be necessary to describe the category \( \text{For}(\mathcal{L}) \) of formulas of the language \( \mathcal{L} \). At first this will be useful to precise the functoriality of the \( H_n(\text{Mod}_\mathcal{M}/\mathcal{L},F) \) with respect to \( \phi \) and \( \mu \).

7. The first purpose of this paper was to show precisely how each classical first order formula \( \phi \) of a language \( \mathcal{L} \) determines a "small" homology theory on the category of interpretations of \( \mathcal{L} \). Now, the continuation of this research pass through the description of \( \text{For}(\mathcal{L}) \). With respect to that, I would like to make the following remark: what have to be morphisms between formulas ? it is not so clear a priori; they have to be "demonstrations" or "proofs", but there is no
canonical idea of what is a demonstration.

But if we decide to stay in (or to come back to) the style of sketches, a first picture is easy to give. In fact 𝔇 "is" a sketch 𝒔₀ (i.e. the category of interpretations of 𝔇 is isomorphic to ModAPPING), the formula 𝜙 (or 𝜙) is a sketch 𝒔, and the inclusion of the category of models of 𝜙 (or 𝜙) in the category of interpretations of 𝔇 is induced by a morphism of sketches P : 𝒔₀ → 𝒔. This P is the "proof" that a model of 𝜙 (or 𝜙) is an interpretation of 𝔇. In fact P is not a general morphism of sketches, but determines 𝒔 as a 𝒔₀ -sketch (see [6] p.10 for the precise definition). So we choose to say now that a formula for 𝒔₀ (in the place of a 𝔇-formula) is nothing but such a P, a 𝒔₀ -sketch. In [6] the boolean calculus of 𝒔₀ -sketches (conjunctions, disjunctions, complements) is exposed as construction in the category of sketches. Then we can defined the category For(antino) as being the category of 𝒔₀ -sketches, as objects, with morphisms from P to P' the morphisms of sketches f : σ → σ' which determine σ' as a σ-sketch, such that f.P = P'.

At this level of language, we can change our notations, replacing Mod 𝜙 by Mod 𝜙, or even, more precisely, by P, and the H_n(Mod 𝜙/I,F) will be denoted by H_n(P/I,F). Of course for general mixed sketches (and not only for those associated to first order formulas) the result in §5 works, and the abelian groups H_n(P/I,F) are smalls. Now

PROPOSITION. The functoriality of these H_n, with respect to P, I and F are trivial facts.

8. In a dual way, given a functor F : Mod 𝜙 → Ab and an interpretation I of 𝔇, the cohomology of I with coefficient in F is defined by considering the cochain complexe

\[
\begin{array}{cccc}
& & C^0(I,F) & \rightarrow C^1(I,F) & \rightarrow C^2(I,F) & \rightarrow \ldots \\
& d^0 & & d^1 & & \\
\end{array}
\]

which is
with

\[ d(x)(l4 M \xrightarrow{\lambda} M_0) - x(1, M_1) \]

and so on, and we define \( H^n(I, F) = \ker d^n/\text{Im } d^{n-1} \).

For these cohomology groups, the same result is true, that is to say that they are small. But now, the computation is based on the effective construction of a "relatively filtered locally free diagram" (r.f.l.f.d.) (in the category \( \text{Mod } \phi_{\mu} \)) generated by \( I \). These cohomology groups will be denoted by \( H^n((I/\text{Mod } \phi_{\mu})^{\text{op}}, F) \).

9. Collecting the results of §5, §7 and §8, we get:

**Theorem:** The abelian groups \( H_n(\text{Mod } \phi/I, F) \) and \( H^n((I/\text{Mod } \phi_{\mu})^{\text{op}}, F) \) are small, i.e. they are elements of the category \( \text{Ab} \), they are functorial with respect to \( I, F, \mu \) and \( \phi \), and if \( I = \phi \), then \( H_n(\text{Mod } \phi/I, F) = 0 \), for every \( n > 0 \), and \( H^n((I/\text{Mod } \phi_{\mu})^{\text{op}}, F) = 0 \), for \( n > 0 \). In fact, more precisely, we have \( H_n(\text{Mod } \phi/I, F) = 0 \), for every \( n > 0 \), if there is a cofree model generated by \( I \), and we have \( H^n((I/\text{Mod } \phi_{\mu})^{\text{op}}, F) = 0 \), for \( n > 0 \), if there is a free model generated by \( I \). So they are small obstructions to the satisfaction of \( \phi \) in \( I \) "up to co-equations" and "up to "equations".

**References**


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