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ON AUTOMORPHISM GROUPS OF $p$-ADIC SCHOTTKY CURVES

by Lothar GERRITZEN

SCHOTTKY curves over $p$-adic fields have first been studied by MUMFORD [8] in 1972. Further work on this subject has been done by DRINFELD, MANIN, NYERS, and myself (see [7], [9], [3], [4]). In the sequel I will give an introduction to some topics of this theory.

1. Discontinuous groups.

Let $k$ be a locally compact non-archimedean ground field and $G = \text{PGL}_2(k)$ the group of fractional linear transformations on the projective line $\mathbb{P}(k) = k \cup \{\infty\}$ over $k$. By a discontinuous group $\Gamma$, we mean a subgroup $\Gamma$ of $G$ for which there exists a disk $D$ in $\mathbb{P}$ such that

$$\gamma(D) \cap D = \text{empty}, \text{ for almost all } \gamma \in \Gamma.$$ 

By a disk $D$ in $\mathbb{P}$, we understand either a disk on the affine line

$$D = \{z \in k; \ |z - m| < r\}$$

or the complement of a disk on the affine line

$$D = \{z \in \overline{k}; \ |z - m| \geq r\}.$$ 

The question that one would like to answer is: What are the discontinuous subgroups of $G$? Although the situation is certainly simpler than in the classical case $k = \mathbb{C}$, a satisfying answer seems to be far away.

The most important method to construct discontinuous groups dates back to 1887 and has been given by F. SCHOTTKY (see [11]).

**Construction 1:** Let $D_1, D_1', \ldots, D_r, D_r'$ be a system of pairwise disjoint disks of $\mathbb{P}$ and fix centers $m_i$ of $D_i$ and $m_i'$ of $D_i'$. Then it is well defined what the boundaries $\partial D_i, \partial D_i'$ of $D_i$ and $D_i'$ with respect to these centers are. Now let $\gamma_i$ be a transformation in $G$ that maps $\partial D_i$ onto $\partial D_i'$ and that sends the interior $\text{int} D_i := D_i - \partial D_i$ onto the complement of $D_i'$. This is always possible in more than one way. The group $\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle$ is then a free group of rank $r$ and the set $\{\gamma_1, \ldots, \gamma_r\}$ is a system of free generators of $\Gamma$. Also $\Gamma$ operates discontinuously on

$$\Omega := \bigcup_{\gamma \in \Gamma} \gamma(F)$$

where $F = \mathbb{P} - (\bigcup_{i=1}^r D_i \cup \bigcup_{i=1}^r D_i')$.

Any group that can be constructed in such a way will be called a Schottky group.

**Construction 2:** The above method to construct discontinuous groups can be gene-
ralized to obtain so-called combination groups. For this, we use the concept of isometric circles. If the transformation $\gamma$ in $G$ is represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and if $c \neq 0$, we call the disk
$$U_\gamma := \{ z \in \mathbb{P} ; |cz + d| \leq \sqrt{\det \gamma} \}$$
the isometric disk of $\gamma$.

Let now $\Gamma_1, \ldots, \Gamma_r$ be the discontinuous subgroups of $G$, and assume that no $\gamma \in \Gamma_i$, $\gamma \neq \text{id}$, $1 \leq i \leq r$, has the point $\infty$ as a fixed point. Suppose that
$$U_{\gamma_i} \cap U_{\gamma_j} = \text{empty}$$
for each $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$, $\gamma_i \neq \text{id} \neq \gamma_j$, if the index $i$ is different from $j$. Then the group $\Gamma$ generated by $\Gamma_1, \ldots, \Gamma_r$ is discontinuous and is the free product $\Gamma_1 * \ldots * \Gamma_r$.

This can be verified by using Ford's method of isometric circles (see [3], § 1).

If for example $\Gamma_1 = \langle \alpha \rangle$, ord $\alpha = 2$, and $\Gamma_2 = \langle \beta \rangle$, ord $\beta = 3$, we obtain a group $\Gamma$ that is isomorphic as an abstract group to the classical modular group $\text{SL}_2(\mathbb{Z})/(\pm 1)$.

The problem of classifying the discontinuous groups can be answered if the group contains no elements of finite order.

**Theorem 1.** - Any discontinuous, finitely generated group which has no elements $\neq \text{id}$ of finite order is a Schottky group.

For the proof, see [3], § 2 or [9].

It seems likely that any finitely generated discontinuous group contains a subgroup of finite index which is a Schottky group.

2. Automorphic functions.

Any discontinuous subgroup $\Gamma \subset G$ also operates on $\mathbb{P}(\overline{k})$, if $\overline{k}$ is any algebraically closed, complete extension field of $k$.

**Theorem 2.** - There is a largest Stein domain $\Omega = \mathcal{O}(\Gamma) \subset \mathbb{P}(\overline{k})$ on which $\Gamma$ acts discontinuously, i.e. $\gamma(D) \cap D = \text{empty}$ for almost all $\gamma \in \Gamma$ and every disk $D \subset \Omega$.

If $\Gamma$ is finitely generated there is an affinoid domain $F \subset \Omega$ such that
$$\Omega = \bigcup_{\gamma \in \Gamma} \gamma(F).$$

**Proof.** - If $\Gamma$ is a Schottky group, proofs can be found in [3], § 3, [7], [9]. A proof for the general case can be given along the same lines as in [3].

The orbit space $S(\Gamma) = \mathcal{O}/\Gamma$ is a 1-dimensional non-singular analytic space and a projective curve if $\Gamma$ is finitely generated. The genus of this curve depends
only on the structure of the group $\Gamma$.

**Theorem 3.** Let $\Gamma$ be finitely generated. Then: genus of $S(\Gamma) = \mathbb{Z}$-rank of the commutator factor group $\overline{\Gamma}$ of $\Gamma$.

**Proof.** Assume that $a, b \in \Omega$ and let $a, b$ be two points in $\Omega$, not contained in $\Gamma^\infty$. Then the possibly infinite product

$$f(a, \ldots, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}$$

converges uniformly on every affinoid domain $\mathcal{O}$, as $|\gamma(a) - \gamma(b)| \to 0$ (see [4], § 2). Therefore we get a meromorphic function $f(a, \ldots, b; z)$.

Now

$$f(\alpha, b; z) = \rho_{\alpha} f(a, b; z)$$

where $\rho_{\alpha} \in \kappa^* = \kappa - \{0\}$.

As $f(\alpha \beta(z)) = \rho_{\alpha \beta} f(z) = \rho_{\alpha} f(\beta(z)) = \rho_{\alpha} \cdot \rho_{\beta} f(z)$, the map $\alpha \mapsto \rho_{\alpha}$ is a homomorphism of $\Gamma$ into the multiplicative group $\kappa^*$ of $\kappa$.

Consider the simplest case $\Gamma = 0$: then $f(a, b; z)$ is a $\Gamma$-invariant meromorphic function which as function on the orbit space $S$ has exactly one simple pole if $b$ is no fixed point of a transformation $\neq \text{id}$ of $\Gamma$. Thus $S$ has genus $0$.

Next consider the case $\Gamma$ finite. Let $\pi : \Omega \to S$ denote the canonical mapping and fix a point $b \in \Omega$. We then find $a_1, a_2 \in \Omega$ such that $\pi(a_1) \neq \pi(a_2)$ and such that $f(a_1, b; z)$, $f(a_2, b; z)$ have same factors of automorphy $\rho$.

Then

$$f(a_1, a_2; z) = \frac{f(a_1, b; z)}{f(a_2, b; z)}$$

is a meromorphic function on $S$ with one simple pole, if $a_2$ is not a fixed point of a nontrivial transformation in $\Gamma$. Thus $S$ has genus $0$.

Finally, we consider the general case, where $\Gamma = \mathbb{Z}^\infty$ finite group: with the help of Poincaré series as in the proof of [4], Satz 8, we can show that to any $\rho \in \text{Hom}(\Gamma, \kappa^*) = P$

there is a meromorphic form $g(z)$ such that

$$g(\gamma(z)) = \rho(\gamma) \cdot g(z).$$

If $L$ is the lattice of all $\rho \in P$ such that there is an analytic automorphic form without zeros and factor of automorphy $\rho$, we get that $J(\Gamma) = P/L$ is the Jacobian variety of $S$ (see [7], § 3). On the other hand $P/L$ is an analytic torus of dimension $r$. As $\dim J = \text{genus } S$, we are done.

3. Automorphism groups of Schottky curves.

A projective curve $S$ is called a Schottky curve if it is of the form $S(\Gamma)$,
where \( \Gamma \) is a Schottky group. What can be said about the automorphism group \( \text{Aut } S \)?

**THEOREM 4.** - \( \text{Aut } S = N/\Gamma \) where \( N \) is the normalizer of \( \Gamma \) in \( G \), i.e. \( N = \{ u \in G : u\Gamma u^{-1} = \Gamma \} \).

**Proof.** - Any automorphism \( \alpha : S \to S \) can be lifted to an analytic automorphism \( \tilde{\alpha} : \Omega \to \Omega \) as in [1], § 9. Now \( \tilde{\alpha} \) can be continued to an analytic map \( \mathbb{P} \to \mathbb{P} \), [4], Kor. 2 to Satz 5. Therefore \( \tilde{\alpha} \) must be a fractional linear transformation. On the other hand, it is clear that any transformation in \( N \) maps \( \Omega \) onto itself as it maps the set of limit points of \( \Gamma \) onto itself and thus induces an automorphism of \( S \).

Q.E.D.

If \( \Gamma \) is a Schottky group of rank \( r \), then \( \Gamma \cong \mathbb{Z}^r \). The group \( N/\Gamma = \text{Aut } S \) acts by inner automorphisms on \( \Gamma \).

**THEOREM 5.** - \( \text{Aut } S \) acts on \( \Gamma \) effectively and thus \( \text{Aut } S \) can be considered as a finite subgroup of \( \text{GL}_r(\mathbb{Z}) \).

**Proof.** - The action of \( \text{Aut } S \) on \( \Gamma \) induces an action of \( \text{Aut } S \) on the Jacobian variety \( P/L = J(S) \). But this action on \( J(S) \) is the canonical action on \( J(S) \) considered as the Picard variety of \( S \). This action is always effective. This can be proved as follows: Let \( s_0 \in S \) and \( j : S \to J(S) = J \) the canonical embedding such that

\[
j(s_0) = \text{neutral element } 0 \text{ of the abelian variety } J.
\]

The universal property of \( j \) gives to any automorphism \( \alpha \in \text{Aut } S \) a biregular mapping \( \alpha^* : J \to J \) such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{j} & J \\
\downarrow{\alpha} & & \downarrow{\alpha^*} \\
S & \xrightarrow{j} & J 
\end{array}
\]

is commutative.

Define \( \tilde{\alpha} \) by setting \( \tilde{\alpha}(x) := \alpha^*(x) - \alpha^*(0) \). Then \( \tilde{\alpha}(0) = 0 \) and \( \tilde{\alpha} \) is therefore a group automorphism of \( J \). Obviously \( \tilde{\alpha}_1 \tilde{\alpha}_2 = \tilde{\alpha}_1 \circ \tilde{\alpha}_2 \), and we have thus a group representation of \( \text{Aut } S \) into the automorphism group \( \text{Aut } J \) of the abelian variety \( J \). This representation does not depend on \( s_0 \in S \).

We want to see that this representation is faithfull.

Let \( \alpha \in \text{Aut } S \) and \( \tilde{\alpha} = \text{id} \). If \( \alpha \) has a fixed point \( s_0 \in S \), then \( \tilde{\alpha} = \alpha^* = \text{id} \) and therefore \( \alpha = \text{id} \) as \( \alpha^* \) is a continuation of \( \alpha \). If \( \alpha \) has no fixed point, we consider the quotient curve \( S' = S/\langle \alpha \rangle \) of \( S \) by the subgroup generated by \( \alpha \). By the genus formula of Hurwitz, we get that the genus \( r' \) of \( S' \) is smaller than the genus \( r \) of \( S \) if \( \alpha \neq \text{id} \).
Now, \( \text{ord } \alpha^*(0) = \text{ord } \alpha \), and the quotient variety

\[ J' = J/\langle \alpha^*(0) \rangle \]

of \( J \) by the subgroup generated by \( \alpha^*(0) \) is an abelian variety of the same dimension as \( J \).

Thus the canonical composition mapping \( S \rightarrow J' \) induces a regular mapping \( j : S' \rightarrow J' \).

Since \( j(S) \) generates \( J \), we see that \( j'(S') \) generates \( J' \). The canonical mapping of the Jacobian variety of \( S' \) into \( J' \) must thus be surjective and as the Jacobian variety of \( S' \) has dimension \( r' \) this is possible only if \( r' = r \) and \( \alpha = \text{id} \). The completes the proof of Theorem 5.

**COROLLARY 1.** - If \( r = 2 \), then \( \text{ord}(\text{Aut } S) \leq 12 \). If \( r = 3 \), then \( \text{ord}(\text{Aut } S) \leq 48 \).

**Proof.** - Any finite subgroup of \( \text{GL}_2(\mathbb{Z}) \) (resp. \( \text{GL}_3(\mathbb{Z}) \)) has order less or equal than 12 (resp. 48).

**COROLLARY 2.** - If \( \text{char } k = 0 \) and \( r = 4 \), then \( \text{ord}(\text{Aut } S) \leq 240 \).

**Proof.** - By Hurwitz estimation, we get that \( \text{ord}(\text{Aut } S) \leq 84.3 = 252 \) (see [6], Section 7). But the order of any finite subgroup of \( \text{GL}_4(\mathbb{Z}) \) divides \( \frac{81}{7} = 2.7.3^2.5 \) (see [10], Chap IX, exercise 2). Therefore \( \text{ord}(\text{Aut } S) \) must divide one of the following values:

\[ 2^7 = 132, \quad 2^6.3 = 192, \quad 2^4.3^2 = 144, \quad 2^5.5 = 160, \quad 2^4.3.5 = 240, \quad 2^2.3^2.5 = 180. \]

**Remark.** - It seems likely that much better estimates can be given for \( r \geq 3 \).

4. **An example.**

Let \( \alpha \) and \( \beta \) be elliptic transformations of order 2 and 3 whose fixed points lie in the ground field \( k \). If \( n \) is the transformation in \( G \), that permutes the fixed points of \( \alpha \) as well as those of \( \beta \), then

\[
\begin{align*}
n\alpha n &= \alpha \\
n\beta n &= \beta^{-1} \\
n^2 &= \text{id}.
\end{align*}
\]

Let \( N \) be the group generated by \( \alpha, \beta \) and \( n \), and \( \Delta \) the group generated by \( \alpha \) and \( \beta \). If the isometric disks of \( \alpha \) and \( \beta \) do not intersect, then \( \Delta \) is discontinuous. The commutator subgroup \( \Gamma \) of \( \Delta \) is freely generated by \( \alpha \beta \beta^{-1} \), \( \alpha^2 \beta^{-1} \alpha \beta \) which is thus a Schottky group of rank 2. \( N \) is the normalizer of \( \Gamma \) in \( G \) as \( \text{ord } N/\Gamma = 12 \) and because of Corollary 2 to Theorem 5.

Thus \( N/\Gamma = \text{Aut } S(\Gamma) = \text{dihedral group of order } 12 \).

Therefore \( S(\Gamma) \) is the curve of the equation \( y^2 = (x^3 - 1)(x^3 - \lambda) \) with a cer-
tain parameter $\lambda$ (see [5], I, (4)).

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LITERATURE
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