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p-ADIC WHITTAKER GROUPS

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An algebraic curve (non singular, irreducible and complete) over $\underset{\sim}{C}$ which is hyperelliptic can be uniformized by a Whittaker group (see [1], p. 247-249). We will treat the rigid analytic case for complete non-archimedean valued fields k with characteristic $\neq 2$. In order to avoid rationality problems the field k is supposed to be algebraically closed. A part of the results in this paper was independently proved by G. Van STEEN.

1. Combinations of discontinuous groups.

Let $\Gamma \subset PGl(2, k)$ be a discontinuous group. We will assume that $\infty \in \underset{\sim}{P}^1(k) = \underset{\sim}{P}^1$ is an ordinary point for Γ. A fundamental domain F for Γ, containing ∞, is a subset F of $\underset{\sim}{P}^1$ satisfying :

(i) $\underset{\sim}{P}^1 - F$ is a finite union of open spheres $B_1, \ldots, B_n$ in k such that the corresponding closed spheres $B_1^+, \ldots, B_n^+$ are disjoint,

(ii) The set $\{\gamma \in \Gamma ; \gamma F \cap F \neq \emptyset\}$ is finite,

(iii) if $\gamma \neq 1$ and $\gamma F \cap F \neq \emptyset$ then $\gamma F \cap F \subseteq \bigcup_{i=1}^n (B_i^+ - B_i)$,

(iv) $\bigcup_{\gamma \in \Gamma} \gamma F = \Omega =$ the set of ordinary points of Γ.

We will write $\mathring{F}$ for $\underset{\sim}{P}^1 - \bigcup_{i=1}^n B_i^+$.

One can show that a fundamental domain for Γ exists if Γ is finitely generated (see [2] and [3]).

PROPOSITION. - *Let $\Gamma_1, \ldots, \Gamma_m$ be discontinuous groups with fundamental domains containing the point ∞, $F_1, \ldots, F_m$. Suppose that $\mathring{F}_i \supset \underset{\sim}{P}^1 - F_j$ for all $i \neq j$. Then the group Γ generated by $\Gamma_1, \ldots, \Gamma_m$ is discontinuous. Moreover $\Gamma = \Gamma_1 * \ldots * \Gamma_m$ (the free product) and $\bigcap F_i$ is a fundamental domain for Γ.*

Proof. - Put $F = \bigcap_{i=1}^m F_i$ and $\mathring{F} = \bigcap_{i=1}^m \mathring{F}_i$. Let $W = \delta_s \delta_{s-1} \ldots \delta_1$ be a reduced word in $\Gamma_1 * \ldots * \Gamma_n$, i. e. each $\delta_i \in \bigcup \Gamma_j - \{1\}$ and if $\delta_i \in \Gamma_\ell$ then $\delta_{i+1} \notin \Gamma_\ell$. Then $W(\mathring{F}) \subseteq \underset{\sim}{P}^1 - F$. Hence Γ is equal to $\Gamma_1 * \ldots * \Gamma_n$. Further $W(F) \cap F \neq \emptyset$ implies that $W \in \bigcup \Gamma_j$. So we have shown that F satisfies the conditions (i), (ii) and (iii). Let $\delta > 0$, then there are finite sets $W_1 \subset \Gamma_1, \ldots, W_m \subset \Gamma_m$ such that the complement of $\bigcup_{\gamma \in W_i} \gamma F_i$ consists of

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finitely many spheres of radii $< \delta$.

Given $\varepsilon > 0$ then there is $\delta > 0$ and some $n \gg 0$ such that the complement of $\bigcup_{\gamma \in W} \gamma F$, where W consists of all reduced words in $W_1, \dots, W_m$ of length $\leqslant n$, is a finite union of spheres of radii $< \varepsilon$.

This shows that the set of limit points of Γ is equal to the compact set

$$\underset{\sim}{P}^1 - \bigcup_{\gamma \in \Gamma} \gamma F .$$

2. Example. If each $\Gamma_i \cong \underset{\sim}{Z}$, so Γ_i is generated by an hyperbolic element, then Γ is a free group on m generators. We will call such a Γ a Schottky group of rank m . It can be shown that any group Γ , which satisfies :

(i) Γ discontinuous ;

(ii) Γ is finitely generated ;

(iii) Γ has no elements of finite order $(\neq 1)$,

is a Schottky group of rank m . Moreoever Ω/Γ turns out to be an algebraic curve over k with genus m .

3. Definition of the p-adic Whittaker groups. (characteristic $k \neq 2$.)

Let s be an element of order two in $PGl(2, k)$. Then s has two fixed points a and b . Moreover s is determined by $\{a, b\}$. Let B be an open sphere in $\underset{\sim}{P}^1$ maximal, w. r. t. the condition $sB \cap B = \emptyset$ and let c be a point of B .

There exists a $\sigma \in PGl(2, k)$ with $\sigma(a) = 1$, $\sigma(b) = -1$, $\sigma(c) = 0$. Then $t = \sigma s \sigma^{-1}$ has the form $z \longmapsto 1/z$; t has $1, -1$ as fixed points and $\sigma(B) = \{z \in \underset{\sim}{P}^1 ; |z| < 1\}$. It follows that $\underset{\sim}{P}^1 - B$ is a fundamental domain for the group $\{1, s\}$.

Let $(g+1)$ elements $s_0, \dots, s_g$ of order two in $PGl(2, k)$ be given. Suppose that their fixed points $\{a_0, b_0\}, \{a_1, b_1\}, \dots, \{a_g, b_g\}$ are all finite and are such that the smallest closed spheres $B_0^+, \dots, B_g^+$ in k containing $\{a_0, b_0\}, \{a_1, b_1\}, \dots, \{a_g, b_g\}$, are disjoint.

Choose points $c_i \in B_i^+$ such that the open sphere B_i with center c_i and radius = radius of B_i^+ does not contain a_i and b_i .

According to Prop. 1 the group $\Gamma = \langle s_0, s_1, \dots, s_g \rangle$ generated by $\{s_0, \dots, s_g\}$ is discontinuous, has $F = \underset{\sim}{P}^1 - \bigcup_{i=0}^{g} B_i$ as fundamental domain and is equal to

$$\langle s_0 \rangle * \langle s_1 \rangle * \dots * \langle s_g \rangle \cong \underset{\sim}{Z}/2 * \dots * \underset{\sim}{Z}/2 .$$

Let $\varphi : \Gamma \longrightarrow \underset{\sim}{Z}/2$ be the group homomorphism given by $\varphi(s_i) = 1$ for all i . The kernel W of φ is called a Whittaker group. The group W is generated by $\{s, s_0, s_2 s_0, \dots, s_g s_0\}$. An easy exercise shows that W is a free group on

$\{s_1 s_0 , s_2 s_0 , \dots , s_g s_0\}$. So W is a Schottky group of rank g .

The groups W and Γ have the same set $\mathcal{L}$ of limit points. Let $\Omega = \underset{\sim}{P}^1 - \mathcal{L}$. Then Ω/W and Ω/Γ have a canonical structure of an algebraic curve over k . The natural map $\Omega/W \xrightarrow{f} \Omega/\Gamma$ is a morphism of algebraic curves of degree 2 .

4. PROPOSITION. $\Omega/\Gamma \cong \underset{\sim}{P}^1$.

Proof. - Consider

$$\theta(a , b , z) = \prod_{\gamma\in\Gamma} \frac{z - \gamma(a)}{z - \gamma(b)} ,$$

where $a , b \in \Omega$ and $a \notin \Gamma b$ and $\infty \notin \Gamma a \cup \Gamma b$.

This function converges uniformly on the **affinoid** subsets of Ω since

$$\lim |\gamma(a) - \gamma(b)| = 0 .$$

So $F(z) = \theta(a , b , z)$ is a meromorphic function on Ω . For any $\delta \in \Gamma$ we have $F(\delta z) = c(\delta)\, F(z)$ where $c(\delta) \in k^*$. Clearly $c : \Gamma \to k^*$ is a group homomorphism and hence $c(\delta) = \pm 1$.

For given a one can take b close to a such that $|F(\infty) - F(s_0 \infty)| < \frac{1}{2}$. For this choice of a and b , we find that F is invariant under Γ . So F defines a morphism $\tilde{F} : \Omega/\Gamma \to \underset{\sim}{P}^1$. This morphism has only one pole. Hence $\tilde{F}$ is an isomorphism.

Second proof (G. Van STEEN). - If Γ is finitely generated then Ω/Γ is an algebraic curve of genus = rank of $\Gamma_{ab} = \Gamma/[\Gamma , \Gamma]$.

In our case the rank is clearly zero.

5. THEOREM. Ω/W is an hyperelliptic curve of genus g . The affine equation of Ω/W is $y^2 = \prod_{i=0}^{g} (x - F(a_i))(x - F(b_i))$.

Proof. - It follows from 3 and 4 that Ω/W is indeed hyperelliptic of genus g . Therefore Ω/W must have $2g + 2$ ramification points over Ω/Γ . A point $p \in \Omega/W$, image of $e \in \Omega$, is a ramification if, and only if, $s_0 \in We$. The points $a_0 , b_0 , \dots , a_g , b_g$ satisfy this condition, and their images in Ω/Γ are different. So the equation follows.

6. COROLLARY. - Let $s_0 , \dots , s_g \in PGl(2 , k)$ be elements of order 2 such that the group Γ generated by them satisfies : Γ is discontinuous and

$$\Gamma = \langle s_0 \rangle * \langle s_1 \rangle * \dots * \langle s_g \rangle .$$

Then there are elements $s_0^* , \dots , s_g^*$ of order 2 in $PGl(2 , k)$ with the $(2g + 2)$ fixed points in the position required in 3, and such that $\Gamma = \langle s_0^*, \dots, s_g^* \rangle$.

Proof. - In 3, 4 and 5, the position of the $(2g + 2)$ fixed points of $\{s_0 , \dots , s_g\}$ is only used to prove that Γ is discontinuous and equal to

$\langle s_0 \rangle * \cdots * \langle s_g \rangle$. So we can also form $W = \langle s_0 s_1, s_0 s_2, \ldots, s_0 s_g \rangle \subset \Gamma$ and conclude that $\Omega/W \xrightarrow{f} \Omega/\Gamma = \mathbb{P}^1$ has degree 2 and has $2g+2$ ramification points, called $A_1, \ldots, A_{2g+2}$. Let $\sigma : \Omega/W \to \Omega/W$ be the automorphism of order 2 defined by f. Then $A_1, \ldots, A_{2g+2}$ are the fixed points of σ.

Write $t_1 = s_0 s_1, \ldots, t_g = s_0 s_g$. Every element in Γ of order 2 must have the form $a s_i a^{-1}$ ($a \in \Gamma$; $i = 0, \ldots, g$) (see [5]). Further $a \in \Gamma$ has the form $w s_0$ or w, with $w \in W$. Since $s_0 s_i s_0 = t_i s_i t_i^{-1}$, we find that every element in Γ of order 2 has the form $w s_i w^{-1}$, with $w \in W$ and $i \in \{0, \ldots, g\}$. It is easily verified that this presentation is unique.

Further $\Omega \xrightarrow{\pi} \Omega/W$ is a universal covering (see [4]). Hence for any $e, f \in \Omega$ with $\sigma(\pi(e)) = \pi(f)$, there exists a unique lifting $s : \Omega \to \Omega$ of σ with $s(e) = f$. Moreover $s \in \Gamma$.

Take now $e \in \pi^{-1}(A_j)$ and a lifting s of σ with $s(e) = e$. Then $s^2 = 1$. Hence $s = w s_i w^{-1}$ for some $i \in \{0, \ldots, g\}$ and $w \in W$. The i does not depend on the choice of $e \in \pi^{-1}(A_j)$. Hence we have constructed a map

$$\tau : \{A_1, \ldots, A_{2g+2}\} \to \{0, 1, \ldots, g\}.$$

Further any $w s_i w^{-1}$ has at most two fixed points in $\pi^{-1}(\{A_1, \ldots, A_{2g+2}\})$. It follows that $\tau^{-1}(i)$ consists of at most two points. Hence τ is surjective and every s_i has both fixed points in $\pi^{-1}(\{A_1, \ldots, A_{2g+2}\}) \subset \Omega$. The generators for Γ can be changed into $s_0, t_2^n s_1 t_2^{-n}, s_2, \ldots, s_g$. With a sequence of changes of this type one finds generators $s_0^* \ldots s_g^*$ for Γ with their $(2g+2)$ fixed points in the required position.

7. THEOREM. – <u>Suppose that</u> X <u>is a hyperelliptic curve of genus</u> g <u>over</u> k <u>which is totally split. Then there exists a Whittaker group</u> W, <u>unique up to conjugation in</u> $PGl(2, k)$, <u>with</u> $X \cong \Omega/W$.

<u>Proof.</u> – We will use freely the results of [3] and [4]. We know that

$$\Omega \xrightarrow{\pi} \Omega/W \cong X$$

exists where W is a Schottky group of rank g, unique up to conjugation. We have to show that W is in fact a Whittaker group.

Let σ be the automorphism of X with order two such that $\tau : X \to X/\sigma \cong \mathbb{P}^1$. Then σ has $A_1, \ldots, A_{2g+2} \in X$ as fixed points. Let Γ denote the set of all lifts $s : \Omega \to \Omega$ of $\sigma : X \to X$ and of $id : X \to X$. Then Γ is a group and W has index 2 in Γ. The set

$$K = \overline{\pi^{-1}(\{A_1, \ldots, A_{2g+2}\})} \subset \mathbb{P}^1$$

is a compact set with limit points $= \mathcal{L} = \mathbb{P}^1 - \Omega =$ the limit points of $W =$ the limit points of Γ. Let $\overline{\Omega}$ denote the reduction of Ω with respect to K. Then

$\overline{\Omega}/\Gamma$ is a reduction of $\underset{\sim}{P}^1$ and it is in fact the reduction of $\underset{\sim}{P}^1$ with respect to the finite set $\{\tau(A_1) , \dots , \tau(A_{2g+2})\}$.

Let $\overline{X}$ denote the reduction induced by $\overline{\Omega}$, i. e. $\overline{\Omega}$ is given with respect to a pure covering $\mathfrak{U}$, and $\overline{X}$ is the reduction with respect to $\pi(\mathfrak{U})$.

One easily sees that $\overline{X} = \overline{\Omega}/W$ and consists of projective lines over the residue field $\overline{k}$ of k . The intersection graph $G(\overline{X})$ is defined by :

vertices = the components of $\overline{X}$ and edges = the intersection points.

The map σ induces an automorphism of $\overline{X}$ and $G(\overline{X})$, again denoted by σ . Further $\overline{X} \xrightarrow{\overline{\tau}} \overline{X}/\sigma \cong \overline{\Omega}/\Gamma$ and $G(\overline{X})/\sigma \cong G(\overline{\Omega}/\Gamma)$ = a connected finite tree.

Through the image $\overline{A}_1$ of A_1 on $\overline{X}$ goes only one component of $\overline{X}$ since $\overline{\tau}(A_1)$ lies on only one component of $\overline{\Omega}/\Gamma$. Call this vertex of $G(\overline{X})$ the vertex g_1 . Then $\sigma(g_1) = g_1$ and the homeomorphism σ of $G(\overline{X})$ induces an automorphism $\hat{\sigma}$ of $\pi_1 (G(\overline{X}) , g_1)$ = the fundamental group of $G(\overline{X})$.

We know further that $\pi_1(G(\overline{X}) , g_1)$ is in a natural way isomorphic to W . Suppose that we can find a base for the fundamental group, $t_1 , \dots , t_g$ such that $\hat{\sigma}(t_i) = t_i^{-1}$ for all i . Then we can lift this situation to Ω as follows : Choose an element $e \in \pi^{-1}(A_1)$; let s_0 be the lift of σ satisfying $s_0(e)=e$; let h_0 be the component of $\overline{\Omega}$ on which e lies ; let the curve in $G(\overline{\Omega})$ with begin point h_0 and lying above ℓ_i have endpoint $h_i \in G(\overline{\Omega})$; let $T_i \in W$ be defined by $T_i(e)$ lies on h_i .

Then $W = \langle T_1 , \dots , T_g \rangle$ and $s_0 \, T_i \, s_0 = T_i^{-1}$ for all i . Put

$$s_1 = s_0 \, T_1 \, , \dots \, s_g = s_0 \, T_g \, .$$

Then $\Gamma = \langle s_0 , s_1 , \dots , s_g \rangle$ and easy inspection yields

$$\Gamma = \langle s_0 \rangle * \langle s_1 \rangle * \dots * \langle s_g \rangle \, .$$

According to Corollary 6, we have shown that W is a Whittaker group.

Finally we have to show the following lemma :

8. LEMMA. – Let G be a finite connected graph with Betti number g . Let σ be an homeomorphism of G such that :

(i) σ has order 2 ;

(ii) G/σ is a tree ;

(iii) σ fixes a vertex $p \in G$.

Then the fundamental group $\pi_1(G , p)$ has generators $t_1 , \dots , t_g$ such that the induced automorphism $\hat{\sigma}$ of $\pi_1(G , p)$ has the form $\hat{\sigma}(t_i) = t_i^{-1}$ for all i .

Proof. – Induction on the number of vertices of G .

(1) p <u>is the only vertex of</u> G . - Then G is a wedge of g circles. As generators for π_1 we take the g circles together with an orientation. Call them $t_1, \dots, t_g$. Since σ is an homeomorphism we must have

$$\hat{\sigma}(t_i) \in \{t_1 , \dots , t_g , t_1^{-1} \dots t_g^{-1}\}$$

for all i . Since G/σ has a trivial fundamental group, one finds that $\hat{\sigma}(t_i)=t_i^{-1}$ for all i .

(2) <u>Induction step</u>. - Choose an edge λ of G with endpoints p and $q \neq p$. If $\sigma(\lambda) = \lambda$ then we make a new graph G^* by identifying p and q and deleting the edge λ .

If $\sigma(\lambda) \neq \lambda$, but $\sigma(\lambda)$ has also endpoints p and q , then we make G^* by identifying p and q and also identifying λ on $\sigma(\lambda)$.

If $\sigma(\lambda)$ has endpoints p , r with $r \neq q$, then we make G^* by identifying q and r with p and deleting λ and $\sigma(\lambda)$.

In all cases, G^* is homotopic to G ; σ acts again on G^* and induces the same automorphism of the fundamental group.

9. <u>Remarks</u>.

1° An easy calculation gives that the number of moduli for Whittaker groups of rank g is $2g - 1$. This is the same as the number of moduli for hyperelliptic curves of genus g .

2° Is it possible to give an explicit calculation of the numbers $F(a_i)$, $F(b_i)$ in theorem 5 ?

3° Hyperelliptic curves and Whittaker groups in characteristic 2 will be treated by G. Van STEEN.

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