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$p$-adic Whittaker groups

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p-ADIC WHITTAKER GROUPS

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An algebraic curve (non singular, irreducible and complete) over \( \mathbb{C} \) which is hyperelliptic can be uniformized by a Whittaker group (see [1], p. 247-249). We will treat the rigid analytic case for complete non-archimedean valued fields \( K \) with characteristic \( \neq 2 \). In order to avoid rationality problems the field \( K \) is supposed to be algebraically closed. A part of the results in this paper was independently proved by G. Van STEEN.

1. Combinations of discontinuous groups.

Let \( \Gamma \subset \text{PGL}(2, K) \) be a discontinuous group. We will assume that \( \omega \in \mathbb{P}^1(K) \) is an ordinary point for \( \Gamma \). A fundamental domain \( F \) for \( \Gamma \), containing \( \omega \), is a subset \( F \) of \( \mathbb{P}^1 \) satisfying:

(i) \( \mathbb{P}^1 - F \) is a finite union of open spheres \( B_1, \ldots, B_n \) in \( K \) such that the corresponding closed spheres \( B_1^+, \ldots, B_n^+ \) are disjoint,

(ii) The set \( \{ \gamma \in \Gamma; \gamma F \cap F \neq \emptyset \} \) is finite,

(iii) if \( \gamma \neq \gamma' \) and \( \gamma F \cap F \neq \emptyset \) then \( \gamma F \cap F \subset \bigcup_{i=1}^n (B_i^+ - B_i) \),

(iv) \( \bigcup_{\gamma \in \Gamma} \gamma F = \Omega = \) the set of ordinary points of \( \Gamma \).

We will write \( \mathfrak{F} \) for \( \mathbb{P}^1 - \bigcup_{i=1}^n B_i^+ \).

One can show that a fundamental domain for \( \Gamma \) exists if \( \Gamma \) is finitely generated (see [2] and [3]).

PROPOSITION. - Let \( \Gamma_1, \ldots, \Gamma_m \) be discontinuous groups with fundamental domains containing the point \( \omega \), \( F_1, \ldots, F_m \). Suppose that \( \mathfrak{F} = \mathbb{P}^1 - F_i \) for all \( i \neq j \). Then the group \( \Gamma \) generated by \( \Gamma_1, \ldots, \Gamma_m \) is discontinuous. Moreover \( \Gamma = \bigcap_{i=1}^m \Gamma_i \) (the free product) and \( \bigcap_{i=1}^m F_i \) is a fundamental domain for \( \Gamma \).

Proof. - Put \( F = \bigcap_{i=1}^m F_i \) and \( \mathfrak{F} = \bigcap_{i=1}^m \mathfrak{F}_i \). Let \( W = \delta_0 \delta_1 \cdots \delta_{i-1} \delta_i \) be a reduced word in \( \Gamma_1 \cdots \Gamma_n \), i.e. each \( \delta_i \in \bigcup_{j \neq i} \Gamma_j \) and if \( \delta_i \in \Gamma_j \), then \( \delta_{i+1} \notin \Gamma_j \). Then \( W(F) \subset \mathbb{P}^1 - F \). Hence \( \Gamma \) is equal to \( \Gamma_1 \cdots \Gamma_m \). Further \( W(F) \cap F \neq \emptyset \) implies that \( W \in \bigcup_{j \neq i} \Gamma_j \). So we have shown that \( F \) satisfies the conditions (i), (ii) and (iii). Let \( \delta > 0 \), then there are finite sets \( W_1 \subset \Gamma_1, \ldots, W_m \subset \Gamma_m \) such that the complement of \( \bigcup_{\gamma \in W_i} \gamma F_i \) consists of

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Given $\varepsilon > 0$ then there is $\delta > 0$ and some $n >> 0$ such that the complement of $U_{W \in W} W_1$, where $W$ consists of all reduced words in $W_1, \ldots, W_n$ of length $\leq n$, is a finite union of spheres of radii $< \varepsilon$.

This shows that the set of limit points of $\Gamma$ is equal to the compact set $P^1 - \bigcup_{W \in W} W$.

2. Example. If each $\Gamma_i \simeq \mathbb{Z}$, so $\Gamma_i$ is generated by an hyperbolic element, then $\Gamma$ is a free group on $n$ generators. We will call such a $\Gamma$ a Schottky group of rank $n$. It can be shown that any group $\Gamma$, which satisfies:

(i) $\Gamma$ discontinuous;

(ii) $\Gamma$ is finitely generated;

(iii) $\Gamma$ has no elements of finite order ($\neq 1$),

is a Schottky group of rank $n$. Moreover $\mathbb{C}/\Gamma$ turns out to be an algebraic curve over $k$ with genus $n$.

3. Definition of the $p$-adic Whittaker groups. (characteristic $k \neq 2$.)

Let $s$ be an element of order two in $\text{PGl}(2, k)$. Then $s$ has two fixed points $a$ and $b$. Moreover $s$ is determined by $\{a, b\}$. Let $B$ be an open sphere in $P^1$ maximal, w. r. t. the condition $sB \cap B = \emptyset$ and let $c$ be a point of $B$.

There exists a $\sigma \in \text{PGl}(2, k)$ with $\sigma(a) = 1$, $\sigma(b) = -1$, $\sigma(c) = 0$. Then $t = \sigma s \sigma^{-1}$ has the form $z \mapsto 1/z$; $t$ has 1, -1 as fixed points and $\sigma(B) = \{z \in P^1; |z| < 1\}$. It follows that $P^1 - B$ is a fundamental domain for the group $\{1, s\}$.

Let $(g + 1)$ elements $s_0, \ldots, s_g$ of order two in $\text{PGl}(2, k)$ be given. Suppose that their fixed points $\{a_0, b_0\}$, $\{a_1, b_1\}$, ..., $\{a_g, b_g\}$ are all finite and are such that the smallest closed spheres $B^+_0$, ..., $B^+_g$ in $k$ containing $\{a_0, b_0\}$, $\{a_1, b_1\}$, ..., $\{a_g, b_g\}$, are disjoint.

Choose points $c_i \in B^+_i$ such that the open sphere $B_i$ with center $c_i$ and radius = radius of $B^+_i$ does not contain $a_i$ and $b_i$.

According to Prop. 1 the group $\Gamma = \langle s_0, s_1, \ldots, s_g \rangle$ generated by $\{s_0, \ldots, s_g\}$ is discontinuous, has $F = P^1 - \bigcup_{i=0}^g B_i$ as fundamental domain and is equal to

$$\langle s_0 \rangle * \langle s_1 \rangle * \ldots * \langle s_g \rangle \simeq \mathbb{Z}/2 * \ldots * \mathbb{Z}/2.$$  

Let $\varphi: \Gamma \rightarrow \mathbb{Z}/2$ be the group homomorphism given by $\varphi(s_i) = 1$ for all $i$. The kernel $W$ of $\varphi$ is called a Whittaker group. The group $W$ is generated by $\{s, s_0, s_2 s_0, \ldots, s_g s_0\}$. An easy exercise shows that $W$ is a free group on
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The groups $W$ and $\Gamma$ have the same set $E$ of limit points. Let $\Omega = \mathbb{P}^1 - E$.

Then $\Omega/W$ and $\Omega/\Gamma$ have a canonical structure of an algebraic curve over $k$. The natural map $\Omega/W \to \Omega/\Gamma$ is a morphism of algebraic curves of degree 2.

### 4. PROPOSITION

$\Omega/\Gamma \cong \mathbb{P}^1$.

**Proof.** Consider

$$\theta(a, b, z) = \prod_{v \in \Gamma} \frac{z - \gamma(v)}{z - \gamma(b)},$$

where $a, b \in \Omega$ and $a \not\in \Gamma b$ and $a \not\in \Gamma a \cup \Gamma b$.

This function converges uniformly on the affinoid subsets of $\Omega$ since

$$\lim_{z \to \infty} |\gamma(a) - \gamma(b)| = 0.$$

So $F(z) = \theta(a, b, z)$ is a meromorphic function on $\Omega$. For any $\delta \in \Gamma$ we have $F(\delta z) = c(\delta) F(z)$ where $c(\delta) \in k^*$. Clearly $c : \Gamma \to k^*$ is a group homomorphism and hence $c(\delta) = \pm 1$.

For given $a$ one can take $b$ close to $a$ such that $|F(\infty) - F(\infty)| < \frac{1}{2}$. For this choice of $a$ and $b$, we find that $F$ is invariant under $\Gamma$. So $F$ defines a morphism $F : \Omega/\Gamma \to \mathbb{P}^1$. This morphism has only one pole. Hence $F$ is an isomorphism.

**Second proof (G. Van STEEN).** - If $\Gamma$ is finitely generated then $\Omega/\Gamma$ is an algebraic curve of genus $= \text{rank of } \Gamma_{ab} = [\Gamma : \Gamma]$.

In our case the rank is clearly zero.

### 5. THEOREM

$\Omega/W$ is an hyperelliptic curve of genus $g$. The affine equation of $\Omega/W$ is

$$y^2 = \prod_{i=0}^{2g} (x - F(a_i))(x - F(b_i)).$$

**Proof.** It follows from 3 and 4 that $\Omega/W$ is indeed hyperelliptic of genus $g$.

Therefore $\Omega/W$ must have $2g + 2$ ramification points over $\Omega/\Gamma$. A point $p \in \Omega/W$, image of $\omega \in \Omega$, is a ramification if, and only if, $s_0 \in W \omega$. The points $a_0, b_0, \ldots, a_g, b_g$ satisfy this condition, and their images in $\Omega/\Gamma$ are different. So the equation follows.

### 6. COROLLARY

Let $s_0, \ldots, s_g \in \text{PGl}(2, k)$ be elements of order 2 such that the group $\Gamma$ generated by them satisfies: $\Gamma$ is discontinuous and

$$\Gamma = \langle s_0 \rangle \ast \langle s_1 \rangle \ast \cdots \ast \langle s_g \rangle.$$

Then there are elements $s_0^*, \ldots, s_g^*$ of order 2 in $\text{PGl}(2, k)$ with the $(2g + 2)$ fixed points in the position required in 3, and such that $\Gamma = \langle s_0^* \rangle, \ldots, \langle s_g^* \rangle$.

**Proof.** In 3, 4 and 5, the position of the $(2g + 2)$ fixed points of $\{s_0, \ldots, s_g\}$ is only used to prove that $\Gamma$ is discontinuous and equal to
\begin{align*}
\langle s_0 \rangle \times \cdots \times \langle s_g \rangle. \quad & \text{So we can also form } W = \langle s_0, s_1, s_0 s_2, \ldots, s_0 s_g \rangle \subset \Gamma \text{ and conclude that } \Omega/W \to \Omega = \mathbb{P}_1 \text{ has degree 2 and has } 2g + 2 \text{ ramification points, called } A_1, \ldots, A_{2g + 2}. \text{ Let } \sigma : \Omega/W \to \Omega/W \text{ be the automorphism of order 2 defined by } f. \text{ Then } A_1, \ldots, A_{2g + 2} \text{ are the fixed points of } \sigma.
\end{align*}

Write \( t_1 = s_0 s_1, \ldots, t_g = s_0 s_g \). Every element in } \Gamma \text{ of order 2 must have the form } a s_i a^{-1} (a \in \Gamma; i = 0, \ldots, g) \text{ (see [5]). Further } a \in \Gamma \text{ has the form } w s_0 \text{ or } w, \text{ with } w \in W. \text{ Since } s_0 s_i s_0 = t_i s_i t_i^{-1}, \text{ we find that every element in } \Gamma \text{ of order 2 has the form } w s_i w^{-1}, \text{ with } w \in W \text{ and } i \in \{0, \ldots, g\}. \text{ It is easily verified that this presentation is unique.}

Further } \Omega \to \Omega/W \text{ is a universal covering (see [4]). Hence for any } e, f \in \Omega \text{ with } \sigma(\pi(e)) = \pi(f), \text{ there exists a unique lifting } s : \sigma \to \Omega \text{ of } \sigma \text{ with } s(e) = f. \text{ Moreover } s \in \Gamma.

Take now } e \in \pi^{-1}(A_j) \text{ and a lifting } s \text{ of } \sigma \text{ with } s(e) = e. \text{ Then } s^2 = 1. \text{ Hence } s = w s_i w^{-1} \text{ for some } i \in \{0, \ldots, g\} \text{ and } w \in W. \text{ The } i \text{ does not depend on the choice of } e \in \pi^{-1}(A_j). \text{ Hence we have constructed a map }
\begin{align*}
\tau : \{A_1, \ldots, A_{2g + 2}\} & \to \{0, 1, \ldots, g\}.
\end{align*}

Further any } w s_i w^{-1} \text{ has at most two fixed points in } \pi^{-1}(\{A_1, \ldots, A_{2g + 2}\}). \text{ It follows that } \tau^{-1}(i) \text{ consists of at most two points. Hence } \tau \text{ is surjective and every } s_i \text{ has both fixed points in } \pi^{-1}(\{A_1, \ldots, A_{2g + 2}\}) \subset \Omega. \text{ The generators for } \Gamma \text{ can be changed into } s_0, t_1^2 s_1 t_1^2, s_2, \ldots, s_g. \text{ With a sequence of changes of this type one finds generators } s_0, \ldots, s_f \text{ for } \Gamma \text{ with their } (2g + 2) \text{ fixed points in the required position.}

7. THEOREM. - Suppose that } X \text{ is a hyperelliptic curve of genus } g \text{ over } k \text{ which is totally split. Then there exists a Whittaker group } W, \text{ unique up to conjugation in } \text{ PGL}(2, k), \text{ with } X \cong \Omega/W.

\textbf{Proof.} - \text{ We will use freely the results of [3] and [4]. We know that } \Omega \to \Omega/W \cong X \text{ exists where } W \text{ is a Schottky group of rank } g, \text{ unique up to conjugation. We have to show that } W \text{ is in fact a Whittaker group.}

Let } \sigma \text{ be the automorphism of } X \text{ with order two such that } \tau : X \to X/\sigma \cong \mathbb{P}_1. \text{ Then } \sigma \text{ has } A_1, \ldots, A_{2g + 2} \subset X \text{ as fixed points. Let } \Gamma \text{ denote the set of all lifts } s : \Omega \to \Omega \text{ of } \sigma : X \to X \text{ and of } \text{id} : X \to X. \text{ Then } \Gamma \text{ is a group and } W \text{ has index 2 in } \Gamma. \text{ The set }
\begin{align*}
K = \pi^{-1}(\{A_1, \ldots, A_{2g + 2}\}) \subset \mathbb{P}_1
\end{align*}
\text{ is a compact set with limit points } = \mathbb{P}_1 - \Omega = \text{ the limit points of } W = \text{ the limit points of } \Gamma. \text{ Let } \overline{\Omega} \text{ denote the reduction of } \Omega \text{ with respect to } K. \text{ Then}
\( \Omega/\Gamma \) is a reduction of \( \mathbb{P}^1 \) and it is in fact the reduction of \( \mathbb{P}^1 \) with respect to the finite set \( \{ \tau(A_1), \ldots, \tau(A_{2g+2}) \} \).

Let \( \overline{X} \) denote the reduction induced by \( \Omega \), i.e. \( \Omega \) is given with respect to a pure covering \( \mathcal{U} \), and \( \bar{X} \) is the reduction with respect to \( \pi(\mathcal{U}) \).

One easily sees that \( \overline{X} = \Omega/\mathcal{W} \) and consists of projective lines over the residue field \( k \). The intersection graph \( G(\overline{X}) \) is defined by:

- vertices = the components of \( \overline{X} \)
- edges = the intersection points.

The map \( \sigma \) induces an automorphism of \( \overline{X} \) and \( G(\overline{X}) \), again denoted by \( \sigma \).

Further \( \overline{X} \xrightarrow{\tau} \overline{X}/\sigma \cong \Omega/\Gamma \) and \( G(\overline{X})/\sigma \cong G(\overline{\Omega}/\Gamma) \) is a connected finite tree.

Through the image \( \overline{A}_1 \) of \( A_1 \) on \( \overline{X} \) goes only one component of \( \overline{X} \) since \( \overline{\tau}(A_1) \) lies on only one component of \( \overline{\Omega}/\Gamma \). Call this vertex of \( G(\overline{X}) \) the vertex \( e_1 \).

Then \( \sigma(e_1) = e_1 \) and the homeomorphism \( \sigma \) of \( G(\overline{X}) \) induces an automorphism \( \hat{\sigma} \) of \( \pi_1(\overline{X}, e_1) \) = the fundamental group of \( G(\overline{X}) \).

We know further that \( \pi_1(\overline{X}, e_1) \) is in a natural way isomorphic to \( \mathcal{W} \). Suppose that we can find a base for the fundamental group, \( t_1, \ldots, t_g \) such that \( \hat{\sigma}(t_i) = t_i^{-1} \) for all \( i \). Then we can lift this situation to \( \Omega \) as follows:

Choose an element \( e \in \pi_1(A_1) \); let \( s_0 \) be the lift of \( \sigma \) satisfying \( s_0(e) = e \); let \( h_0 \) be the component of \( \overline{\Omega} \) on which \( e \) lies; let the curve in \( G(\overline{\Omega}) \) with begin point \( h_0 \) and lying above \( A_1 \) have endpoint \( h \) \( \in \overline{G(\overline{\Omega})} \); let \( T_i \in \mathcal{W} \) be defined by \( T_i(e) \) lies on \( h \).

Then \( W = \langle T_1, \ldots, T_g \rangle \) and \( s_0 T_i s_0 = T_i^{-1} \) for all \( i \). Put \( s_1 = s_0 T_1, \ldots, s_g = s_0 T_g \).

Then \( \Gamma = \langle s_0, s_1, \ldots, s_g \rangle \) and easy inspection yields \( \Gamma = \langle s_0 \rangle * \langle s_1 \rangle * \ldots * \langle s_g \rangle \).

According to Corollary 6, we have shown that \( W \) is a Whittaker group.

Finally we have to show the following lemma:

8. LEMMA. - Let \( G \) be a finite connected graph with Betti number \( g \). Let \( \sigma \) be an homeomorphism of \( G \) such that:

(i) \( \sigma \) has order \( 2 \);

(ii) \( G/\sigma \) is a tree;

(iii) \( \sigma \) fixes a vertex \( p \in G \).

Then the fundamental group \( \pi_1(G, p) \) has generators \( t_1, \ldots, t_g \) such that the induced automorphism \( \hat{\sigma} \) of \( \pi_1(G, p) \) has the form \( \hat{\sigma}(t_i) = t_i^{-1} \) for all \( i \).

Proof. - Induction on the number of vertices of \( G \).
(1) p is the only vertex of G. Then G is a wedge of g circles. As generators for \( \pi_1 \) we take the g circles together with an orientation. Call them \( t_1, \ldots, t_g \). Since \( \sigma \) is an homeomorphism we must have
\[
\hat{\sigma}(t_i) \in \{t_1, \ldots, t_g, t_i^{-1} \ldots t_i^{-1}\}
\]
for all \( i \). Since \( G/\sigma \) has a trivial fundamental group, one finds that \( \hat{\sigma}(t_i) = t_i^{-1} \) for all \( i \).

(2) Induction step. Choose an edge \( \lambda \) of G with endpoints \( p \) and \( q \neq p \).
If \( \sigma(\lambda) = \lambda \) then we make a new graph \( G^* \) by identifying \( p \) and \( q \) and deleting the edge \( \lambda \).

If \( \sigma(\lambda) \neq \lambda \), but \( \sigma(\lambda) \) has also endpoints \( p \) and \( q \), then we make \( G^* \) by identifying \( p \) and \( q \) and also identifying \( \lambda \) on \( \sigma(\lambda) \).

If \( \sigma(\lambda) \) has endpoints \( p, r \) with \( r \neq q \), then we make \( G^* \) by identifying \( q \) and \( r \) with \( p \) and deleting \( \lambda \) and \( \sigma(\lambda) \).

In all cases, \( G^* \) is homotopic to \( G \); \( \sigma \) acts again on \( G^* \) and induces the same automorphism of the fundamental group.


1° An easy calculation gives that the number of moduli for Whittaker groups of rank \( g \) is \( 2g - 1 \). This is the same as the number of moduli for hyperelliptic curves of genus \( g \).

2° Is it possible to give an explicit calculation of the numbers \( F(a), F(b) \) in theorem 5?

3° Hyperelliptic curves and Whittaker groups in characteristic 2 will be treated by G. Van STEEN.

REFERENCES