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An application of Newton iteration procedure to $p$-adic differential equations


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AN APPLICATION OF NEWTON ITERATION PROCEDURE
TO p-ADIC DIFFERENTIAL EQUATIONS

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This report is based on the author's lectures at Strasbourg, Padova, Grenoble, Groningen and Paris. The motivations of this research were explained in the papers to appear ([3],[5]) and the lecture-notes [4] (joint with S. SPERBER). Therefore, in this paper, we will report only on the technical part.

1. Preliminaries.

Let \( K \) be a field of characteristic zero complete with respect to an absolute value \( |\cdot| \) which is non-trivial and ultrametric. The field of rational number, \( \mathbb{Q} \), is a subfield of \( K \), and we require that the restriction of \( |\cdot| \) to \( \mathbb{Q} \) is a \( p \)-adic absolute value for some prime number \( p \). We normalize \( |\cdot| \) so that \( |p| = 1/p \).

For \( \varphi = \sum_{m=0}^{\infty} a_m x^m \in K[[x]] \), we set

\[
|\varphi|_0(r) = \sup_{m \geq 0} |a_m| r^m.
\]

If \( |\varphi|_0(r_0) < +\infty \) for some positive constant \( r_0 \), then \( \varphi \) is convergent for \( |x| < r_0 \). The following lemma is fundamental throughout this report.

**Lemma 1.** Assume that \( \varphi_j = \sum_{m=0}^{\infty} a_{j,m} x^m \in K[[x]] \), \( j = 1, 2, \ldots \), with the properties:

(i) \( \lim_{j \to \infty} a_{j,m} = a_m \) exists for every \( m \);

(ii) \( |\varphi_j|_0(r) < M(r) \) for \( 0 < r < r_0 \), \( j = 1, 2, \ldots \), where \( r_0 \) is a positive number, and \( M(r) \) is a non-negative number which depends only on \( r \). Then, \( \varphi = \sum_{m=0}^{\infty} a_m x^m \) is convergent for \( |x| < r_0 \), and \( \lim_{j \to \infty} |\varphi_j - \varphi|_0(r) = 0 \) for \( 0 < r < r_0 \). (Cf. B. DWORK [1].)

2. An example (a rough sketch).

Let us consider a non-linear differential equation
(2.1) \[ x \frac{du}{dx} + au = f(x) + u^2 g(x, u), \]

where \( \alpha \in \mathbb{K}, f, g \in \mathbb{K}[[x]],\) and \( f, g \) are convergent. We want to find a convergent power series \( \varphi \in \mathbb{K}[[x]] \) which satisfies the equation (2.1). To do this, we try to construct \( \varphi \) in the following form

(2.2) \[ u = \varphi = \sum_{j=0}^{\infty} \varphi_j, \quad \varphi_j \in \mathbb{K}[[x]]. \]

**Step 1.** First of all, \( \varphi_0 \) is determined by the linear differential equation

(2.3) \[ x \frac{d\varphi_0}{dx} + \alpha \varphi_0 = f. \]

**Step 2.** Change \( u \) by \( u = \varphi_0 + v. \) Then (2.1) becomes

(2.1') \[ x \frac{dv}{dx} + \alpha v = \varphi_0(x)^2 g(x, \varphi_0(x)) + \varphi_0(x) G(x) v + v^2 g_1(x, v), \]

where

\[
\varphi_0(x) = 2g(x, \varphi_0(x)) + \varphi_0(x) g_u(x, \varphi_0(x)) \quad (g_u = \partial g/\partial u),
\]

\[
v^2 g_1(x, v) = \varphi_0(x)^2 [g(x, \varphi_0(x) + v) - g(x, \varphi_0(x)) - g_u(x, \varphi_0(x)) v] + 2 \varphi_0(x) v[g(x, \varphi_0(x) + v) - g(x, \varphi_0(x))] + v^2 g(x, \varphi_0(x) + v).
\]

We determine \( \varphi_1 \) by the linear part of (2.1')

(2.4) \[ x \frac{d\varphi_1}{dx} + \alpha \varphi_1 = \varphi_0^2 g(x, \varphi_0) + \varphi_0 G(x) \varphi_1. \]

The other \( \varphi_j \) will be determined successively in a similar manner.

This is our Newton iteration procedure.

A closer look at equation (2.3). If \( f = \sum_{m=0}^{\infty} c_m x^m \) \((c_m \in \mathbb{K})\), then \( \varphi_0 \) is given by

(2.5) \[ \varphi_0 = \sum_{m=0}^{\infty} \frac{c_m}{m + \alpha} x^m. \]

Assuming that \( |f|_0^0(r) \leq M \) for \( 0 \leq r < r_0 \), where \( r_0 \) and \( M \) are some positive numbers, we want to derive

(2.6) \[ |\varphi_0|_0^0(r) \leq M \text{ for } 0 \leq r < r'_0, \]

for \( r'_0 \) a positive number, as large as possible, such that \( 0 < r'_0 < r_0 \). To do this, we introduce two assumptions

(2.7) \[ c_m = 0 \text{ for } m < m_0, \]

(2.8) \[ |m + \alpha|^{-1} \leq c_m^{m-\delta} \quad (m \geq m_0), \]

where \( m_0 \) is a positive integer, \( C \) is a positive number greater than one, and \( \delta \) is a positive number smaller than one, i.e. \( C > 1, 0 < \delta < 1 \).

The assumption (2.8) may be called "non-Liouville property" of the exponent \( \alpha \). The condition (2.7) may be written

(2.7') \[ f = 0 \pmod{x^{m_0}}. \]
Note that, if equation (2.1) admits a formal power series solution, then, we can change (2.1) so that condition (2.7) may be satisfied for any prescribed \( m_0 \). Also note that any algebraic number \( \alpha \) satisfies condition (2.3) for any \( \delta \) if we choose \( C \) and \( m_0 \) suitably.

Under assumption (2.3), set

\[
(2.5) \quad \rho_0 = (1/C)^{m_0^{-\delta}}
\]

Then \( 0 < \rho_0 < 1 \), and

\[
\rho_0^m = (\rho_0^m)^{1-\delta} = (C^{-m/m_0})^{1-\delta} \leq (1/C)^{m_0^{-\delta}} \leq |m + \alpha| \quad \text{if} \quad m \geq m_0.
\]

Hence, under assumptions (2.7) and (2.8), we have

\[
|\varphi_0(\rho_0)| \leq \sup_{m \geq m_0} |m + \alpha|^{-1} \leq \sup_{m \geq m_0} \left| c_m \right| \rho_0^m \leq \sup_{m \geq m_0} \left| c_m \right| r^m = |f|_0(r),
\]

and

\[
(2.6') \quad |\varphi_0|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0.
\]

Equation (2.4) without \( \varphi_0 \) \( g(t) \varphi_1 \). To simplify the explanation, we remove \( \varphi_0 \) \( g(x) \varphi_1 \) from the right-hand member of equation (2.4); i.e., we consider the equation

\[
(2.10) \quad x \varphi_1 dx + \varphi_1 = \varphi_0 g(x, \varphi_0).
\]

We know already that

\[
(2.11) \quad \varphi_0 \equiv 0 \pmod{x^{m_0}},
\]

and that \( \varphi_0 \) satisfies (2.6'). First of all, (2.11) implies that

\[
(2.12) \quad \varphi_0^2 g(\cdot, \varphi_0) \equiv 0 \pmod{x^{2m_0}}.
\]

Hence, if we assume that \( g \) satisfies the condition

\[
(2.13) \quad \left| \varphi_0^2 g(\cdot, \varphi_0) \right|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0,
\]

we have

\[
(2.14) \quad \left\{ \begin{array}{l}
\varphi_1 \equiv 0 \pmod{x^{2m_0}}, \\
|\varphi_1|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1,
\end{array} \right.
\]

where \( \rho_1 = (1/C)^{(2m_0)^{-\delta}} = \rho_0^{-\delta} \).

Suppose that, proceeding inductively as above, we have defined for all \( j \geq 0 \),

\[
\left\{ \begin{array}{l}
\varphi_j \equiv 0 \pmod{x^{2m_0}}, \\
|\varphi_j|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1, \ldots, \rho_j,
\end{array} \right.
\]

and

\[
(2.15) \quad \left| \varphi_j \right|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1, \ldots, \rho_j,
\]

we have

\[
(2.16) \quad \left| \varphi_{j+1} \right|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1, \ldots, \rho_j.
\]

Suppose that, proceeding inductively as above, we have defined for all \( j \geq 0 \),

\[
\left\{ \begin{array}{l}
\varphi_j \equiv 0 \pmod{x^{2m_0}}, \\
|\varphi_j|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1, \ldots, \rho_j.
\end{array} \right.
\]

and

\[
(2.17) \quad \left| \varphi_j \right|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1, \ldots, \rho_j,
\]

we have

\[
(2.18) \quad \left| \varphi_{j+1} \right|_0(r) \leq M \quad \text{for} \quad 0 \leq r < \rho_0 \rho_0 \rho_1, \ldots, \rho_j.
\]
where \( \rho_j = \rho_{j-1}^{2^{-\delta}} = \rho_0^{2^{-j\delta}} \); set

\[ \psi_j = \sum_{j=0}^{\infty} \rho_j^2 , \quad \rho_\infty = \left( \sum_{j=0}^{\infty} \rho_j \right)^{1/2} = \rho_0^{(1-2^{-\delta})^{-1}} > 0 . \]

Then \( |\psi_j|_0 < M \) for \( 0 < r < r_0 \rho_\infty \), and \( \psi_j \) converges x-adically to

\[ \rho = \sum_{j=0}^{\infty} \rho_j . \]

Therefore, by virtue of lemma 1, we conclude that \( \rho \) is convergent for \( |x| < r_0 \rho_\infty \).

The argument of this section is not strictly speaking correct, since we removed \( \rho_0 G(x) \rho_1 \) from the right-hand member of equation (2.4). A correct treatment of equation (2.1) is given in SIBUYA-SPERBER ([2], [4]).

3. Typical results.

In this section, we shall give a rigorous treatment of a problem which is more general than the problem of section 2. We assume that \( K \) contains an element \( \pi \) such that

\[ |\pi| = \left( \frac{1}{p} \right)^{\psi-1} . \]

We consider the following situation.

(i) We are given \( \alpha_1 , \ldots , \alpha_n \in K \) such that

\[ |\alpha_j| < 1 , \quad |m + \alpha_j|^{-1} \leq C m^{1-\delta} \quad \text{and} \quad |m + \alpha_1 - \alpha_j|^{-1} \leq C m^{1-\delta} . \]

for \( m \geq 2^k \) and \( i, j = 1, \ldots , n \), where \( k \) is a non-negative integer, and \( C \) and \( \delta \) are positive numbers such that \( C > 1 \), \( 0 < \delta < 1 \).

(ii) We are also given \( a_1 , \ldots , a_n \in K[[X]] \) such that

\[ a_j \equiv 0 \pmod{X} , \quad \left| \int_0^{r_0} a_j(t) \, dt \right|_0 < |\pi| , \]

for \( 0 < r < r_0 \) and \( j = 1, \ldots , n \), where \( r_0 \) is a positive number, and where, for \( a = \sum_{m=1}^{\infty} a_m X^m \), we have denoted \( \sum_{m=1}^{\infty} (a_m / m) X^m \) by \( \int_0^{r_0} a(t) \, dt \).

We define two sequences of numbers, \( \{ \sigma_n \} \) and \( \{ \tau_n \} \) by

\[ \begin{cases} \sigma_1 = 1/C , & \tau_1 = (1/C)^{2(1-2^{-\delta})^{-1}} \\ \sigma_n = \sigma_{n-1}^2 \tau_{n-1} , & \tau_n = (\sigma_n \sigma_1)^{2(1-2^{-\delta})^{-1}} . \end{cases} \]

Note that

\[ 0 < \tau_n < \sigma_n < \tau_{n-1} < 1 . \]

In this section, we shall prove the following two theorems.
**THEOREM 1.** Assume that a differential operator $H = \sum_{j=0}^{n-1} b_j(x) \partial^j$ ($\partial = xd/dx$) satisfies the following conditions:

\[
\begin{cases}
  b_j \in K[[x]] \text{ and } b_j \equiv 0 \pmod{x^k}, \\
  |b_j|_0(r) < |\pi| \text{ for } 0 \leq r < r_0.
\end{cases}
\]

Then, there exists $\eta_1, \ldots, \eta_n \in K[[x]]$ such that

\[
\begin{cases}
  \eta_j \equiv 0 \pmod{x^k}, \\
  |\int_0^r t^{n-1} \eta_j(t) \, dt|_0(r) < |\pi| \text{ for } 0 \leq r < r_0 \gamma_n^{-k_5}, \ j = 1, \ldots, n,
\end{cases}
\]

and that

\[
(3.3) \quad (\partial + \alpha_1 + a_1) \ldots (\partial + \alpha_n + a_n) - H = (\partial + \alpha_1 + a_1 - \eta_1) \ldots (\partial + \alpha_n + a_n - \eta_n).
\]

**THEOREM 2.** Assume that

\[
f \in K[[x]], \ f \equiv 0 \pmod{x^k}, \ |f|_0(r) < 1 \text{ for } 0 \leq r < r_0,
\]

and that

\[
G = \sum_{\mu_0+\cdots+\mu_{n-1} \geq 2} g_{\mu_0^{\mu_0} \cdots \mu_{n-1}}^{\mu_0^{\mu_0} \cdots \mu_{n-1}}(x) v_0^{\mu_0} \cdots v_{n-1}^{\mu_{n-1}} \in K[[x, v_0, \ldots, v_{n-1}]],
\]

with $g_{\mu_0^{\mu_0} \cdots \mu_{n-1}}^{\mu_0^{\mu_0} \cdots \mu_{n-1}}(x) \in K[[x]]$,

\[
|g_{\mu_0^{\mu_0} \cdots \mu_{n-1}}^{\mu_0^{\mu_0} \cdots \mu_{n-1}}|_0(r) \leq |\pi| \text{ for } 0 \leq r < r_0.
\]

Then, there exists a unique $\phi \in K[[x]]$ such that

\[
(3.11) \quad \phi \equiv 0 \pmod{x^k},
\]

and that

\[
(3.12) \quad (\partial + \alpha_1 + a_1) \ldots (\partial + \alpha_n + a_n)(\phi) = f + G(x, \phi, \partial \phi, \ldots, \partial^{n-1} \phi).
\]

Furthermore, this power series $\phi$ also satisfies the condition

\[
(3.13) \quad |\phi|_0(r) < 1 \text{ for } 0 \leq r < r_0 \gamma_n^{-k_5}.
\]

**Remark 1.** The power series $\phi$ is a solution of a non-linear differential equation with purely Fuchsian linear part. This is a prototype of the most difficult situations in the study of $p$-adic non-linear problems. The most important part of theorem 2 is the estimate (3.13), i.e., the $r$-interval in which $|\phi|_0(r) < 1$ holds.
Remark 2. - Theorem 1 is a Hensel-type lemma. The problem of factorization of a linear differential operator is naturally reduced to a non-linear problem such as that of theorem 2. For example, if the order of the operator is two, the corresponding non-linear problem is a Riccati equation. In general, if the order of the operator is \( n \), the order of the corresponding non-linear problem is \( n - 1 \). Taking advantage of this situation, we can prove theorem 1 and 2 simultaneously by an induction on \( n \). Since the case \( n = 1 \) was treated in SIBUYA-SPERBER [2], we shall prove these theorems for \( n \geq 2 \). (Cf. also SIBUYA-SPERBER [4].)

4. Proof of theorem 1 for \( n \).

In this section, assuming theorem 2 for \( n - 1 \), theorem 1 for \( n = 1 \), and theorem 1 for \( n - 1 \), we shall prove theorem 1 for \( n \). Set

\[
\begin{align*}
L &= (\partial + a_1 + a_n) \cdots (\partial + a_{n-1} + a_n), \\
\eta &= \dot{\eta} + a_n.
\end{align*}
\]

We want to find \( \eta \in K[[x]] \) and \( \tilde{L} = \sum_{j=0}^{n-2} y_j \sigma^j \) \((y_j \in K[[x]])\) such that

\[
(4.2)
L\dot{\eta} - H = (L - \tilde{L})(\dot{\eta} - \eta).
\]

The relation (4.2) is equivalent to the assertion that

\[
(4.2')
L\dot{\eta}(u) - H(u) = 0
\]

for all \( u \) belonging to a sufficiently large extension of \( K[[x]] \) such that

\[
(\dot{\eta} - \eta)(u) = 0.
\]

Therefore, (4.2) is equivalent to the assertion that

\[
(4.3)
L(u\eta) = H(u) \text{ for all such } u \text{ satisfying } \dot{\eta}(u) = u\eta.
\]

Observe that

\[
(\partial + \alpha_j + a_j)(uv) = u(\partial + (\alpha_j - \alpha_n) + (a_j - a_n) + \eta)(v),
\]

if \( \dot{\eta}(u) = u\eta \). Hence

\[
(4.4)
L(u\eta) = u(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) + \eta) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) + \eta)(\eta),
\]

if \( \dot{\eta}(u) = u\eta \). We can write

\[
(4.4')
(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) + \eta) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) + \eta)(\eta)
= (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(\eta)
- F(x, \eta, \cdots, \sigma^{n-2} \eta),
\]

where \( F \) is a function of \( x, \eta, \cdots, \sigma^{n-2} \eta \).
where

\[ \tilde{F} = \sum_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}} \tilde{F}_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}}(x)^{\mu_0} \cdots \mu_{n-2} \in K[[x]][v_0, \ldots, v_{n-2}], \]

On the other hand, if \( u = \mu \), we have

\[ \exists \nu = u(\alpha_n - \alpha_n + \eta), \quad \delta^2 u = u \{(-\alpha_n - \alpha_n + \eta)^2 + \delta(-\alpha_n - \alpha_n + \gamma)\}, \quad \text{etc.} \]

Hence, \( H(u) \) has the following form

\[ (4.5) \quad H(u) = uF(x, \eta, \ldots, \delta^{n-2} \eta), \]

where

\[ F = \sum_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}} F_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}}(x)^{\mu_0} \cdots \mu_{n-2} \in K[[x]][v_0, \ldots, v_{n-2}], \]

\[ F_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}} \in K[[x]], \quad F^{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}} = 0 \quad (\text{mod} \ x^k), \]

\[ |F^{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}}(r)| < 4^{2} \quad \text{for} \quad 0 \leq r < r_0. \]

Thus, we derive from \( (4.3) \) the equation for \( \eta \):

\[ (\delta + (\alpha_1 - \alpha_n) + (\alpha_1 - \alpha_n)) \cdots (\delta + (\alpha_{n-1} - \alpha_n) + (\alpha_{n-1} - \alpha_n))(\eta) = F + \tilde{F}. \]

Set \( \eta = \pi w \), and \( \tilde{f}(x) = F_{\alpha=0}(x), \quad \tilde{H} = \sum_{j=0}^{n-2} b_j(x) \eta^j \), where

\[ \sum_{j=0}^{n-2} b_j(x) v_j = \sum_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}} F_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}}(x)^{\mu_0} \cdots \mu_{n-2}, \]

and

\[ \tilde{G}(x, v_0, \ldots, v_{n-2}) = \sum_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}} \{F_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}}(x) + \tilde{F}_{\mu_0^{\cdot}\mu_2^{\cdot}\mu_{n-2}^{\cdot}}(x)\}^{\mu_0} \cdots \mu_{n-2}. \]

Then the equation for \( w \) is given by

\[ (4.5) \quad (\delta + (\alpha_1 - \alpha_n) + (\alpha_1 - \alpha_n)) \cdots (\delta + (\alpha_{n-1} - \alpha_n) + (\alpha_{n-1} - \alpha_n))(w) = (1/\eta) \tilde{f} + \tilde{H}(w) + (1/\eta) \tilde{G}(x, \pi w, \pi^2 w, \ldots, \pi^{n-2} w) \]

Utilizing theorem 1 for \( n - 1 \), we find \( \tau_1, \ldots, \tau_{n-1} \in K[[x]] \) such that
\[ \eta_j = 0 \pmod{x^2^k}, \quad |\int_0^t \eta_j(t) \, dt|_0 < |\eta| \quad \text{for} \quad 0 < r < r_0 (\sigma_{n-1} \tau_{n-1})^{2^{-k_0}}, \]

and that
\[
\begin{align*}
& (\alpha + (\alpha_1 - \alpha_n) + (\alpha_1 - \alpha_n)) \cdots (\alpha + (\alpha_{n-1} - \alpha_n) + (\alpha_{n-1} - \alpha_n)) - \eta \\
& = (\alpha + (\alpha_1 - \alpha_n) + (\alpha_1 - \alpha_n) - \eta) \cdots (\alpha + (\alpha_{n-1} - \alpha_n) + (\alpha_{n-1} - \alpha_n) - \eta) - \eta_{n-1}.
\end{align*}
\]

Then, applying to (4.6) theorem 2 for \( n - 1 \), we find a unique solution \( w(x) \) such that
\[
\begin{align*}
& \begin{cases}
\psi = 0 \pmod{x^2^k}, \\
|\psi|_0(r) < 1 \quad \text{for} \quad 0 < r < r_0 (\sigma_{n-1} \tau_{n-1})^{2^{-k_0}}.
\end{cases}
\end{align*}
\]

Thus, we constructed \( \eta \) so that (4.3) is satisfied and
\[
\begin{align*}
& \begin{cases}
\eta = 0 \pmod{x^2^k}, \\
|\eta|_0(r) < |\eta| \quad \text{for} \quad 0 < r < r_0 (\sigma_{n-1} \tau_{n-1})^{2^{-k_0}}.
\end{cases}
\end{align*}
\]

To compute \( \tilde{L} \), we derive \( \tilde{L}(\xi - \eta) = H - L\eta \). Putting
\[
H - L\eta = \sum_{j=0}^{n-1} \hat{b}_j(x) \cdot b^j, \quad b_j \in K[[x]],
\]
we get
\[
\begin{align*}
& \begin{cases}
\hat{b}_j = 0 \pmod{x^2^k}, \\
|\hat{b}_j|_0(r) < |\eta| \quad \text{for} \quad 0 < r < r_0 (\sigma_{n-1} \tau_{n-1})^{2^{-k_0}}.
\end{cases}
\end{align*}
\]

Furthermore,
\[
(4.3) \quad Y_{n-2} = \hat{b}_{n-1}, \quad Y_\mu = \hat{b}_{\mu+1} - \sum_{j=\mu+1}^{n-2} f_{j,\mu+1} Y_j, \quad \mu = 0, \ldots, n - 3,
\]
where \( f_{j,\mu} \in K[[x]] \), and
\[
|f_{j,\mu}|_0(r) \leq 1 \quad \text{for} \quad 0 < r < r_0 (\sigma_{n-1} \tau_{n-1})^{2^{-k_0}}.
\]

Finally, applying to \( L - \tilde{L} \) theorem 1 for \( n - 1 \), and to \( \xi - \eta \) theorem 1 for \( n = 1 \), and utilizing the inequality \( \sigma_{n-1} < \sigma_1 \), we complete the proof.

5. Proof of theorem 2 for \( n \).

In this section, assuming theorem 1 for \( n \), and theorem 2 for \( n = 1 \), we shall prove theorem 2 for \( n \). Setting
\[
(5.1) \quad \psi_j = \sum_{\lambda=0}^j \psi_{\lambda} \quad \psi_j = \psi_{j-1} + \psi_j,
\]
we determine \( \psi_j \in K[[x]] \) by

\[
\tilde{\psi}_j = 0 \pmod{x^2^k}, \quad |\tilde{\psi}_j|_0(r) < |\psi| \quad \text{for} \quad 0 < r < r_0 (\sigma_{n-1} \tau_{n-1})^{2^{-k_0}}.
\]
(5.2) \((\alpha + \alpha_1 + a_1) \cdots (\alpha + \alpha_n + a_n) (\psi_j)\)

\[= f + \varrho(x, \psi_{j-1}, \partial \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) + \sum_{i=0}^{n-1} G_{vi}(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) \partial^i \psi_j ,\]

where \(G_{vi} = \partial G/\partial v_1\). This means that the \(\phi_j\) are determined by linear differential equations:

(5.3) \(L_j(\phi_j) = f_j \quad (j = 0, 1, \ldots)\),

where

(5.4) \[
\begin{align*}
L_0 &= (\alpha + \alpha_1 + a_1) \cdots (\alpha + \alpha_n + a_n), \\
L_j &= L_0 - \sum_{i=0}^{n-1} G_{vi}(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) \partial^i (j \geq 1)
\end{align*}
\]

(5.5) \[
\begin{align*}
f_0 &= f \\
f_j &= G(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) - G(x, \psi_{j-2}, \ldots, \partial^{n-1} \psi_{j-2}) - \sum_{i=0}^{n-1} G_{vi}(x, \psi_{j-2}, \ldots, \partial^{n-1} \psi_{j-2}) \partial^i \psi_{j-1}, \quad (j \geq 1)
\end{align*}
\]

where \(\psi_j = 0\) if \(\lambda < 0\).

We want to construct the \(\phi_j\) so that

(5.6) \[
\begin{align*}
\phi_j &= \equiv 0 \quad (\bmod x^{k+j}) \\
|\phi_j|_0(r) &< 1 \quad \text{for} \quad 0 < r < r_0 \sigma \prod_{k=0}^{j-1} (\sigma_1 \cdots \sigma_n)^2(k+j) \delta
\end{align*}
\]

To do this, set

(5.7) \(L_j = L_{j-1} - H_j \quad (j \geq 1)\),

where by (5.4)

(5.8) \[
H_j = \sum_{i=0}^{n-1} G_{vi}(x, \psi_{j-1}, \ldots, \partial^{n-1} \psi_{j-1}) - G_{vi}(x, \psi_{j-2}, \ldots, \partial^{n-1} \psi_{j-2}) \partial^i \psi_{j-1}.
\]

Using an induction on \(j\), we can achieve a factorization of \(L_j\) into linear factors, by virtue of theorem 1 for \(n\), if

\(|x| < r_0 \prod_{k=0}^{j-1} (\sigma_1 \cdots \sigma_n)^2(k+j) \delta
\]

Then, by using theorem 2 for \(n = 1\) (n-times), we can achieve (5.6).
Thus, we get
\[ |\psi_j|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0 \tau_n^{-k_5}, \quad j = 0, 1, \ldots, \]
and \( \gamma_j \) converges \( x \)-adically to \( \gamma = \sum_{z=0}^{\infty} c_z \). Hence, by lemma 1 of section 1,
\[ |\gamma|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0 \tau_n^{-k_5}. \]

Finally, letting \( j \) tend to infinity on both sides of (5.2), we complete the proof.

Results for more general cases, applications, and treatments of systems of differential equations were given in SIBUYA–SPERBER ([3],[4]).

REFERENCES


