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About $p$-adic interpolation of continuous and differentiable functions


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About p-adic Interpolation of Continuous and Differentiable Functions

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0. Introduction.

In 1958, Mahler proved that \( \{ (\frac{x}{p^n}); n \in \mathbb{N} \} \) form a normal base for \( C(\mathbb{Z}, \mathbb{Q}) \). Since then, a number of different proofs of this theorem were given (cf. [1], [2], [4], [5], [8]).

In section 1, we show that the method used by Yvette Amice [1] can be generalized to prove that \( \{ (\frac{x}{p^n})^s; n \in \mathbb{N} \} \) form a normal base, for each \( s \in \mathbb{N}^* \). This leads to a generalization of Mahler's formula (1.2). It is a remarkable fact that some polynomials (e.g., \( x \)) get an infinite expansion. So the linear space spanned by the \( (\frac{x}{p^n})^2 \) lays dense in \( C(\mathbb{Z}, \mathbb{Q}) \); however, it does not lay dense in \( C^1(\mathbb{Z}, \mathbb{Q}) \).

In section 2, we prove that there exist polynomials \( R_n \), with \( \deg R_n = 2n + 1 \), such that the polynomials \( R_n (\frac{x}{p^n})^2 \) together with the \( R_n \) form a normal base of \( C^1(\mathbb{Z}, \mathbb{Q}) \). A close relation with van der Put's base, consisting of locally constant and locally linear functions should be noted.

1. Normal bases for \( C(\mathbb{Z}, \mathbb{Q}) \).

1.1 Theorem. - For each \( s \in \mathbb{N}^* \), \( \{ q_n = (\frac{x}{p^n})^s; n \in \mathbb{N} \} \) form a normal base of \( E = C(\mathbb{Z}, \mathbb{Q}) \).

Proof. - In view of [2] no 3.1.5, or [7] lemme 1, it is sufficient to prove that \( \{ q_n; n \in \mathbb{N} \} \) form a vectorial base of \( E = C(\mathbb{Z}, \mathbb{Q}) \). Let \( E_h \) be the space of \( \mathbb{F} \)-valued functions constant on each ball

\[ B_{p^{-h}}(a) = \{ x \in \mathbb{Z}; \ |x - a| < p^{-h} \} \]

Since \( E = \bigcup E_h \), our proof will be finished if we can show that \( \{ q_i; i < p^h \} \) form a base of \( E_h \).

For \( i < p^h \) and \( |x - y| < p^{-h} \), we have

\[ |(\frac{x}{p^i}) - (\frac{y}{p^i})| < 1 \]

([2], 3.2.2.3).

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hence
\[(x)_i^s - (y)_i^s = \frac{(x)_i^s - (y)_i^s}{\sum_{m=0}^{s} x_i^m (y)_i^{s-m-1}} < 1,\]
so \(\bar{q}_1(x) = \bar{q}_1(y)\). It follows that \(\bar{q}_1 \in \mathbb{E}_h\), and
\[
\bar{q}_1 = \sum_{j=1}^{p^h-1} \bar{q}_1(j) x_j.
\]
So the transition matrix form \(\{x_j; i < p^h\}\) to \(\{\bar{q}_i; i < p^h\}\) is triangular; the desired result follows.

1.2 COROLLARY. - Let \(s \in \mathbb{N}^+\). Each continuous \(f: \mathbb{Z}_p \rightarrow \mathbb{Q}\) can be written as a uniformly convergent series
\[f(x) = \sum_{n=0}^{\infty} a_n(s)_n x_s\]
where
\[a_n(s) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{n-k}(s) f(k)\]
and
\[b_0(s) = 1, b_m(s) = \frac{\sum_{i_1=1}^{m} \ldots \sum_{i_r=1}^{m}}{\sum_{i_1=1}^{m} \ldots \sum_{i_r=1}^{m}} (-1)^{r+m} \binom{m}{k_1 \ldots k_r} x_s^{r+m}.\]

Proof. - We have to calculate the interpolation coefficients \(a_n(s)\). They are determined by the formulas
\[a_n(s) = f(0), a_n(s) = f(n) - \sum_{i=0}^{n-1} a_i(s) q_i(n).\]

We prove the formula using induction on \(n\). Suppose true for \(n < N\), then we have:
\[a_{N+1}(s) = f(N+1) - \sum_{n=0}^{N} a_n(s)_n (N+1)_s\]
\[= f(N+1) - \sum_{n=0}^{N} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_{n-k}(s) f(k) \binom{N+1}{n}_s\]
\[= f(N+1) - \sum_{k=0}^{N} \sum_{n=k}^{N} (-1)^{n-k} f(k) \sum_{i=1}^{r} \sum_{i_1+i_2=\ldots+i_r=k} (-1)^{r+n-k} \binom{n-k}{k} s_{i_1} \ldots s_{i_r} \binom{N+1}{n}_s.\]
Putting \(z_{r+1} = N + 1 - n\), we get
\[ a_{N+1} = f(N+1) + \sum_{k=0}^{\infty} f(k) \sum_{i=1}^{r} (-1)^{r-i} \binom{N+1-k}{i} \binom{N+1}{k} \]

\[ = f(N+1) + \sum_{k=0}^{\infty} f(k) (-1)^{N+1-k} \binom{s}{k} (N+1)^{s} , \]

this finishes the proof.

1.3 Note. - We can write down explicit formulas for the \( \beta_m(s) \):

\[ \beta_0(s) = \beta_1(s) = 1 , \]

\[ \beta_2(s) = 2^5 - 1 , \]

\[ \beta_3(s) = 6^5 - 2 \cdot 3^5 + 1 , \]

\[ \beta_4(s) = 24^5 - 3 \cdot 12^5 + 6^5 + 2 \cdot 4^5 - 1 . \]

It is easy to tabulate the \( \beta_m(s) \):

<table>
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<td>4</td>
<td>1</td>
<td>211</td>
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<td>271375</td>
</tr>
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</table>

1.4 Note. - Comparing the case \( s = 1 \) with Mahler's formula, we get the following arithmetic formula:

\[ \beta_m^{(1)} = \left( \prod_{i=1}^{m} \frac{z_i}{z_i} \right) \binom{m}{\sum_{i=1}^{m} \frac{z_i}{z_i}} = 1 . \]

1.5 Note (due to L. VAN HAMME). - One can determine the \( \beta_m(s) \) also, by using generating functions. One has the following identity between formal power series:

\[ \left( \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{s} \right) \left( \sum_{n=0}^{\infty} (-1)^n \frac{b_n}{s} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{s} , \]

if, and only if, \( c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \). Put \( b_n = 1 \), \( c_n = f(n) \), \( a_n = a_n(s) \).

Then it follows that
\( f(n) = \sum_{k=0}^{\infty} \binom{n}{k}^s a_k \)

if, and only if,

\[
\sum_{n=0}^{\infty} (-1)^n a(s) \frac{z^n}{(n!)^s} = \sum_{n=0}^{\infty} \frac{\beta_n(s) z^n}{(n!)^s} \left( \sum_{n=0}^{\infty} (-1)^n f(n) \frac{z^n}{(n!)^s} \right) \]

if, and only if,

\[
a_n(s) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \beta_{n-k}(s) f(k),
\]

where

\[
\sum_{n=0}^{\infty} \frac{\beta_n(s) z^n}{(n!)^s} = \sum_{n=0}^{\infty} \frac{1}{(-1)^n \frac{z^n}{(n!)^s}}
\]

this last condition determines the \( \beta_n(s) \).

1.6 Note. - Applying corollary 1.2, we can obtain a lot of \( p \)-adically convergent series, e.g.

\[
x = x^2 - \frac{1}{2} x^2 (x - 1)^2 + \frac{1}{3} x^2 (x - 1)^2 (x - 2)^2 + \ldots
\]

\[
= x^3 - \frac{3}{4} x^3 (x - 1)^3 + \frac{23}{36} x^3 (x - 1)^3 (x - 2)^3 + \ldots
\]

It is a remarkable fact that these series converge \( p \)-adically for each prime number \( p \). Also note that derivation of the series yields an apparent contradiction after putting \( x = 0 \); this shows that the series do no converge in \( C^1(\mathbb{Z}_p, \mathbb{Q}_p) \)-norm. The same phenomenon happens with Van der Put's base. We return to this problem in section 2.

1.7 Note. - The proof of theorem 1.1 is merely based on the proof of Mahler's theorem as given in [2].

One could try to adapt the proof given by BOJANIC [4], MAHLER [5] or VAN ROOY [8] to prove the theorem; however, it seems that these kinds of argument do not work here.

We can generalise theorem 1.1 if we replace \( \mathbb{Z}_p \) by regular compact part \( M \) of a local field \( K \) and the interpolation sequence \( \mathbb{N} \) by a very well distributed sequence \( u : \mathbb{N} \rightarrow M \). For more details about very well distributed sequences, we refer to the work of Yvette AMICE [1]. We denote, following the notations in [1],

\[
P_n(X) = (X - u_0)(X - u_1) \ldots (X - u_{n-1}), \quad Q_n(X) = P_n(X)/P_{n}(u_n).
\]

Given \( \alpha_1, \alpha_2, \ldots \) in \( K \), with \( |\alpha_i| \leq 1 \) and \( \sum_{i=1}^{\infty} \alpha_i = 1 \), we define \( q_n = \sum_{i=1}^{\infty} \alpha_i Q_n^i \). We omit the proof of the following theorem, since it is merely the same as the proof of theorem 1 in [1], up to one modification as in theorem 1.1.
1.8 PROPOSITION. - If $u : \mathbb{N} \to M$ is a very well distributed sequence in a regular compact part $M$ of the local field $K$, and the $q_n$ are defined as above, then $\{q_n ; n \in \mathbb{N}\}$ form a normal base of $C(M, K)$.

2. A normal base for $C^1(\mathbb{Z}_p, \mathbb{Q}_p)$.

For details about $p$-adic differentiability, we refer to [6]. Recall that a function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is called $C^1$ (or continuously differentiable) if the difference quotient $\frac{f(x)}{x}$ defined by

$$\frac{f(x, y)}{x} = \frac{f(x) - f(y)}{x - y}$$

can be extended to a continuous function $\frac{f}{x}$ on $\mathbb{Z}_p^2$. The space of $C^1$-functions becomes the Banach space $C^1 = C^1(\mathbb{Z}_p, \mathbb{Q}_p)$ under the norm

$$||f||_1 = \max(|f(0)|, \sup(|\frac{f(x, y)}{x}| ; x \neq y}).$$

It is known ([3], [6], [9]) that the following sets form normal bases for $C^1$:

$$\{\gamma_n (\frac{x}{n}) ; n \in \mathbb{N}\} \quad \text{(Mahler's base)}$$

$$\{\gamma_n (x) ; n \in \mathbb{N}\} \cup \{\gamma_n (x)(x - n) ; n \in \mathbb{N}\} \quad \text{(Van der Put's bases)}$$

We remind of the fact that $\gamma_n$ is defined by

$$\gamma_0 = 1$$

$$\gamma_n = a_s p^s \text{ if } n = a_s p^s + a_{s-1} p^{s-1} + \cdots + a_0, a_s \neq 0.$$  

So $\nu(\gamma_n) = s$, and $|\gamma_n|_{-1} = \max(|m|^{-1} ; 0 < m < n)}$.

$\gamma_n$ is the characteristic function of $\{x ; |x - n| < |\gamma_n| \}$.

Define $R_n = \gamma_n (\frac{x}{n})^2$; it will then follow from lemma 2.2 that $||R_n||_1 = 1$; however, the $R_n$ do not form a base for $C^1$, as we already know from 1.6. Can we choose polynomials $\tilde{R}_n$ such that $\deg \tilde{R}_n = 2n + 1$ and the $\tilde{R}_{n-1}$ form a normal base for $C^1$? Inspired by Van der Put's base, we could try $\tilde{R}_n = R_n (x - n)$.

After normalisation, we get $\tilde{R}_n = \gamma_{n+1} (\frac{x}{n+1})$. Unfortunately, it turns out that $\{R_n, \tilde{R}_n ; n \in \mathbb{N}\}$ are not orthogonal in $C^1$. This comes from the fact that

$$|\tilde{R}_n'(n)| = |\frac{\gamma_{n+1}}{n+1}| < 1 \text{ for some } n.$$

An answer to our question is furnished by following theorem.

2.1 THEOREM. - Let

$$R_n = \gamma_n (\frac{x}{n})^2$$


then \( \{ R_n; \tilde{R}_n; \ n \in \mathbb{N} \} \) form a normal base for \( C^1(\mathbb{R}, \mathbb{R}) \).

Note that for \( n = a_s p^s - 1, \ 0 < a_s < p \), \( \tilde{R}_n = \gamma_{n+1} \binom{x}{n+1} \). We need some lemmas.

2.2 **Lemma**. \( ||R_n||_1 = ||\tilde{R}_n||_1 = 1 \).

**Proof.** For all \( x \neq y \), we have

\[
\left| \frac{R_n(x) - R_n(y)}{x - y} \right| \leq \frac{|\gamma_n|}{|x - y|} \left| \binom{x}{n} - \binom{y}{n} \right| \max(|\binom{x}{n}|, |\binom{y}{n}|) \leq 1
\]

because \( ||\gamma_n \binom{x}{n}||_1 = 1 \) and \( ||\binom{x}{n}|| = 1 \).

Furthermore

\[
\left| \frac{R_n(n) - R_n(n - \gamma_n)}{n - \gamma_n} \right| = 1
\]

In quite a similar way, we prove that \( ||\tilde{R}_n||_1 \leq 1 \); finally

\[
||\tilde{R}_n||_1 \geq |R_n(n)| = \frac{\gamma_{n+1}}{\gamma_n} = 1.
\]

2.3 **Lemma**. If \( 0 \leq m < n \), then \( |\tilde{R}_n(m)| < 1 \).

**Proof.**

\[
\tilde{R}_n(m) = \gamma_{n+1} \frac{d}{dx} \binom{x}{n} \bigg|_{x=m} \frac{m - (n + 1 - \gamma_{n+1})}{(n+1) - (n + 1 - \gamma_{n+1})} \frac{m - (\gamma_{n+1} - 1)}{n + 1 - \gamma_{n+1}}.
\]

If \( m \geq n + 1 - \gamma_{n+1} \), then \( \tilde{R}_n(m) = 0 \). Suppose \( m < n + 1 - \gamma_{n+1} \). If

\[
|\gamma_{n+1}| < |\gamma_n|,
\]

the result follows easily from the fact that \( ||\binom{x}{n}||_1 = |\gamma_n|^{-1} \). So we can suppose that \( \gamma_n = \gamma_{n+1} = a_s p^s \).

We introduce the notation

\[
\text{Schiff}(a_s p^s + a_{s-1} p^{s-1} + \cdots + a_0) = a_s + a_{s-1} + \cdots + a_0.
\]

We remind of the fact that

\[
\text{Schiff } m + \text{Schiff } (n - m - 1) + 1 - \text{Schiff } n \leq (p-1) \nu(\gamma_n), \text{ for } 0 \leq m < n.
\]

This follows from the fact that \( ||\binom{x}{n}||_1 = 1 \), but it can also be proved directly.

Now, let \( n = a_s p^s + \cdots + a_0 \), then \( m < a_{s-1} p^{s-1} + \cdots + a_0 \), and \( n-m-1 \geq a_s p^s \).

We have
25-07

\[ v\left( \frac{d}{dx} \left( \frac{x^n}{n} \right) \right) = v\left( \frac{(n - m - 1)!}{n!} \right) = (p - 1)^{-1} (\text{Schiff}(n) - \text{Schiff}(m) - \text{Schiff}(n - m - 1) + 1) = (p - 1)^{-1} (a_s + \text{Schiff}(n - a_s p^3) - \text{Schiff}(m) - \text{Schiff}(n - m - 1 - a_s p^3) - a_s + 1) \geq -v(\gamma_n - a_s p^3) > -v(\gamma_n), \]

hence

\[ \left| \frac{d}{dx} \left( \frac{x^n}{n} \right) \right| < \frac{1}{|\gamma_n|} = \frac{1}{\gamma_{n+1}}, \]

the result follows.

**Proof of theorem 2.1.** - The polynomials form a dense subspace of $C^1$ (cf. Mahler's base). Since the $R_n$ and $\tilde{R}_n$ generate the polynomials, it only remains to show that $\{R_n, \tilde{R}_n; n \in \mathbb{N}\}$ form an orthogonal system.

Using [8], 5.1.(e), it is sufficient to show that for each $m \in \mathbb{N}$:

- $R_n$ is orthogonal to the linear hull of $\{\tilde{R}_n, R_{n+1}, \tilde{R}_{n+1}, \ldots\}$
- $\tilde{R}_n$ is orthogonal to the linear hull of $\{R_{n+1}, \tilde{R}_{n+1}, R_{n+2}, \ldots\}$.

This follows from the fact that for all $\alpha_j, \beta_j \in K$, we have

\[ \|R_n - \sum_{j<n} \alpha_j R_j - \sum_{j \geq n} \beta_j \tilde{R}_j\|_1 \geq |\delta_1 (R_n - \sum_{j>n} \alpha_j R_j - \sum_{j \geq n} \beta_j \tilde{R}_j)(n, n - \gamma_n)| = 1 = \|R_n\|_1, \]

and

\[ \|\tilde{R}_n - \sum_{j<n} \alpha_j R_j - \sum_{j \geq n} \beta_j \tilde{R}_j\|_1 \geq |\tilde{R}_n(n) - \sum_{j>n} \alpha_j R'_j(n) - \sum_{j \geq n} \beta_j \tilde{R}'_j(n)| = |\tilde{R}'_n(n)| = 1 = \|\tilde{R}_n\|_1, \]

using the fact that $R'_j(n) = 0$,

\[ |R'_j(n)| < 1 \text{ for } j > n. \]

**2.4 Note.** - Our proof is merely inspired by Van Rooij's proof of Mahler's theorem ([8], 5.27). It is also possible to give a proof using the residue class space (as in 1.1), which is, however, considerably longer.
REFERENCES


