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FAMILIES OF MUMFORD CURVES

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In this lecture, we consider the following :

k : algebraically closed, p-adic field with residue field $\tilde{k}$.

$\pi : X \to S$ family of Mumford curves, i. e. a proper, flat, k-analytic morphism such that all fibres $X_s := \pi^{-1}(s)$ are Mumford curves with genus $g(X_s)$.

0. General facts

0.1. The Euler-Poincaré characteristic $EP(\mathcal{O}, \pi, s) = g(X_s)$ is locally constant on S in the sence of Grothendieck topologies. If S is connected, then $g(X_s)=g$ is constant on S .

0.2. If $g = g(X_s) \geq 2$, the relative-tricanonical linebundle $\omega_{X/S}^{\otimes 3}$ is very ample on all fibres. The direct image $\xi := \pi_* \omega_{X/S}^{\otimes 3}$ is a holomorphic vectorbundle on S of rank $5g - 5$ and gives an embedding of the family

$$\begin{array}{ccc} X & \hookrightarrow & \underset{\sim}{P}(\xi) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

If $g = 1$, one gets the same, if one takes $\mathcal{O}(-3D)$ instead of $\omega_{X/S}^{\otimes 3}$, where $D \subset X$ is a divisor, finite over S .

0.3. Now by the GAGA-theorem in the relative case one knows, that for all affinoid subdomains $U = SpA \subset S$ the restricted family $\pi : X|U \to U$ is an algebraic morphism.

I. Uniformization of families

All fibres X_s in the family have an uniformization. Now one wants to get a

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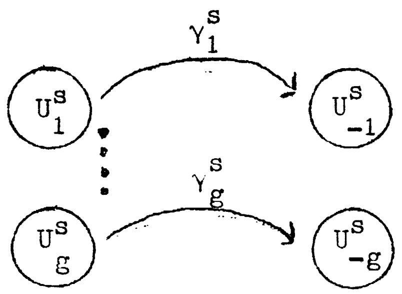
simultaneous uniformization of the family locally on the base S. To explain what I mean, I will repeat the definition of Mumford curve : In our situation, we know

$X_s \simeq \Omega_s/\Gamma_s$,

$\Gamma_s = \langle \gamma_1^s , \dots , \gamma_g^s \rangle \subset PGL(2 , k)$ free, discontinuous group,

$\Omega_s \subset \underset{\sim}{P}^1$ the set of ordinary points.

Moreover one can assume that $\underset{\sim}{\gamma}^s$ is a geometric base, i. e. there are closed balls U_i^s which are pairwise disjoint such that $\gamma_i^s(\underset{\sim}{P} - U_i^s) = \mathring{U}_{-i}^s$ for $i = 1 , \dots , g$.



$U_i^s = \{z \in \underset{\sim}{P}^1 ; \ |z - a_i^s| \leqslant |r_i^s|\}$

$\mathring{U}_i^s = \{z \in \underset{\sim}{P}^1 ; \ |z - a_i^s| < |r_i^s|\}$

"Simultaneous Uniformization" means that all γ_i^s , a_i^s , r_i^s depend holomorphically on $s \in S$.

THEOREM I. - Let S be a reduced affinoid space, $\pi : X \longrightarrow S$ an analytic family of Mumford curves of genus g (with $g + 1 < p = \text{char } \tilde{k}$ or $0 = \text{char } \tilde{k}$), then there is a finite base extension $S' \longrightarrow S$ and a finite covering $\{S'_1 , \dots , S'_r\}$ of S' by affinoids S'_i , such that $X'_i := X \times_S S'_i$ have simultaneous uniformizations, i. e. :

$$\underset{\sim}{P}^1_{S'_i} \supset \Omega'_i \longrightarrow X'_i \quad S'_i\text{-morphism},$$

$$\underset{\sim}{\gamma}^i : \ S'_i \longrightarrow (PGL(2 , k))^g \ \text{holomorphic},$$

such that

$\Gamma'_i := \langle \underset{\sim}{\gamma}^i \rangle$ act holomorphically, discontinuously on Ω'_i by the canonical action $\underset{\sim}{P}^1_{S'_i}$,

$\Omega'_i/\Gamma'_i \simeq X'_i$ isomorphic over S'_i ,

$\underset{\sim}{\gamma}^i(s)$ is a geometric base for $\Gamma'_i(s)$.

To prove this theorem, one makes the following steps :

I.1. There is a finite base extension $S' \longrightarrow S$, a finite covering $\{S'_1 , \dots , S'_r\}$ of S' by affinoids S'_i , and a finite covering $\{X'_{ij}\}_j$ von S'_i with

$$X'_{ik} \simeq \{(s\ ,\ z) \in S'_i \times \underset{\sim}{P}^1\ ;\quad |z - a^0_{ik}(s)|^{m_0} \leq |r^0_{ik}(s)|\ ,$$

$$|z - a^\nu_{ik}(s)|^{m_\nu} \leq |r^\nu_{ik}(s)| \quad \text{for} \quad 1 \leq \nu \leq n_{ik}\}$$

where $m_\nu \geq 1$ are natural members, and $a^\nu_{ik}\ ,\ r^\nu_{ik} \in \mathcal{O}(S'_i)$ with

$$|a^\nu_{ik}(s) - a^0_{ik}(s)|^{m_0} < |r^0_{ik}(s)| \quad \text{for} \quad 1 \leq \nu \leq n_{ik}$$

$$|a^\nu_{ik}(s) - a^\mu_{ik}(s)|^{m_\nu} > |r^\nu_{ik}(s)| \quad \text{for} \quad 1 \leq \nu \neq \mu \leq n_{ik}$$

$$0 < |r^\nu_{ik}(s)|^{m_0} < |r^0_{ik}(s)|^{m_\nu} \quad \text{for} \quad 1 \leq \nu \leq n_{ik}$$

For the components U_ν of $X_{ik} \cap X_{i\ell}$ one has :

$$U_\nu \simeq \{(s\ ,\ z) \in S'_i \times \underset{\sim}{P}^1\ ;\quad |r^-_\nu(s)| \leq |z - a_\nu(s)|^{m_\nu} \leq |r^+(s)|\}$$

where $r^+_\nu\ ,\ r^-_\nu\ ,\ a_\nu \in \mathcal{O}(S'_i)$ satisfy $0 < |r^-_\nu(s)| < |r^+_\nu(s)|$ for all $s \in S'_i$.

Sketch of proof : After finite base extension $S^* \to S$ one finds a section σ to $\pi^* := \pi \times_S S^*$. Then by the theorem of Riemann-Roch, one finds a finite morphism

$$\begin{array}{ccc} X^* & \overset{\varphi}{\dashrightarrow} & \underset{\sim}{P}^1_{S^*} \\ & \searrow \quad \swarrow & \\ & S^* & \end{array}$$

with $\deg \varphi \leq g + 1 < p$.

Now after finite base extension $S^+ \to S^*$ the ramification set of $\varphi \times_{S^*} S^+$ "essentially" splits into divisors of degree 1 . So there are holomorphic sections $\tau_i\ :\ S^+ \to \underset{\sim}{P}^1_{S^+}$ such that

$$\mathrm{Ramif}(\varphi \times_{S^*} S^+) = \tau_1 S^+ \cup \dots \cup \tau_r S^+ \ .$$

Then one constructs the coverings $\{S'_i\}$ of S' and $\{X'_{ij}\}$ of X'_i by using these sections $\{\tau_\rho\}$ and by making a further finite base extension $S' \to S^+$. The main tool here is the method of constructing a stable reduction of an algebraic curve M , which admits a morphism $\psi\ :\ M \to \underset{\sim}{P}^1$ with $\deg \psi < p$, by the ramification points of ψ .

I.2. By the covering $\{X'_{ij}\}_j$ one constructs a k-analytic S'_i-morphism $\Omega'_i \to X'_i$ such that $\Omega'_i(s)$ is the universal covering of $X'_i(s)$ for all $s \in S'_i$. Next one proves an embedding theorem

$$\begin{array}{ccc} \Omega'_i & \hookrightarrow & \mathbb{P}^1_{S'_i} \\ & \searrow \quad \swarrow & \\ & S'_i & \end{array}$$

By this embedding the group $\mathrm{Deck}(\Omega'_i/X'_i)$ can be regarded as a subgroup of $PGL(2, \mathcal{O}(S'_i))$. From this, one easily gets the theorem.

COROLLARY. - Let $\mathcal{B}_g$ be the space of geometric bases for Mumford curves, $\mathfrak{M}_g$ the space of Mumford curves and $g \geq 2$.

For every family $X \to S$ of Mumford curves of genus $g \geq 2$, the canonical map $S \to \mathfrak{M}_g$ admits locally a holomorphic lifting after finite base extension

$$\begin{array}{cccc} X & S' \supset & \longrightarrow & \mathcal{B}_g \\ \downarrow & \downarrow & & \downarrow \\ S & S & \longrightarrow & \mathfrak{M}_g \end{array}$$

II. Rigidity of algebraic families

An algebraic family of Mumford curves $\pi : X \to S$ is a proper, flat, algebraic morphism where S is a connected scheme of finite type over k, such that all fibres X_s are Mumford curves. Let g be the genus of the fibres.

THEOREM II. - Every algebraic family of Mumford curves is constant, i. e. the map $S \to \mathfrak{M}_g$, $s \mapsto [X_s]$, is constant (if $g + 1 < p = \mathrm{char}\, \tilde{k}$ or $0 = \mathrm{char}\, \tilde{k}$).

Idea of the proof. - It is enough to consider affine, irreducible, algebraic curves for S. Similar as in § I, one can show, that there is a finite extension $S' \to S$ by an algebraic curve such that $X' = X \times_S S'$ has locally on S' a simultaneous uniformization. If $\Omega' \to S'$ is the universal covering, then the canonical map can be lifted

$$\begin{array}{ccc} \Omega' & \longrightarrow & \mathcal{B}_g \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathfrak{M}_g \\ & & \\ s & \longmapsto & [X_s] \end{array}$$

Now $\mathcal{B}_g$ is bounded in some sense, then $\Omega' \to \mathcal{B}_g$ has to be constant by the :

PROPOSITION. - Bounded holomorphic functions on the universal covering of an algebraic curve are constant.

COROLLARY. - Analytic families $X \longrightarrow S$ over an connected, complete, algebraic curve are constant (GAGA).

Remarks.

(1) All considerations are made only for $g \geq 2$, because the case $g = 0$ is trivial and, in the case $g = 1$, one gets rigidity for analytic families over connected, affine, algebraic curves by considering the j-invariants of the fibres as holomorphic function on the base.

(2) The assumption concerning the genus " $g + 1 < \text{char } \tilde{k}$ " should not be necessary, like a new research by the author has shown.

(3) Theorem II should be true for analytic families of Mumford curves over connected schemes of finite type over k as in the case $g = 1$.

This lecture is a very short version of my paper "Ein globaler Starrheitssatz für Mumford Kurven" which will come out in "Journal für reine und angewandte Mathematik" in the next months.