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PUISEUX EXPANSIONS

by Bernard M. DWORK (*)

The object of this note is to discuss p -adic convergence of Puiseux expansions of algebraic functions. We shall review joint work [D-R] with ROBBA on this question and shall discuss the problem of lifting Puiseux expansions in characteristic p .

Notation.

K = field of characteristic zero complete under a discrete nonarchimedean valuation with residue class field of characteristic p .

k = residue class field of K .

\mathcal{O} = ring of integers of K .

$R = \mathcal{O}[x]$, $\bar{R} = k[x]$.

E = completion of $K(x)$ under the Gauss norm.

$\hat{R} = \mathcal{O}[[x]]$, $\hat{\bar{R}} = k[[x]]$.

\hat{E} = quotient field of completion of \hat{R} under the sup norm on $D(0, 1^-)$.

An element $\xi \in K((x^{1/m}))$ will be said to "converge" in $D(x, r^-)$ if for suitable $N \in \mathbb{N}$, $x^N \xi(x^N)$ is a power series converging in $D(0, (r^N)^-)$.

A series $\xi(x) = \sum_{j=-\infty}^{\infty} A_j x^{j/m}$ will be said to be a Puiseux Laurent series "convergent au bord" if $\xi(x^N)$ converges in an annulus $\Delta_{r,1} = \{x ; r < |x| < 1\}$.

Let $f \in R[y]$, \bar{f} its image in $\bar{R}[y]$ under the natural mapping. We say that $\xi \in K((x^{1/m}))$ (resp. $k((x^{1/m}))$) is a Puiseux expansion for f (resp. \bar{f}), if $f(x, \xi)$ (resp. $\bar{f}(x, \xi) = 0$).

We refer to the union of the zeros of the discriminant and the zeros of the leading coefficient as the singular locus of f .

We consider two questions :

Question I : Let ξ be a Puiseux expansion for f . Does ξ converge in $D(0, 1^-)$?

Question II : Let $\bar{\xi}$ be a Puiseux expansion for \bar{f} . Can $\bar{\xi}$ be lifted to a Puiseux expansion for f ?

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We observe that liftability implies not only convergence on $D(0, 1^-)$ of the lifted expansion but also boundedness by unity.

It is clear that if $\deg_y f = n = \deg_y \bar{f}$ and if \bar{f} has n distinct Puiseux expansions, and if the answer to II is affirmative, then the answer to I is also affirmative.

We shall have occasion to consider various conditions :

(H₁): The valuation induced on $K(x)$ by the Gauss norm is at worst tamely ramified in the splitting field of f .

(H₂): The singular locus of f has no element in the punctured disk $D(0, 1^-) - \{0\}$.

(H₃): \bar{f} and $\bar{f}_y (= \frac{\partial \bar{f}}{\partial y})$ have no common factor in $\bar{R}[y]$.

(H₄): $\deg_y f = \deg_y \bar{f}$.

THEOREM [D-R]. - Assume (H₁), (H₂), then Question I has an affirmative response.

For proof see [D-R]. The condition (H₂) is clearly necessary.

Example. - $f = y^3 - x(x + p)$.

A Puiseux expansion $x^{1/3} \sum_{j=0}^{\infty} A_j x^{j/3}$ satisfying I would imply

$$\sum A_j x^j = (x^3 + p)^{1/3},$$

and differentiating shows that $x^2/(x^3 + p)^{2/3}$ is analytic in $D(0, 1^-)$, which is impossible.

Remark. - We attributed to HIRONAKA [Dw] the statement (for $f \in \mathbb{Z}[x, y]$).

(F 1) if f is irreducible over $\mathbb{Q}(x)$, and if the discriminant of f and of \bar{f} have the same degrees as polynomials in x , then the Puiseux expansions of \bar{f} lift to Puiseux expansions of f .

This example demonstrates the inaccuracy of (F 1).

LEMMA. - Assume (H₃). We conclude that each Puiseux expansion $\bar{\eta}$ of \bar{f} has a lifting to a Puiseux Laurent series for f "convergent au bord".

Note. - We do not assume (H₂). We do not affirm a positive response to Question II.

Proof. - We may assume that $\bar{\eta} \in K[[x]]$. Let $\hat{\eta}$ be a lifting of $\bar{\eta}$ in $\mathcal{O}[[x]]$. Hence letting (\mathfrak{U}) be the prime ideal of K ,

$$f(x, \hat{\eta}) \in \mathfrak{U} \mathcal{O}[[x]] \text{ while } f_y(x, \hat{\eta}) \equiv \bar{f}_y(x, \bar{\eta}) \neq 0,$$

i. e. $f_y(x, \hat{\eta})$ is an element of $\mathcal{O}[[x]]$ with at least one unit coefficient.

Choose a positive real θ , $1 > \theta > |\Pi|$, then there exists $r < 1$ such that

$$|f_y(x, \hat{\eta})| > (|\Pi|/\theta)^{1/2}, \quad \forall x \in \Delta_{r,1}$$

while

$$|f(x, \hat{\eta})| \leq |\Pi|, \quad \forall x \in D(0, 1^-).$$

We now put $y = \hat{\eta} + w$, so

$$f(x, y) = f(x, \hat{\eta}) + wf_y(x, \hat{\eta}) + \sum_{j=2}^{\infty} w^j/j! f^{(j)}(\hat{\eta})$$

and then put $w = zf(x, \hat{\eta})/f_y(x, \hat{\eta})$, so that

$$(1) \quad \frac{f(x, y)}{f(x, \hat{\eta})} = 1 + z + \sum_{j=2}^{\infty} z^j \frac{f^{(j)}(\hat{\eta})}{j!} \frac{f(x, \hat{\eta})^{j-1}}{f_y(x, \hat{\eta})^j} = 1 + z + A_2 z^2 + \dots$$

where

$$A_j = \frac{f^{(j)}(\hat{\eta})}{j!} \frac{f(x, \hat{\eta})^{j-1}}{f_y(x, \hat{\eta})^j}, \quad j = 2, 3, \dots$$

Thus on $\Delta_{r,1}$, A_j is bounded by $\theta < 1$ and hence $z \mapsto -1 - \sum_{j=2}^{\infty} A_j z^j$ is a contractive map on the space of functions analytic and bounded by unity on $\Delta_{r,1}$. It is clear that the unique fixed point z_0 then gives a solution of (1) by setting $\eta = \hat{\eta} + z_0 f(x, \hat{\eta})/f_y(x, \hat{\eta})$, that η converges on $\Delta_{r,1}$ and that the Laurent series $\sum_{j=-\infty}^{\infty} B_j x^j$ representing $z_0 f(x, \hat{\eta})/f_y(x, \hat{\eta})$ is bounded by $(|\Pi|/\theta)^{1/2} < 1$, and hence

$$|B_j| < 1 \quad j \geq 0$$

$$|B_j r^j| < 1 \quad j < 0$$

which shows that B_j has zero image in K , i. e. η is a lifting of $\bar{\eta}$ as asserted.

COROLLARY 1. - Assume (H_1) , (H_2) , (H_3) , then question II has an affirmative response.

COROLLARY 2. - Assume (H_2) , (H_3) , (H_4) , then question II has an affirmative response.

Proof. - Assumptions (H_3) , (H_4) imply (H_1) and hence the first corollary implies the second. The theorem shows that f has a full set of Puiseux expansions converging in $D(0, 1^-)$. If $\bar{\xi}$ is a Puiseux expansion of \bar{f} then by the lemma $\bar{\xi}$ has a Laurent series lifting, ξ , "convergent au bord". This ξ must coincide with one of the previously mentioned solutions and so converges in $D(0, 1^-)$.

We now disprove :

(F₂) Assumptions (H₂), (H₃) imply question II has an affirmative solution.

To construct a counter-example, it seems useful to consider a polynomial not satisfying (H₁). For this reason, we consider $y^{p+1} + xy + p$ which over E has factors $f_1 \equiv y^p + x \pmod{p}$, and f_2 of degree 1 in y . It is more convenient to write $y = pz$ and so consider

$$p^p z^{p+1} + xz + 1 = f(z) .$$

Mod p , we have the solution $z = -x^{-1}$ which clearly cannot lift to a Puiseux expansion at $x = 0$ in characteristic zero since in that characteristic $x = 0$ is not a singularity. Trivially (H₃) is satisfied. To check (H₂), we must compute the discriminant. We recall that for $y^N + A \frac{x}{y} + B = g(y)$, the discriminant is

$$(-1)^{\binom{N+1}{2}} [(-1)^N N^N B^{N-1} - (N-1)^{N-1} A^N] .$$

Thus, for $z^{p+1} + (x/p^p)z + (1/p^p)$, the discriminant is

$$\pm [(-1)^{p+1} (p+1)^{p+1} \left(\frac{1}{p^p}\right)^p - p^p \left(\frac{x}{p^p}\right)^{p+1}] = \pm p^{-p^2} [(-1)^{p+1} (p+1)^{p+1} - x^{p+1}] ,$$

i. e. the zeros are outside of $D(0, 1^-)$.

We now discuss in detail a well known example.

$$(E) \quad f(y) = y^p - y - \frac{1}{x} .$$

Here the discriminant is given by our previous formula to be

$$(-1)^{p(p+3)/2} [p^p \left(-\frac{1}{x}\right)^{p-1} - (p-1)^{p-1}] .$$

Hence the singular locus consists of

$$\left\{ p^{p/(p-1)} \frac{\omega}{1-p} \right\}_{\omega^p = \omega}$$

a set of p points.

(E 1) There are no Puiseux expansions at $x = 0$ in characteristic p .

Proof. - If \wp is the prime above $x = 0$, then $\text{ord}_{\wp} y = -1/p$.

Hence a Puiseux expansion, if it exists, must be of the form

$$(1.1) \quad y = A_{-1} z^{-1} + A_0 + A_1 z + \dots, \quad z = x^{1/p}, \quad \text{with all } A_j \text{ in } \mathbb{F}_{\wp}$$

but then

$$y = y^p - \frac{1}{x} = \sum_{j=-1}^{\infty} A_j^p z^{pj} - \frac{1}{x} \in k((x))$$

a contradiction as is well known [Ch], (p. 64).

(E 2) No Puiseux expansion at zero in characteristic zero converges for
 $|x| \geq |p|^{p/p-1} = r_0$.

Proof. - A Puiseux expansion convergent for $x \in D(0, r^-) - \{0\}$ means that we obtain an element of $K((z))$ with $z = x^{1/p}$ which converges for $0 < |x| < r$. For $r \leq 1$, we then have point wise

$$|y^p - y| = |x|^{-1} > 1, \text{ and so } |y^p| > |y| > 1.$$

Thus $|y| = |x|^{-1/p}$ point wise, and so if $r > r_0$, then y^{p-1} will assume the value $1/p$ at suitable values of x such that $|x| = r_0$.

Since $dy/dx(py^{p-1} - 1) = -1/x^2$, and since

$$dy/dx = (dy/dz)/(dx/dz) = (1/pz^{p-1}) dy/dz$$

is analytic as function of z for $x \in D(0, r^-) - \{0\}$, we obtain a contradiction if $r > r_0$. The same analysis shows that convergence for $|x| = r_0$ is also impossible.

(E 3) There are p distinct Puiseux expansions at infinity in characteristic p .

Proof. - Let $\bar{y}_0 = -\frac{1}{z} - (\frac{1}{z})^p - (\frac{1}{z})^{p^2} - \dots$, then $\bar{y}_0^p - \bar{y}_0 = 1/z$.

The p solutions are $\{\bar{y}_0 + a\}_{0 \leq a < p}$.

(E 4) The p distincts Puiseux expansions at infinity (in characteristic zero) converge and are bounded by unity for $|z| > 1$, but do not converge for $|z| = 1$.

Proof. - At $x = \infty$ condition (H_2) is satisfied. The global conditions (H_3) , (H_4) are also satisfied. Hence by the corollary the Puiseux expansions in characteristic p may be lifted. These then are $\{y_a\}_{a=0,1,\dots,p-1}$ where $y_a \pmod{p} = a + \bar{y}_0$.

This shows that $y_a \in z_p[[1/z]]$, but if we write

$$y_a = \sum_{j=0}^{\infty} B_j \frac{1}{z^j},$$

$|B_j| = 1$ for an infinite set of j . This shows that the domain of convergence is precisely $|z| > 1$.

This concludes our discussion of the example.

Generalizations.

1°. Let $f \in \hat{R}[y]$. The theorem and the lemma generalize replacing (H_2) by H_2' , (H_3) by (H_3') as indicated below.

(H_2') : The valuation induced on the quotient field of $\mathcal{O}[[x]]$ by the Gauss norm is at worst tamely ramified in the splitting field of f

(H₃') : \bar{f} and \bar{f}_y have no common factor in $\hat{R}[y]$.

2°. Let G be an $n \times n$ matrix with coefficients in the quotient field of \hat{R} . We assume that the differential equation $dy/dx = Gy$ has no singularity in $D(0, 1^-)$ except for a regular singularity at $x = 0$ with rational exponents. We assume that at the generic point t (in the sense of ROBBA [Ro 1]) the equation has n independent solutions bounded and analytic on $D(t, 1^-)$. We conclude that the solution matrix at the origin is of the form Dx^H where H is a constant diagonal matrix and D is a bounded matrix converging on $D(0, 1^-)$.

The proof is omitted since it is so close to that of [D-R]. The key point is that the argument of ROBBA [Ro 2] shows that the hypothesis of boundedness on the generic disk implies the semi-simplicity of H .

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