SIEGFRIED BOSCH

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STABLE REDUCTION AND RIGID ANALYTIC UNIFORMIZATION OF ABELIAN VARIETIES

by Siegfried BOSCH (*)

[Univ. Münster]

This is a report on joint work with W. Lütkebohmert [BL], where we have given a purely analytic approach to the uniformization of curves and abelian varieties.

In classical uniformization theory over the field of complex numbers, one shows that each non-singular projective curve can be analytically represented as a quotient $\mathbb{C}/\Gamma$ with an open subset $\mathbb{C} \subset \mathbb{P}^1$ and a subgroup $\Gamma \subset \text{PGL}(2, \mathbb{C})$ acting discontinuously on $\mathbb{C}$. Similarly, each abelian variety can be represented as an analytic torus which, from the multiplicative point of view, is defined as a quotient $(\mathbb{C}^\times)^G/\Gamma$ by a discrete subgroup $\Gamma \subset (\mathbb{C}^\times)^G$, free of rank $g$. Over a non-archimedean ground field $k$, the situation is quite different. The curves and abelian varieties which admit a good uniformization as above constitute only a small open part in the corresponding moduli spaces (see [M1] and [M2]). On the other hand, there are curves and abelian varieties admitting no non-trivial uniformization at all. This behavior is related to the phenomenon of good reduction, which has no classical counterpart. In general, good uniformization and good reduction occur in a mixed way. It is for this reason that uniformization theory is substantially more complicated in the non-archimedean case.

1. Uniformization via algebraic geometry.

The basic tools in uniformization theory over discretely valued fields are the stable reduction theorem of Deligne-Mumford [DM] and the semi-abelian reduction theorem of Grothendieck [SGA7]. In order to explain these results, consider a discrete valuation ring $R$, and let $k$ be its field of fractions.

Let $A$ be an abelian variety over $k$, and denote by $\mathfrak{A}$ its Néron model ([N], [R1]). Then the theorem of Grothendieck says that (modulo finite separable extension of the ground field) the identity component of the special fibre $\mathfrak{A}_s$ of $\mathfrak{A}$ is semi-abelian; i.e. $\mathfrak{A}^0_s$ is an extension of an abelian variety by a multiplicative group. Likewise, if $C$ is a smooth geometrically connected projective curve of genus $\geq 2$ over $k$, one can consider its minimal model $\mathfrak{C}$ over $R$. (See [Ab] or [Li] for the existence of a regular model and [Sh] for the minimality.) The result of Deligne-Mumford asserts (again, modulo finite extension of the ground field) that the special fibre $\mathfrak{C}_s$ of $\mathfrak{C}$ has only ordinary double points as...
singularities. Both results are more or less equivalent. In fact, the proof of the stable reduction theorem in \([EM]\) uses the semi-abelian reduction of the Jacobian \(J\) of \(C\) and the fact that the "identity component" of the Néron model of \(J\) represents the functor \(\text{Pic}^0(C/\text{Spec } R)\) (see \([R3]\)). Apart from uniformization theory, there are far-reaching applications of both theorems in the theory of moduli and also in number theory.

Assuming that \(k\) is complete, the universal covering \(\hat{C}\) of \(C\) can be easily constructed as follows. Consider \(\hat{C}\), the formal completion of \(C\). Then \(\hat{C}\) is a formal scheme over \(R\) or, from the rigid analytic viewpoint, a formal analytic variety over \(k\). In fact, \(\hat{C}\) may be interpreted as the analytification of \(C\) (again denoted by \(\hat{C}\)), and the formal structure of \(\hat{C}\) gives rise to a reduction map \(\bar{\pi} : C \to \overline{\hat{C}} = \overline{C}\). The formal fibre \(\bar{\pi}^{-1}(x)\) over a closed (rational) point \(x \in \hat{C}\) is an open disc if \(x\) is non-singular, and an open annulus if \(x\) is an ordinary double point. This leads to a geometric description of \(C\). Namely, the \(\pi\)-inverse of the non-singular locus \(\hat{C} \setminus \text{Sing } \hat{C}\) is a disjoint union of components which are smooth over \(R\) and thus simply connected (in the sense of rigid analysis; see \([BL]\), II, 8.12). The curve \(\hat{C}\) is obtained by connecting these components by means of the annuli \(\bar{\pi}^{-1}(x)\), \(x \in \text{Sing } \hat{C}\). Likewise, the universal covering \(\hat{C}\) of \(C\) is constructed by resolving all loops which are generated by this process. If \(\overline{\hat{C}}\) has only rational components (this is the case of Mumford's split degenerate reduction \([M1]\), \(C\) has a good uniformization as discussed above. Namely, \(\hat{C}\) can be viewed as an open analytic subvariety of \(\mathbb{P}^1\), the automorphisms of \(\hat{C}\) over \(C\) being fractional linear transformations. The uniformization of abelian varieties \(A\) is more complicated (see \([R2]\)). Here, analogous to the case of curves, a fundamental role is played by the formal completion \(\hat{A}\) of the Néron model of \(A\), or to be more precise, by the identity component \(\hat{A}^0\) of \(\hat{A}\).

2. Uniformization of curves via rigid analysis.

It is a surprising fact, first realized by VAN DER PUT \([P]\), that the formal completion \(\hat{C}\) of the minimal model \(C\) of \(C\) can be constructed by a direct analytic method (without knowing \(C\)). Thus proceeding as in section 1, a purely analytic approach to the uniformization of curves is obtained by proving the following analytic version of the stable reduction theorem (for arbitrary non-archimedean ground fields, modulo finite separable field extension):

The curve \(C\) can be viewed as a formal analytic variety with associated reduction \(\pi : C \to \hat{C}\) such that \(\hat{C}\) has at most ordinary double points as singularities.

The proof of this result in \([BL]\) uses the key fact that for arbitrary reductions \(\pi : C \to \hat{C}\), the periphery of a fibre \(\pi^{-1}(x)\) is a disjoint union of annuli. Namely, one starts with an arbitrary reduction \(\pi : C \to \hat{C}\) and refines \(\pi\) inductively by using the technique of blowing up, in such a way that all bad singularities of
3. Uniformization of abelian varieties via rigid analysis.

As an application of the analytic stable reduction theorem, it is shown, in [BL], how to obtain the uniformization of abelian varieties (over arbitrary non-archimedean ground fields \( k \), modulo finite separable field extension). Namely, consider the Jacobian \( J \) of a smooth geometrically connected projective curve \( C \).

The stable reduction theorem provides deep insight into the analytic structure of \( C \). Thereby it is possible to construct line bundles with prescribed properties and to carry out explicit computations. One constructs an open analytic subgroup \( \tilde{J} \) of \( J \), the group of normalized line bundles on \( C \); its a canonical reduction \( \tilde{J} \) which is isomorphic to the Jacobian of \( \tilde{C} \). Hence \( \tilde{J} \) is an extension of an abelian variety by a multiplicative group. If the valuation of the ground field \( k \) is discrete, \( \tilde{J} \) may be interpreted as the identity component of the formal completion of the Néron model of \( J \).

In order to construct the universal covering of \( J \), one looks at the analytic cohomology group \( H^1(C, \mathbb{Z}) \). It is free of rank \( r \leq g \) (= genus of \( C \)); in fact, the rank \( r \) reflects the number of loops in \( C \) as discussed in section 1. Using Picard functors, one interprets \( H^1(C, \mathbb{Z}) \) as the \( \mathbb{Z} \)-module of analytic group homomorphisms \( \mathbb{G}_m \rightarrow J \) or \( \mathbb{G}_m \rightarrow \tilde{J} \), where \( \mathbb{G}_m \) denotes the multiplicative group over \( k \) and where \( \mathbb{G}_m \) is its subgroup of "units". Thus there is a closed subgroup \( \mathbb{G}_m^r \subset \mathbb{G}_m \) which reduces to the multiplicative part of \( \tilde{J} \) and which may be extended to an analytic homomorphism \( \mathbb{G}_m^r \rightarrow J \). Then \( \hat{J} := \mathbb{G}_m^r \times \tilde{J} / \text{(diagonal)} \) is the universal covering of \( J \). The projection map \( \hat{J} \rightarrow J \) has a discrete kernel \( \Gamma \) which is free of rank \( r \), so that \( J = \hat{J} / \Gamma \).

In particular, the following assertions are equivalent:

(i) \( C \) is a Mumford curve.

(ii) \( \text{rank } H^1(C, \mathbb{Z}) = g \).

(iii) \( J \) is an analytic torus.

Furthermore, \( J \) has good reduction, if \( C \) has good reduction.

Since, up to isogeny, any abelian variety \( A \) can be embedded into a product of Jacobian varieties, the above uniformization of Jacobians implies the uniformization of \( A \). Namely, one constructs the analogues \( \hat{A} \) and \( \hat{\tilde{A}} \) of the groups \( \tilde{J} \) and \( \hat{J} \), and shows \( A = \hat{A} / \Gamma \). In this case, the rank of \( \Gamma \) has to be interpreted as the rank of the cohomology group \( H^1(A', \mathbb{Z}) \), where \( A' \) is the dual abelian variety of \( A \).

4. Further applications of the analytic method.

Using simple algebraization techniques, the methods of [BL] seem to yield new
proofs for the facts from algebraic geometry mentioned in section 1, namely for the following results:

- Existence of minimal models for curves,
- Existence of Néron models for abelian varieties,
- Stable reduction theorem of Deligne–Mumford,
- Semi-abelian reduction theorem of Grothendieck.

BIBLIOGRAPHY


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