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Differentials of the second kind for families of Mumford curves


DIFFERENTIALS OF THE SECOND KIND FOR FAMILIES OF MUMFORD CURVES

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The space of everywhere meromorphic differentials on a Mumford curve $M$ of genus $g$ which can be integrated on the universal covering of $M$ is a space of codimension $g$ in the full space of meromorphic differentials on $M$. This fact allows to conclude that the Gauss-Manin connection associated to an analytic family of Schottky groups has $g$ linearly independent horizontal elements which are defined everywhere on the parameter space of the family. I will give a sketch of the proof for this result.

1. $\xi$-functions and differentials of the second kind.

Let $K$ be an algebraically closed field together with a complete non-archimedean valuation. Let $\Gamma$ be a Schottky subgroup of the group $PGL_2(K)$ of fractional linear transformations of the Riemann surface $\mathcal{P} = K \cup \{\infty\}$ over $K$. Let $Z$ be the domain of ordinary points of $\Gamma$, see [GP], Chap. I, § 4.

**Theorem 1.** - Let $h(z)$ be a rational function on $\mathcal{P}$, whose poles all lie in $Z$ and let $z_0 \in Z$ be an ordinary point for $\Gamma$. Then the series

$$\xi(h; z_0; z) := \sum_{\gamma \in \Gamma} \frac{h(\gamma(z)) - h(\gamma(z_0))}{h(\gamma(z))}
+ \sum_{\gamma \in \Gamma} \frac{h(\gamma(z))}{h(\gamma(z_0)) = \infty}$$

is as a function of $z$ uniformly convergent on any affinoid subdomain of $Z$. Its limit is a meromorphic function on $Z$.

A proof of this result appears in [G], (1).

Let now $I$ be the $K$-vector space of those meromorphic functions $f(z)$ on $Z$ for which

$$f(\gamma z) - f(z) \in K$$

for all $\gamma \in \Gamma$.

The differential $df$ of a function from $I$ is $\Gamma$-invariant and is thus a differential of the Mumford curve $M = Z/\Gamma$.

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Denote by $H$ the $K$-vectorspace of rational functions on $P$ whose poles all lie in $Z$. One can show that any $f \in I$ is obtained as $\zeta(h, z_0; z)$ with $h \in H$, see [G], (2).

Let $\text{Hom}(\Gamma, K)$ be the $K$-vectorspace of group homomorphisms $\zeta : \Gamma \to K$. If we fix a basis $\alpha_1, \ldots, \alpha_g$ of the free group $\Gamma$, we obtain a canonical isomorphism $\text{Hom}(\Gamma, K) \cong K^g$ when we map $\zeta$ onto the $g$-tuples $(\zeta(\alpha_1), \ldots, \zeta(\alpha_g))$.

For any $f \in I$ we denote by $P(f)$ the group homomorphism $\Gamma \to K$ given by

$$P(f)(\gamma) = f(\gamma z) - f(z).$$

Then $P(f)(\gamma)$ is the period of the differential $df$ with respect to the "cycle" $\gamma$.

The mapping

$$P : I \to \text{Hom}(\Gamma, K)$$

is $K$-linear whose kernel consists of the field of $\Gamma$-invariant meromorphic functions on $Z$ which is the field of rational functions on the curve $\mathbb{M}$. One can prove that the mapping $P : I \to \text{Hom}(\Gamma, K)$ is surjective, see [G], (3).

**Theorem 2.** Let $\alpha_1, \ldots, \alpha_g$ be a basis of $\Gamma$. Then there exist functions $f_1, \ldots, f_g \in I$ such that $P(f_i)(\alpha_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

A meromorphic differential $\omega = fdz$ on $Z$ is called to be of the second kind if for any point $a \in Z$ there is a meromorphic function $h_a(z)$ on $Z$ such that $\omega - dh_a$ is analytic in $a$.

Denote by $\Omega_2$ the $K$-vectorspace of $\Gamma$-invariant differentials on $Z$ of the second kind. The proof of the following theorem is given in [G], (4).

**Theorem 3.** $\Omega_2 = \Omega_1 \otimes dI$ where $\Omega_1$ is the $g$-dimensional $K$-vectorspace of analytic differentials on $\mathbb{M}$.

2. Families of Schottky groups.

Let $S$ be a rigid analytic space over $K$, see [BGR], Chap. 9. We consider the projective line over $S$, namely the product space $\mathbb{P} \times S$ together with the projection $\pi$ onto the second factor.

Denote by $\text{Aut}_S(\mathbb{P} \times S)$ the group of those bianalytic mapping $\gamma : \mathbb{P} \times S \to \mathbb{P} \times S$ which are compatible with $\pi$ (i.e. $\gamma \circ \pi = \pi$).

One can prove that there is an admissible covering $\mathcal{G} = (S_i)_{i \in I}$ of $S$ such $\gamma|_{S_i}$ is a fractional-linear transformation over $S_i$ which means that there is a matrix

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \text{GL}_2(\mathbb{O}(S_i)),$$
where $\theta(S_1)$ is the $K$-algebra of analytic functions on $S_1$ such that

$$\left(\gamma|_{S_1}\right)(s, z) = \frac{a_i(s) + b_i(s)}{c_i(s) + d_i(s)}.$$ 

For any point $s \in S$ we obtain a canonical homomorphism $\text{Aut}_S(P \times S) \to \text{PGL}_2(K)$ by restricting $\gamma \in \text{Aut}_S(P \times S)$ to the subspace $P \times \{s\}$ of $P \times S$. We denote the restriction of $\gamma$ to $P\{s\}$ by $\gamma_s$.

**Definition.** - A subgroup $\Gamma \subseteq \text{Aut}_S(P \times S)$ is called a Schottky group over $S$ (or a family of Schottky groups parametrized by $S$) if for any point $s \in S$ the restriction of the canonical homomorphism $\text{Aut}_S(P \times S) \to \text{PGL}_2(K)$ to $\Gamma$ gives an isomorphism from $\Gamma$ to a Schottky group $\Gamma_s$ of $\text{PGL}_2(K)$.

Let now $\Gamma$ be a Schottky group over $S$. The proof of the following result will be given elsewhere.

**Theorem 4.** - There exists an admissible subdomain $Z$ of $P \times S$ such that for any $s \in S$ the intersection $Z \cap (P \times \{s\})$ is the domain of ordinary points for the Schottky groups $\Gamma_s$. If $S$ is an affinoid space there is an affinoid subdomain $F \subseteq Z$ such that $U_{\gamma \in \Gamma} \gamma(F) = Z$.

If $S$ is irreducible, then so is the domain $Z$.

**Corollary.** - $Z/\Gamma \to S$ is an analytic family of Mumford curves.

From now on let $S$ be irreducible and $H$ be the $\theta(S)$-algebra of meromorphic functions on $P \times S$ whose poles and points of indeterminancy all lie in $Z$.

Let $z_0 : S \to Z$ be an analytic mapping such that $\pi \circ z_0 = \text{id}_S$ and $h \in H$. Let $h_s$ be the restriction of $h$ onto $P \times \{s\}$. Then there is a meromorphic function $\xi(h ; z_0 ; s, z)$ on $Z$ such that the restriction of $\xi(h ; z_0 ; s, z)$ onto $P \times \{s\}$ equals $\xi(h_s ; z_0(s), z)$. Let $I_s$ be $\theta(S)$-module of meromorphic functions $f(s, z)$ on $Z$ for which $f \circ \gamma - f \in \theta(S)$ for all $\gamma \in \Gamma$. Let $\text{Hom}(\Gamma, (S))$ be the free $\theta(S)$-module of rank $g$ of all group homomorphisms $c : \Gamma \to \theta(S)$.

Let $P(f)(\gamma) := f \circ \gamma = f$. Then $P(f) \in \text{Hom}(\Gamma, \theta(S))$.

**Theorem 5.** - Let $\alpha_1, \ldots, \alpha_g$ be a basis of $\Gamma$. There is an admissible covering $(S_i)_{i \in I}$ of $S$ and for any $i$ there are functions $f_1, \ldots, f_g \in I_{S_i}$ such that $P(f_j)(\alpha_i) = \delta_{ij}$.

Let $\mathcal{F}_2 = \mathcal{F}_{2i/S}$ denote the sheaf on $S$ whose set of sections on an admissible
open domain $U \subseteq S$ are the $K$-vectorspace of $\Gamma$-invariant differentials relative to $Z \to S$, of the second kind on $Z_U = Z \cap (P \times U)$.

Let $\Omega^1_{ex}$ be the subsheaf of $\Omega^1_2$ of exact differentials and $H^1_{DR}$ be the quotient sheaf $\Omega^1_2/\Omega^1_{ex}$.

**Theorem 6.** - $H^1_{DR}$ is a free coherent module over the structure sheaf $\mathcal{O}_S$ on $S$ of rank $2g$. There is a canonical decomposition

$$H^1_{DR} = \bar{\Omega} \oplus \Omega^1$$

where $\Omega^1$ is the subsheaf of $\Omega^1_2$ of analytic differentials and $\bar{\Omega}$ is the sheaf of cohomology classes of differentials of the form $df$ with $f \in I$. $\bar{\Omega}$ and $\Omega^1$ are free modules of rank $g$ over $\mathcal{O}_S$.

**Sketch of proof:** In order to prove that $\Omega^1$ is free of rank $g$, we have to observe that for any $\gamma \in \Gamma$ there is a canonical differential $\omega_\gamma = (du_\gamma/u_\gamma) \in \Omega^1$, where $u_\gamma$ is defined on $Z$ as in [GP], Chap. 2. While the $u_\gamma$ are unique up to a unit from $\mathcal{O}_S$, the differential $\omega_\gamma$ is unique. If $\alpha_1, \ldots, \alpha_g$ is a basis of $\Gamma$, then $\omega_\alpha_1, \ldots, \omega_\alpha_g$ is a basis for $\Omega^1$.

The result concerning $\bar{\Omega}$ follows from Theorem 4. While the function $f^{(i)}$ depend on the index $i$, we find that $df^{(i)}_j - df^{(i)}_j$ are in the intersection $S_i \cap S_j$, the differential of a $\Gamma$-invariant function and thus the cohomology class of $df^{(i)}_j$ equals the cohomology class of $df^{(i)}_j$. Thus they constitute a basis element of $\bar{\Omega}$.

**Theorem 7.** - The restriction $\nabla|\bar{\Omega}$ of $\nabla$ onto $\bar{\Omega}$ is trivial, i.e. there is a basis of horizontal elements in $\bar{\Omega}$.

**Sketch of proof:** The result is local in nature. If $\mathcal{E} = (S_i)$ is an admissible covering of $S$ and if we have proved the result for the family over $S_i$ for all $i$, the proof is complete.

Using Theorem 5 we may therefore assume that there are function $f_1, \ldots, f_g \in I$ such that $P(f_j)(\alpha_j) = \delta_{ij}$, where $\alpha_1, \ldots, \alpha_g$ is a basis of $\Gamma$. We have to show that $\nabla_D(df^{(i)}_i) = 0$ where $df^{(i)}_i$ is the cohomology class of $df^{(i)}_i$ in $H^1_{DR}$. Now by the very definition of $\nabla_D$ we know that $\nabla_D(df^{(i)}_i) = d(df^{(i)}_i)$ where $D$ is an extension of the derivation $D$ to the field of meromorphic function on $M$ with $D(x) = 0$ for a meromorphic function $x$ on $M$ which is not a meromorphic function on $S$. (is not constant on all the curves of the family $M \to S$).
We are done if we can show that $\hat{\Delta}f_i$ is $\Gamma$-invariant. This seems obvious as 

$$(\hat{\Delta}f_i) \cdot \alpha_j = \hat{\Delta}(f_i \cdot \alpha_j) = \hat{\Delta}(f_i + \delta_{ij}) = \hat{\Delta}(f_i).$$

The problem with this argument is that $\hat{\Delta}$ is defined only on the field of meromorphic functions of $M$ and $f_i$ is not in it. But one can define a unique extension of $\hat{\Delta}$ to a vector field on $Z$ which does justify the above line of argument as soon as we have shown

$$(\hat{\Delta}f_i) \cdot \alpha = \hat{\Delta}(f_i \cdot \omega).$$

But $D'(f) := (\hat{\Delta}(f \cdot \omega)) \cdot \alpha^{-1} - \hat{\Delta}(f)$ is an analytic vector field on $Z$ with $D'(f) \equiv 0$ for all meromorphic functions on $M$. Thus $D' \equiv 0$ and

$$(\hat{\Delta}f_i) \cdot \alpha = \hat{\Delta}(f_i \cdot \omega).$$

4. Elliptic case.

The first nontrivial example is the family of Tate curves which has been studied by a number of authors, see for example [R], [Rb], [K], [DR].

Assume that $\text{char } K \neq 2$.

$$S = \{q \in K : 0 < |q| < 1\}$$

$$Z = \{(q, z) \in K^2 : q \in S, z \in K - \{0\}\}$$

$\alpha(q , z) := (q , qz)$ is a bi analytic map $Z \rightarrow Z$. Let $\Gamma$ be the transformation group generated by $\alpha$. Then $M = Z/\Gamma \rightarrow S$ is the universal family of Tate curves.

The de Rham cohomology space $H^1_{\text{DR}}$ for the family $M \rightarrow S$ is freely generated over the structure sheaf on $S$ by the class $\tau_1$ of the analytic differential $(dz/z)$ and by the class $\tau_2$ of the meromorphic differential $d\xi$ where

$$\xi(q , z) = \frac{1}{1 - z} + \sum_{n=1}^{\infty} \left( \frac{1}{1 - q^n z} - \frac{1}{1 - q^n z^{-1}} \right)$$

$$= \frac{1}{1 - z} + \sum_{n=1}^{\infty} \left( \frac{q^n z}{1 - q^n z} - \frac{q^n z^{-1}}{1 - q^n z^{-1}} \right)$$

for which holds

$$\xi(q , qz) - \xi(q , z) = 1$$

$$\xi(q , z^{-1}) = 1 - \xi(q , z)$$

$$\xi(q , -1) = \frac{1}{2} \text{ if char } K \neq 2$$

$$\xi(q , q^n) = 1 \text{ if } q^n = q.$$
\[ \tilde{\phi} = z \frac{\partial \tilde{\phi}}{\partial z} \]

\[ \tilde{\phi}' = z \frac{\partial \tilde{\phi}}{\partial z} . \]

Then \( \tilde{\phi} \), \( \tilde{\phi}' \) are \( \Gamma \)-invariant meromorphic functions on \( Z \) and the following equation holds

\[ \tilde{\phi}'^2 = 4(\tilde{\phi} - e_1)(\tilde{\phi} - e_2)(\tilde{\phi} - e_3) \]

where \( e_1 = \tilde{\phi}(q, -1) \), \( e_2 = \tilde{\phi}(q, \pi) \), \( e_3 = \tilde{\phi}(q, -\pi) \) with \( \pi \) a fixed square root of \( q \).

If we put

\[ x := \frac{\tilde{\phi} - e_1}{e_2 - e_1} \]

\[ y := \frac{\tilde{\phi}'}{2(e_2 - e_1)^{3/2}} \]

then

\[ y^2 = x(x - 1)(x - \lambda) \]

with

\[ \lambda = \frac{e_3 - e_1}{e_2 - e_1} = x(q, -\pi) \]

which is the Legendre normal form for the family of Tate curves.

Let

\[ D_q := \frac{\partial \tilde{\phi}}{\partial q} - \frac{x}{x'} \frac{\partial \tilde{\phi}}{\partial z} \]

where

\[ \tilde{x} = \frac{\partial x}{\partial q} , \quad \tilde{x}' = \frac{\partial x}{\partial z} . \]

We claim that the vector field \( D_q \) coincides with the vector field \( \hat{D} \) for \( D = (\partial / \partial q) \) in the proof of Theorem 7.

\[ D_q (x) = 0 = \hat{D}(x) \]

\[ D_q (f) = \frac{\partial f}{\partial q} = \hat{D}(f) \] if \( f \) is analytic on \( S \).

Thus \( \hat{D} = D_q \).

Let \( v = \left( \nabla q / \partial q \right) \). Then \( v(f dx) = D_q (f) dx \) by definition of \( \left( \nabla q / \partial q \right) \).

One can by direct computation show that

\[ v(df) = d(D_q (f)) \]

and that \( D_q (\xi) = (\partial \xi / \partial q) - (\xi / x')(\partial \xi / \partial z) \) is \( \Gamma \)-invariant.
This proves that $V(T_2) = 0$ which gives a more direct proof of Theorem 7 for the family of Tate curves.

Let $\sigma_1$ (resp. $\sigma_2$) be the cohomology class of $(dx/2y)$ (resp. $x(dx/2y)$).

Then $\sigma_1, \sigma_2$ is a basis of $H^1_{\text{DR}}$. Let

$$\frac{d\xi}{\xi} = \tau_2 = A\sigma_1 + B\sigma_2.$$ 

**THEOREM 8.**

$$A = \frac{\psi(q, -1)}{\sqrt{\psi(q, n) - \psi(q, -1)}}$$

$$B = \sqrt{\psi(q, n) - \psi(q, -1)}$$

and $\frac{A}{B}$ as a function of $\lambda$ can be given by

$$\frac{A}{B} = 2\lambda[(1 - \lambda) \frac{F'}{F} + \frac{1}{2}]$$

where

$$F(\lambda) = F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \lambda\right)$$

$$= \sum_{n=0}^{\infty} \left(-\frac{(1/2)_n}{n!}\right)^2 (1 - \lambda)^n.$$ 

Sketch of proof: The proof of the first part is given by a small computation. One can use the characterization of elements $\tau$ in $H^1_{\text{DR}}$ with $V(\tau) = 0$ given in [P], (7.11), (ii), to prove the second part.

We find that $\tau_2 = \lambda(1 - \lambda) \frac{\partial F}{\partial \lambda} \sigma_1 - \lambda(1 - \lambda) F(\sigma_1)$ where $f$ satisfies the hypergeometric equation

$$\lambda(1 - \lambda) \frac{\partial^2 f}{\partial \lambda^2} + (1 - 2\lambda) \frac{\partial f}{\partial \lambda} - \frac{1}{4} f = 0.$$ 

Here one has to use the fact that the map $\pi : \lambda(\tau) = x(q, -\tau)$ gives a bi-analytic map from $S$ onto $\{\lambda : |1 - \lambda| < 2\}$.

Thus the inverse mapping $\pi(\lambda)$ is an analytic function of $\lambda$.

Now we conclude that $f = cF(\lambda)$ as $f$ is analytic on $\{\lambda : |1 - \lambda| < 2\}$ with a constant $c \in K^*$ which can be determined by letting $\lambda - 1$ (i.e. $\tau = 0$).

**REFERENCES**


